4/17/15 MAE182A TA Session #3

Chap3 2nd order D.E.

Goals:
- Homogeneous eqs
  - Superposition
  - Wronskian
  - Complex roots
  - Repeated roots
- Nonhomogeneous Eqs
  - Undetermined Coeff.
  - Variation of parameters
- Mechical Vibrations
- Forced Vibrations

- 2nd order D.E. : $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$

linear. $y'' = f(t, y, y') = g(t) - p(t)\,y' - q(t)\,y$

$\Rightarrow \quad y'' + p(t)\,y' + q(t)\,y = g(t)$ or $P(t)\,y'' + Q(t)\,y' + R(t)\,y = G(t)$ (1)

$\begin{cases} g(t) = 0 \Rightarrow \text{homogeneous} \\ g(t) \neq 0 \Rightarrow \text{non homogeneous} \end{cases}$

ICs: $y(t_0) = y_0$

$y'(t_0) = y_0'$

- Constant coeff^s.

$(1) \Rightarrow \quad ay'' + by' + cy = 0$

$\boxed{y = e^{rt}} \Rightarrow (ar^2 e^{rt} + bre^{rt} + ce^{rt}) = 0$

$\Rightarrow (ar^2 + br + c)\, \underset{\neq 0}{e^{rt}} = 0$

$\Rightarrow ar^2 + br + c = 0$ (char. eq.)

$\Rightarrow r_1, r_2 = ?$ (2 roots → 2 solutions)

$y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t} \xrightarrow{\text{superposition}} \boxed{y = c_1 y_1 + c_2 y_2}$

- Linear diff. operator L

$L(y) = y'' + p(t)\, y' + q(t)\, y$

[Theorem] Existence and Uniqueness (p.146)

I.V.P. $\quad y'' + p(t)\, y' + q(t)\, y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$

if p, q, g are continuous on interval I containing t_0,

there is exactly one solution axisting throughout the interval I

[Theorem] Superposition (p.147)

if y_1 & y_2 are 2 solutions of $L[y] = y'' + p(t)\, y' + q(t)\, y = 0$ —(2)

$y = c_1 y_1 + c_2 y_2$ is also a solution for $\forall c_1, c_2$

$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$ (result of linearity of D.E.!)

- Wronksian

$$y = c_1 y_1(t) + c_2 y_2(t) \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

$$y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \quad \text{— ①}$$

$$y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \quad \text{— ②}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix} = W_1}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix} = W_2}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W}$$

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1 y_2' - y_1' y_2$$

if $W \neq 0$, ① & ② have a unique solution $(c_1, c_2) \Rightarrow y = c_1 y_1 + c_2 y_2$

if $W = 0$, $(W_1 \neq 0$ or $W_2 \neq 0)$, no solutions

if $W = 0$, $(W_1 = 0$ & $W_2 = 0)$, many solutions

$y = c_1 y_1 + c_2 y_2$ - general solution

y_1 & y_2 form a fundamental set of solutions iff $W \neq 0$

• Complex roots of char. eq.

$$ay'' + by' + cy = 0$$

$$\boxed{y = e^{rt}} \Rightarrow ar^2 + br + c = 0$$

$$\Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

if $b^2 - 4ac < 0$, $r_{1,2} = \lambda \pm i\mu$; $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

Euler's formula : $e^{it} = \cos t + i \sin t$

$$e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos\mu t + i \sin\mu t)$$

[Theorem 3.2.6] (p.153)

$L(y) = y'' + p(t)y' + q(t)y = 0$, p & q are real-valued fun.

if $y = u(t) + iv(t)$ is a complex valued solution,

$u(t)$ & $v(t)$ are also solutions of it

$$\cancel{y_1 = e^{r_1 t} = e^{(\lambda \pm i\mu)t} = e^{\lambda t}\cos\mu t + i\, e^{\lambda t}\sin\mu t}$$

[Theorem 3.2.7] Abel's Theorem (p.154)

y_1 & y_2 are solutions of $L[y] = y'' + p(t)y' + q(t)y = 0$,

p & q are continuouse on I,

$$W(y_1, y_2)(t) = c\exp\left[-\int p(t)\,dt\right]$$

Abel's Theorem

<pf>

$$y_1'' + p(t) y_1' + q(t) y_1 = 0 \quad — ①$$

$$y_2'' + p(t) y_2' + q(t) y_2 = 0 \quad — ②$$

$$① \times (-y_2) \Rightarrow \quad -y_2 y_1'' + p(t) y_1' (-y_2) + q(t) \cancel{y_1 (-y_2)} = 0$$

$$② \times y_1 \Rightarrow +) \quad y_1 y_2'' + p(t) y_1 y_2' + q(t) \cancel{y_1 y_2} = 0$$

$$\boxed{\begin{array}{l} W = y_1 y_2' - y_1' y_2 \\ \Rightarrow W' = y_1 y_2'' - y_1'' y_2 \end{array}} \quad (\underbrace{y_1 y_2'' - y_2 y_1''}) + (\underbrace{y_1 y_2' - y_1' y_2}) p(t) = 0$$

$$\Rightarrow \quad W' + p(t) W = 0$$

$$\Rightarrow \int \frac{dW}{W} = \int -p(t)\, dt \Rightarrow \ln|W| = \int -p(t)\, dt \Rightarrow \boxed{W = c \exp\left[-\int p(t) dt\right]}$$

$$y_1 = e^{r_1 t} = e^{(\lambda \pm i\mu)t} = e^{\lambda t}(\cos\mu t + i \sin\mu t) = \underbrace{e^{\lambda t}\cos\mu t}_{u(t)} + i \underbrace{e^{\lambda t}\sin\mu t}_{v(t)}$$

$$\Rightarrow \begin{cases} y_1 = u(t) + i v(t) \\ y_2 = u(t) - i v(t) \end{cases} \Rightarrow u(t) \,\&\, v(t) \text{ are also solutions}$$

$$W(u,v) = uv' - u'v = (e^{\lambda t}\cos\mu t)(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t)$$
$$- (\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) e^{\lambda t}\sin\mu t$$

$$\Rightarrow \quad W(u,v) = \mu e^{2\lambda t}$$

$\mu \neq 0 \Rightarrow W(u,v) \neq 0 \Rightarrow u \,\&\, v$ form a fundamental set of solutions

$\Rightarrow$ general solution : $y = c_1 u + c_2 v = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$

$$\Rightarrow y = e^{\lambda t}(c_1 \cos\mu t + c_2 \sin\mu t)$$

Repeated roots $r_1 = r_2$, $b^2-4ac=0$ (p. 169) F6

$\Rightarrow r_1 = r_2 = -\frac{b}{2a}$, $\Rightarrow y_1 = e^{-\frac{b}{2a}t}$

$y = e^{r_1 t}$, $y = v(t)\, e^{r_1 t}$ also a solution

$y' = v' e^{r_1 t} + v r_1 e^{r_1 t}$

$y'' = v'' e^{r_1 t} + \underbrace{v' r_1 e^{r_1 t} + v' r_1 e^{r_1 t}}_{2 v' r_1 e^{r_1 t}} + v r_1^2 e^{r_1 t}$

$a y'' + b y' + c y = 0$

$\Rightarrow \Big[\underset{\neq 0}{a} v'' + (\underbrace{2 a r_1 + b}_{=0}) v' + (\underbrace{a r_1^2 + b r_1 + c}_{=0}) v\Big] \underset{\neq 0}{e^{r_1 t}} = 0$

$\Rightarrow \begin{cases} 2ar_1 + b = 2a\left(-\frac{b}{2a}\right) + b = 0 \\ ar_1^2 + br_1 + c = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} = \frac{-(b^2-4ac)}{4a} = 0. \end{cases}$

$\Rightarrow \underset{\neq 0}{a} v'' = 0 \Rightarrow v'' = \frac{d}{dt}\left(\frac{dv}{dt}\right) = 0 \Rightarrow \frac{dv}{dt} = \text{const} \Rightarrow v = c_1 + c_2 t$

$\therefore y = v e^{r_1 t} = (c_1 + c_2 t) e^{r_1 t} = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

· Nonhomogeneous Eqs

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad \text{— (3)}$$

[Theorem 3.5.1] p.176

if Y_1 & Y_2 are solutions to the nonhomog. eqs ($y'' + py' + qy = g$)

~~$Y_1(t) - Y_2(t)$~~ y_1 & y_2 are a fund.l set of solutions

of $y'' + p(t)y' + q(t)y = 0$

then $Y_1 - Y_2 = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homog.}}$

<pf> $L[Y_1] = g(t)$, $L[Y_2] = g(t)$

$L[Y_1] - L[Y_2] = g(t) - g(t) = 0$

$\Rightarrow L[Y_1 - Y_2] = 0.$

[Theorem 3.5.2]

General solution to nonhomog. eq.

$$y(t) = \phi(t) = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homog. solut.}} + \underbrace{Y_p(t)}_{\text{particular solution}}$$

How to fin $Y_p(t)$?

Undetermined coeff.

- choose solution of the same form as nonhomog. term $g(t)$
  - polynomial
  - exponential
  - cos, sin
- if Y_p has the same form as homog solution,

multiply by t until there is no duplication

$L[y]$	$[Y_p]$
$5t^2$	At^2+Bt+C
$\cos 3t$	$A\cos 3t + B\sin 3t$
$8e^{5t}$	Ae^{5t}
$t^2 \sin 2t$	$(At^2+Bt+C)(\cos 2t+\sin 2t)$
$t+e^{5t}$	$(At+B)+Ce^{5t}$

ex Pr. 17. p. 184

$y'' - 2y' + y = te^t + 4.$

1° Homoge. sol: $y'' - 2y' + y = 0$

$y = e^{rt} \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r_{1,2} = 1$

$y_h = c_1 e^t + c_2 te^t$

2° $g_1(t) = te^t$

Guess $Y_{p_1} = (At + B)e^t$, but te^t is a homog. sol.

$\xrightarrow{*t^2}$ $Y_{p_1} = (At^3 + Bt^2)e^t$

$Y_{p_1}' = (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t$

$Y_{p_1}'' = (6At + 2B)e^t + (3At^2 + 2Bt)e^t + (3At^2 + 2Bt)e^t + (At^3 + Bt^2)e^t$

$\Rightarrow Y_{p_1} = \dfrac{t^3 e^t}{6}$ $(A = \frac{1}{6}, B = 0)$

3° $g_2(t) = 4$

$Y_{p_2} = c \Rightarrow 0 - 0 + c = 4 \Rightarrow c = 4$

$\therefore Y = y_h + y_{p_1} + y_{p_2} = c_1 e^t + c_2 te^t + \dfrac{t^3 e^t}{6} + 4.$

• Variation of Parameters

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{—— [4]}$$

Const. coeff. $\Rightarrow$ $ay'' + by' + cy = 0$

general sol. $\Rightarrow$ $y = c_1 y_1(t) + c_2 y(t)$

$*$ $p(t)$ & $q(t)$ are not consts.

[Note] $p(t)$ & $q(t)$ are not consts but the fns of t

[theorem 3.6.1] (p.189)

$$y = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

$\Rightarrow y'', y'$

$$\Rightarrow \quad u_1(t) = -\int \frac{y_2 \cdot g(t)}{W(y_1, y_2)}\, dt + c_1$$

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)}\, dt + c_2$$

a particular solution for [4]

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2\, g(s)}{W(y_1, y_2)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1, y_2)}\, ds$$

General solution:

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

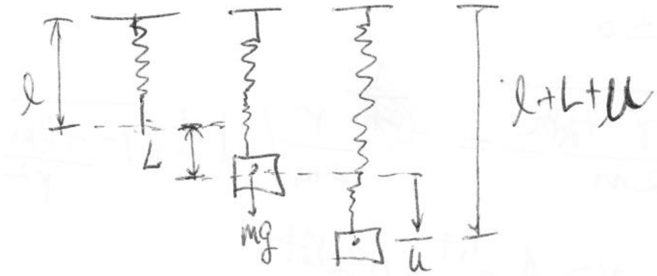
ex. $5y'' + t y' + 7 = t^2 \Rightarrow \cancel{g(t) = \frac{t^2}{5}}$

$$y'' + \left(\frac{t}{5}\right) y' + \frac{7}{5} = \frac{t^2}{5} = g(t)$$

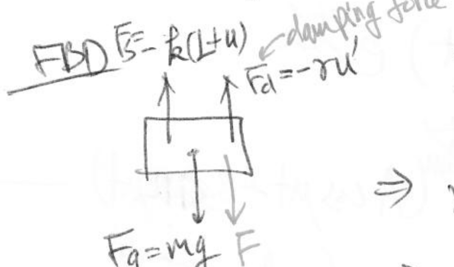
[Note] $g(t) = \frac{t^2}{5}$

$g(t) \neq t^2$!

Mechanical Vibrations



$mg = kL$



$$\sum_i F = \vec{F_g} + \vec{F_s} + \vec{F_d} + \vec{F}$$

$$\Rightarrow m\ddot{u} = mg - k[L+u] - \gamma u' + F(t)$$

$$\Rightarrow m\ddot{u} = -ku - \gamma u' + F(t)$$

$$\Rightarrow m\ddot{u} + \gamma u' + ku = F(t)$$

Free vibration. $F(t) = 0 \Rightarrow m\ddot{u} + \gamma \dot{u} + ku = 0$

[case1] Undamped vibrations $F(t) = 0$, $\gamma = 0$

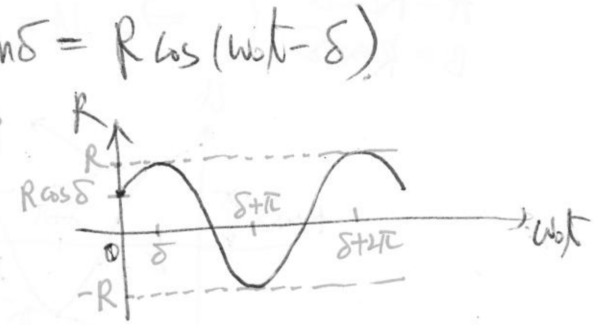
$$m\ddot{u} + ku = 0$$

$$u = e^{rt} \Rightarrow mr^2 + k = 0 \Rightarrow r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0.$$

$$\Rightarrow u = A\cos\omega_0 t + B\sin\omega_0 t \quad \text{where } \omega_0^2 = \frac{k}{m}$$

$$\boxed{\begin{aligned} A &= R\cos\delta \\ B &= R\sin\delta \end{aligned}} \Rightarrow u = R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta = R\cos(\omega_0 t - \delta)$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{amplitude}}}{R} = \sqrt{A^2+B^2}, \quad \tan\underset{\substack{\uparrow \\ \text{phase}}}{\delta} = \frac{B}{A}$$



period: $T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}}$, ω_0 = natural freq.

[Case 2] Damped $\gamma \neq 0$ (p13)

$$m u'' + \gamma u' + k u = 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

$$\begin{cases} \gamma^2 - 4km > 0 \Rightarrow u = Ae^{r_1 t} + Be^{r_2 t} \\ \gamma^2 - 4km = 0 \Rightarrow u = (A + Bt)\, e^{-\gamma t/2m} \\ \gamma^2 - 4km < 0 \Rightarrow u = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t) \end{cases}$$

$$\mu = \frac{(4km - \gamma^2)^{\frac{1}{2}}}{2m} > 0$$

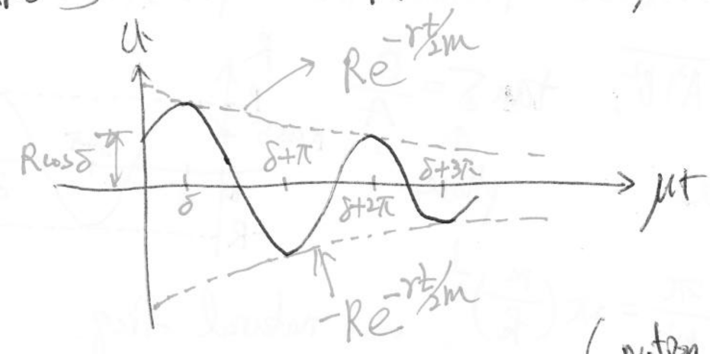
$\gamma^2 - 4km \geq 0$, $\frac{\gamma^2 - 4km}{\gamma^2} < \frac{\gamma^2}{\gamma^2} = 1 \Rightarrow \sqrt{\frac{\gamma^2 - 4km}{\gamma^2}} < 1 \Rightarrow r_1, r_2 < 0$

$\gamma^2 - 4km < 0$ $\quad \frac{-\gamma}{2m} < 0 \Rightarrow$ real part < 0

all 3 solutions: $t \to \infty$, $u \to 0$

$$\gamma^2 - 4km < 0 \Rightarrow u = e^{-\gamma t}(A\cos\mu t + B\sin\mu t)$$

$$\left.\begin{array}{l} A = R\cos\delta \\ B = R\sin\delta \end{array}\right\} \Rightarrow u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$



quasi freq μ

$$\frac{\mu}{\omega_0} = \frac{(4km - \gamma^2)^{\frac{1}{2}}/2m}{(k/m)^{\frac{1}{2}}} = \left(1 - \frac{\gamma^2}{4km}\right)^{\frac{1}{2}} \approx 1 - \frac{\gamma^2}{8km}$$

(motion not period but oscillating at μ)

- Forced Vibration (§3.8)

$$mu''(t) + \gamma u' + k u = F(t) = F_0 \cos\omega t$$

$$u = c_1 u_1(t) + c_2 u_2(t) + A\cos\omega t + B\sin\omega t$$

$$= \underset{\text{transient}}{u_c(t)} + \underset{\text{steady}}{U(t)}$$

- transient homogen. solution dies out but it allows us to satisfy ~~on~~ i.c.s
- steady solution.

$$U(t) = R\cos(\omega t - \delta)$$

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}$$

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad \omega_0^2 = \frac{k}{m}.$$

$$\frac{Rk}{F_0} = \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right]^{1/2}}, \quad \Gamma = \frac{\gamma^2}{mk}$$

[5]

low freq: $\omega \to 0$. $R = \frac{F_0}{K}$

high freq: $\omega \to \infty$. $R \to 0$

Max amplitude, $\omega = \omega_{max} = ?$

$$\frac{d}{dw}\left[\frac{Rk}{F_0}\right] = 0$$

$$\omega_{max}^2 = \omega_0^2 - \frac{r^2}{2m^2} = \omega_0^2\left(1 - \frac{r^2}{2mk}\right)$$

[Note] 1. $\omega_{max} < \omega_0$

2. $r \ll 1 \quad \omega_{max} \approx \omega_0$

3. $R_{max} = \dfrac{F_0}{r\omega_0\sqrt{1 - \dfrac{r^2}{4mk}}} \cong \dfrac{F_0}{r\omega_0}\left(1 + \dfrac{r^2}{8mk}\right)$

4. $\cancel{\dfrac{\omega}{\omega_0} \approx 1}$ $r \ll 1$. $R_{max} \approx \dfrac{F_0}{r\omega_0}$

~~F_0 is small, but R_{max} is quite large~~

r is small $\Rightarrow$ $R_{max} \uparrow$ (Resonance)