4/3/15 MAE 182A TA Session #1

Goals:

1. Basic Math. Models : Direction Fields

 1) How to sketch direction field

 2) Equilibrium solutions

2. Solutions of Some D.E.s

 1) Integral curves

 2) Initial conditions

 3) General solution

3. Classification of D.E.s

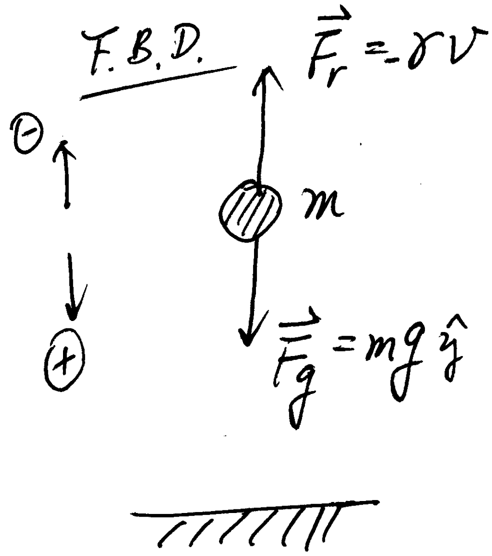
$\begin{cases} \text{O.D.E.} \\ \text{P.D.E.} \end{cases}$

$\begin{cases} \text{linear} \\ \text{non-linear} \end{cases}$ D.E.s

1. Direction fields

Ex 1 Free Fall

F.B.D.



- D.E. to describe the motion of an object falling in the air near sea level.
- Variables: time t, velocity v
- Newton's 2nd law:

$$\sum_i \vec{F}_i = m\vec{a} = m\frac{d\vec{v}}{dt}$$

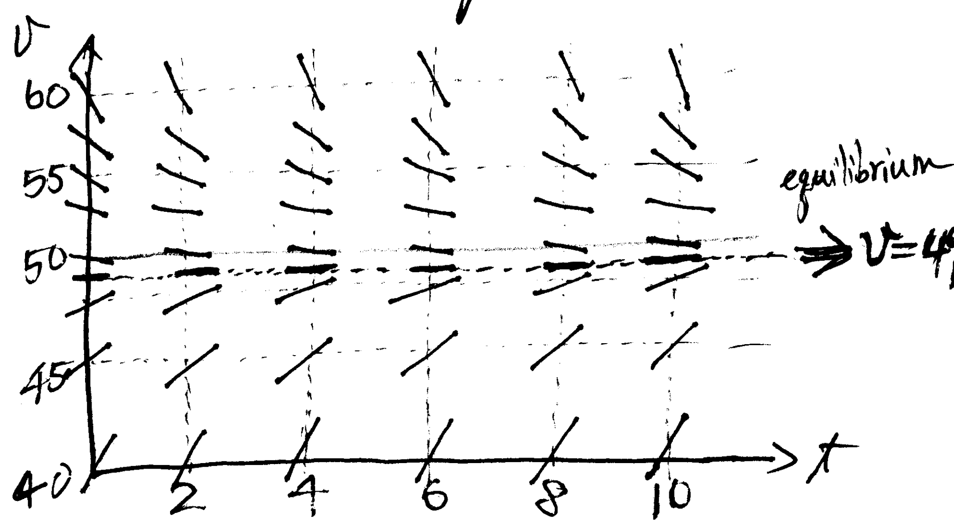
$$\Rightarrow \vec{F}_g + \vec{F}_r = m\frac{d\vec{v}}{dt}$$

$$\Rightarrow mg - \gamma v = m\frac{dv}{dt}$$

$$\boxed{\begin{aligned} m &= 10\,kg \\ \gamma &= 2\,kg/sec \end{aligned}} \Rightarrow \boxed{\underbrace{\frac{dv}{dt}}_{\text{slope}} = g - \frac{\gamma}{m}v = 9.8 - \frac{1}{5}v}$$

1) Sketch direction (slope) field

v	dv/dt
0	9.8
5	8.8
10	7.8
⋮	⋮
40	1.8
45	0.8
50	-0.2
55	-1.2
60	-2.2



2) Equilibrium solution

$$\frac{dv}{dt} = 0 = 9.8 - \frac{1}{5}v \Rightarrow v = 9.8 \times 5 = 49 \ m/s$$

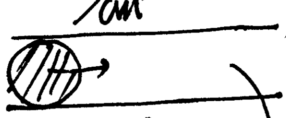
(terminal velocity)

$$\Sigma F = m\frac{dv}{dt} = 0$$

$\Rightarrow$ $v = $ const.

Ex2 drug delivered in patient's bloodstream P.24 (p.9)

density: $\rho = 5 \ mg/cm^3$



volume supply rate: $\gamma = 100 \ cm^3/h$ $\quad \alpha = 0.4/h$

- D.E. for the amount of the drug present in the bloodstream, $\forall t$.
- variables:
  $M(t)$: amount of the drug $\forall t$ (mg)
  t: time
- mass conservation

1°
$$\frac{dM(t)}{dt} = \overbrace{\rho\gamma}^{\text{amount of drug supply/time}} - \overbrace{\alpha M(t)}^{\text{amount of drug absorbed by the body/time}}$$

$\frac{dM(t)}{dt}$: mg/s; ρ: mg/cm³; γ: cm³/h; $\alpha M(t)$: 1/h · mg

$$\Rightarrow \frac{dM(t)}{dt} = 5 \times 100 - 0.4M = 500 - 0.4M(t)$$

2° ~~equili~~ $M = ?$ $\quad t \to \infty$ (equilibrium solution)

$$\frac{dM}{dt} = 0 = 500 - 0.4M$$

$$\Rightarrow M = \frac{500}{0.4} = 1250 \ mg$$

2. Solutions of some D.E.s

free fall: $v' = 9.8 - 0.2v$

drug delivery: $M' = 500 - 0.4M$

mice & owl: $p' = 0.5p - 450$

$\Rightarrow$ general form $\boxed{y' = ay - b}$, $a, b =$ consts. known

$$\frac{dy}{dt} = ay - b = a(y - b/a)$$

Variable separation y, t $\Rightarrow \int \frac{dy}{y - b/a} = \int a\,dt$

$$\Rightarrow \ln|y - b/a| = at + c'$$

$$\Rightarrow y - \frac{b}{a} = \pm e^{at+c'} = \underbrace{\pm e^{c'}}_{=c} e^{at} = ce^{at}$$

$$\Rightarrow \boxed{y = \frac{b}{a} + ce^{at}}$$ (general solution)

c: not unique

1) initial condition $\Rightarrow$ unique solution

$y(0) = y_0 \Rightarrow$ one unknown variable c, one I.C. Don't forget the I.C.s !!!

$$\Rightarrow y(0) = \frac{b}{a} + c = y_0 \Rightarrow c = y_0 - \frac{b}{a}$$

$$y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$$

Ex free fall dropped from 300 m & $v(0)=0$

$$\frac{dv}{dt} = 9.8 - 0.2v = -\frac{1}{5}(v-49)$$

$$\Rightarrow \int \frac{dv}{v-49} = \int -\frac{1}{5}\,dt$$

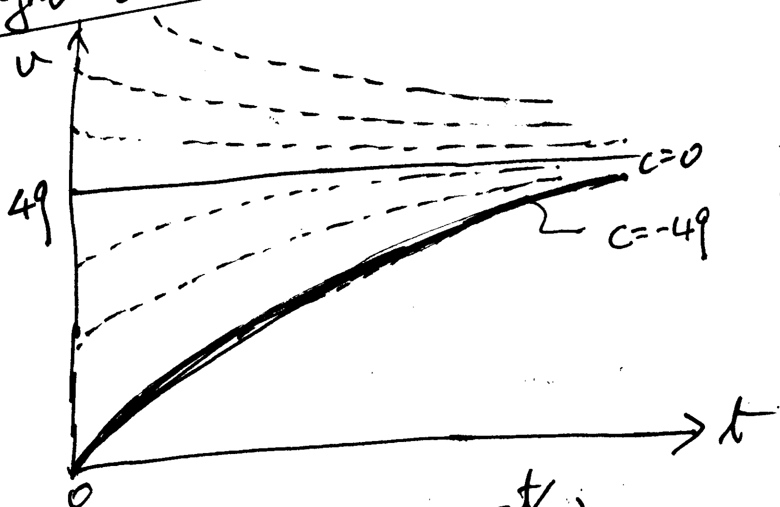
$$\Rightarrow \ln|v-49| = -\frac{1}{5}t + c'$$

$$\Rightarrow v-49 = ce^{-\frac{1}{5}t}$$

$$\Rightarrow v = \cancel{49} \; 49 + \underset{\uparrow}{c}e^{-\frac{1}{5}t}$$

$c=-2,-1,0,1,2$

Integral curves



I.C. $v(0)=0$

$v(0) = 49 + c = 0$

$\Rightarrow c = -49$

$\Rightarrow v = 49(1-e^{-t/5})$

$$v = \frac{dx}{dt} = 49(1-e^{-t/5})$$

$$\Rightarrow \int dx = \int 49(1-e^{-t/5})\,dt$$

$$\Rightarrow x = 49t + 245e^{-t/5} + C$$

I.C. $x(0)=0 \Rightarrow x(0) = 245 + C = 0 \Rightarrow C = -245$

$$\Rightarrow x(t) = 49t + 245e^{-t/5} - 245$$

$x(T) = 300 \Rightarrow 300 = 49T + 245e^{-T/5} - 245 \Rightarrow$ numerical solution

3. Classification of D.E.s

- O.D.E. – fun depends on a single indept. variable

ordinary

$$L \frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{c}Q(t) = E(t)$$

- P.D.E. – fun. depends on several indept. variables

partial

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \quad \text{(heat conduction)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave eq.)}$$

O.D.E: $F(t, y, y', \cdots, y^{(n)}) = 0$

$\Rightarrow$ order $= n$: highest derivative

$$y^{(3)} + 2e^t y'' + yy' = t^4 \quad (\text{order} = 3)$$

- linear v.s. non-linear D.E.s

linear O.D.E: $\boxed{a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)}$

ex. $y'' + y = 0$ $(n=2)$ $\quad y''' + 2y' + 3 = 0$ $(n=3)$

non-linear O.D.E: $y' + \frac{1}{y} = 0$, $\quad y' + y^3 = 0$, $\quad y'' + \cos y = 0$

$$yy' = 2$$