4/10/15 MAE 182A TA Session #2

Goals:

- Integrating factor $\mu(t)$
- Separable Eqs
- Exact Eqs
- Not exact eqs $\xrightarrow{*\mu(t)}$ Exact eqs.

. Integrating factor $\mu(t)$

1st order linear D.E.: $\frac{dy}{dt} + p(t)y = g(t)$ — (1)

* if (1) cannot separate the variables & integrate directly, we can find an integrating factor $\mu(t)$, s.t.

$$\frac{d}{dt}[\mu(t)y] = \mu(t)y' + \frac{d\mu(t)}{dt}y \quad \text{—(2)}$$

ex. $\frac{dy}{dt} + p(t)y = g(t)$

$$\xRightarrow{*\mu(t)} \mu(t)\frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t) \quad \text{— (3)}$$

Compare (2) & (3)

$$\frac{d\mu(t)}{dt} = p(t)\mu(t)$$

$$\Rightarrow \int \frac{d\mu(t)}{\mu(t)} = \int p(t)\,dt \Rightarrow \boxed{\mu(t) = \underset{1}{\underset{\sim}{c}}e^{\int p(t)dt}}$$

(3) $\Rightarrow$ $\frac{d}{dt}[\mu(t)y] = g(t)\mu(t)$

$$\Rightarrow \int d[\mu(t)y] = \int g(t)\mu(t)\,dt$$

ex Pr 15, P.40

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \ t > 0$$

$$t \neq 0 \overset{/t}{\Rightarrow} y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$$

$$\overset{\times \mu(t)}{\Rightarrow} \mu y' + \underbrace{\frac{2}{t}\mu y}_{\shortparallel} = \mu\left(t - 1 + \frac{1}{t}\right) \quad \text{— (1)}$$

$$\frac{d}{dt}[\mu y] = \mu y' + \mu' y$$

$$\Rightarrow \quad \mu' = \frac{2}{t}\mu$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{2}{t} dt$$

$$\Rightarrow \ln|\mu| = 2\ln|t| + c' = \ln(ct^2)$$

$$\Rightarrow \quad \mu = \overset{=1}{c}\, t^2 = t^2$$

$$(1) \Rightarrow \underbrace{t^2 y' + t^2 \frac{2}{t} y}_{\shortparallel} = t^2\left(t - 1 + \frac{1}{t}\right)$$

$$\Rightarrow \quad \frac{d}{dt}(t^2 y) = t^3 - t^2 + t$$

$$\Rightarrow \int d(t^2 y) = \int (t^3 - t^2 + t)\, dt$$

$$\Rightarrow \quad t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

$$\Rightarrow \quad y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

I.C. $y(1) = \frac{1}{4} - \frac{1}{3} + \cancel{\frac{1}{2}} + c = \cancel{\frac{1}{2}} \Rightarrow c = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$$\Rightarrow \quad y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

- Separable Eqs.

1st order DE: $\dfrac{dy}{dx} = f(x, y)$

non-linear eq.: $M(x,y) + \underbrace{N(x,y)}_{\text{non-linear}} \dfrac{dy}{dx} = 0$

if $\left.\begin{matrix} M = M(x) \\ N = N(y) \end{matrix}\right\} \Rightarrow M(x) + N(y)\dfrac{dy}{dx} = 0$

$\Rightarrow M(x)dx + N(y)\,dy = 0$

$\Rightarrow \int M(x)dx = \int -N(y)dy$ (separable)

ex Pr3 P.48

$$y' + \underbrace{y^2}_{\text{non-linear}}\sin x = 0$$

$$\Rightarrow \frac{dy}{dx} = -y^2 \sin x$$

$$\Rightarrow \int \frac{dy}{y^2} = \int -\sin x\, dx$$

$$\Rightarrow y(x) = ?$$

· Modeling w/ 1st order D.E.

ex · Pr 18 p.63

Newton's law of cooling:

$T(t)$ $\boxed{u(t)}$

$$\frac{du}{dt} = -k[u - T(t)] \quad —(1)$$

1° · $\frac{du}{dt} = -k[u - T(t)]$ separable?

$$\int \frac{du}{u - T(t)} \neq \int -k\,dt$$

2° integrating factor $\mu(t)$

$$\frac{d}{dt}[\mu(t)\, u] = \mu \frac{du}{dt} + u \cdot \frac{d\mu}{dt} \quad —(2)$$

$$(1) \overset{*\mu(t)}{\Rightarrow} \mu \frac{du}{dt} = -k\mu[u - T(t)]$$

$$\Rightarrow \mu \frac{du}{dt} + (k\mu) u = k\mu T(t) \quad —(3)$$

compare (2) & (3) $\Rightarrow \frac{d\mu}{dt} = k\mu \Rightarrow \int \frac{d\mu}{\mu} = \int k\,dt \Rightarrow \boxed{\mu = e^{kt}}$

$$(3) \Rightarrow \frac{d}{dt}[\mu u] = k\mu T, \quad T = T_0 + T_1 \cos\omega t$$

$$\Rightarrow \frac{d}{dt}[e^{kt} u] = k e^{kt} T(t) = k[T_0 + T_1 \cos\omega t] e^{kt}$$

$$\Rightarrow e^{kt} u = T_0 e^{kt} + kT_1[k\cos\omega t + \omega \sin\omega t] e^{kt} + C.$$

$$\Rightarrow u(t) = T_0 + kT_1(k\cos\omega t + \omega\sin\omega t) + Ce^{-kt}$$

- Exact eqs

$$M(x,y) + N(x,y)\, y' = 0 \quad -(1)$$

(1) is exact eq $\iff$ $\boxed{M_y(x,y) = N_x(x,y)}$

$$\exists\ \psi(x,y) \ni \quad \psi_x(x,y) = M(x,y)$$
$$\psi_y(x,y) = N(x,y)$$

solution $\Rightarrow$ $\psi(x,y) = c.$

[Note]

$$\frac{d\psi}{dx} = \underbrace{\psi_x}_{M(x,y)} \overbrace{\frac{dx}{dx}}^{1} + \underbrace{\psi_y}_{N(x,y)} \frac{dy}{dx} = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\psi_x = \psi_{xy} \\ N_x\, \cancel{N_y} &= \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\psi_y = \psi_{yx} \end{aligned}\right\} \Rightarrow \psi_{xy} = \psi_{yx}$$

$$\Rightarrow \underline{M_y = N_x}$$

$$\underbrace{(2x+3)}_{M(x,y)} + \underbrace{(2y-2)}_{N(x,y)}y' = 0$$

$$\left.\begin{array}{l} M = 2x+3 \Rightarrow M_y = 0 \\ N = 2y-2 \Rightarrow N_x = 0 \end{array}\right\} \to M_y = N_x \Rightarrow \text{exact eq.}$$

$$\psi_x = \frac{\partial \psi}{\partial x} = M = 2x+3$$

$$\Rightarrow \quad \psi = \int (2x+3)\,dx = x^2+3x+f(y) \quad — ①$$

$$N = \psi_y = \frac{\partial \psi}{\partial y} = 2y-2$$

$$\cancel{\Rightarrow \psi = \int (2y-2)\,dy = y^2-2y+g(x)}$$

$$\psi_y = \frac{\partial \psi}{\partial y} = f'(y) = N = 2y-2$$

$$\Rightarrow \quad f'(y) = 2y-2$$

$$\Rightarrow \quad f(y) = y^2-2y+g(x)$$

$$① \Rightarrow \psi = x^2+3x+f(y) = x^2+3x+y^2-2y+g(x) \quad — ②$$

$$\psi_x = 2x+3+g'(x) = M = 2x+3$$

$$\Rightarrow \quad g'(x) = 0 \Rightarrow g(x) = c''$$

$$② \Rightarrow \psi = x^2+3x+y^2-2y+c'' = c'$$

$$\Rightarrow \quad \underline{x^2+3x+y^2-2y = c}$$

· Not exact eq $\xrightarrow{\mu(t)}$ exact eq.

$$M(x,y) + N(x,y)\,y' = 0 \quad -(1)$$

if $M_y \neq N_x$, not exact eq.

$$(1) \overset{*\mu(t)}{\Longrightarrow} \underbrace{\mu M(x,y)}_{\overset{\shortparallel}{\tilde{M}}} + \mu \underset{\overset{\shortparallel}{\tilde{N}}}{N(x,y)}\, y' = 0 \quad -(2) \text{ (exact eq.)}$$

$\Rightarrow$ (2) is exact eq $\Leftrightarrow$ $\tilde{M}_y = \tilde{N}_x$

$$\Rightarrow (\mu M)_y = (\mu N)_x$$

$$\Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x \quad -(3)$$

$$\Rightarrow \mu_y M - \mu_x N + (M_y - N_x)\mu = 0 \qquad \text{not easy to solve!}$$

[case 1] $\mu = \mu(x)$, $\mu_y = 0$

$$(3) \Rightarrow \mu M_y = \mu_x N + \mu N_x$$

$$\Rightarrow N \frac{d\mu}{dx} = (M_y - N_x)\mu$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{M_y - N_x}{N}\, dx \Rightarrow \mu(x) = ?$$

[case 2] $\mu = \mu(y)$, $\mu_x = 0$

$$(3) \Rightarrow \mu_y M + \mu M_y = \mu N_x$$

$$\Rightarrow M \frac{d\mu}{dy} = \mu (N_x - M_y)$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{N_x - M_y}{M}\, dy \Rightarrow \mu(y) = ?$$

ex Pr 25 P.102

$$\underbrace{(3x^2y+2xy+y^3)}_{M}+\underbrace{(x^2+y^2)}_{N}\,y'=0 \quad \text{—(1)}$$

1°
$$\left.\begin{cases} M(x,y)=3x^2y+2xy+y^3 \Rightarrow M_y=3x^2+2x+3y^2 \\ N(x,y)=x^2+y^2 \Rightarrow N_x=2x \end{cases}\right\} \Rightarrow M_y \neq N_x$$

not exact eq.

2° $\mu(x)$

$$(1) \overset{\times \mu(x)}{\Longrightarrow} \mu(x)(3x^2y+2xy+y^3)\,dx+\mu(x)(x^2+y^2)\,dy=0$$

$$\frac{d\mu}{\mu}=\frac{M_y-N_x}{N}\,dx=\frac{3x^2+\cancel{2x}+3y^2-\cancel{2x}}{x^2+y^2}\,dx=3\,dx$$

$$\Rightarrow \ln|\mu|=3x+c' \Rightarrow \mu=\underset{1}{\underset{\parallel}{c}}e^{3x}=e^{3x}$$

$$\exists\,\psi \ni \psi_x=\mu M=e^{3x}(3x^2y+2xy+y^3)$$

$$\Rightarrow \psi=\int e^{3x}(3x^2y+2xy+y^3)=\left(x^2y+\frac{y^3}{3}\right)e^{3x}+h(y)$$

$$\psi_y=(x^2+y^2)e^{3x}+h'(y)=\mu N=e^{3x}(x^2+y^2)$$

$$\Rightarrow h'(y)=0 \Rightarrow h(y)=c'$$

$$\psi=\left(x^2y+\frac{y^3}{3}\right)e^{3x}+c'=c''$$

$$\Rightarrow (3x^2y+y^3)e^{3x}=c$$