Ch 6 Laplace Transform

Second Order DE

- we now have a third method.

Constant coefficients

- characteristic equation
- $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Variable coefficients

- Series Solution
- $y = \sum_{n=0}^{\infty} C_n (x - x_0)^n$

Laplace Transform useful for equations w/ discontinuous or impulsive forcing terms.

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^{\infty} e^{-st} y(t)\,dt = \lim_{A \to \infty} \int_0^{A} e^{-st} y(t)\,dt$$

- y is piecewise continuous $0 \le t \le A$
- $y(t) \le K e^{at}$

LT exists for $s > a$

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t} = \frac{1}{s-a} \quad s > a$$

$\mathcal{L}$ is a <u>linear operator</u>

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2)$$

Laplace of derivatives

$$\mathcal{L}(y'(t)) = \int_0^A e^{-st} y'(t)\,dt = e^{-sA} y(A) - y(0) + s\int_0^A e^{-st} y(t)\,dt$$

Integration by parts; $e^{-sA}y(A)$ goes to zero as $A \to \infty$; the integral goes to $\mathcal{L}$ as $A \to \infty$

$$= s\mathcal{L}\, y(t) - y(0)$$

$$\mathcal{L}(y''(t)) = s\mathcal{L}(y') - y'(0)$$

$$= s[s\mathcal{L}(y) - y(0)] - y'(0)$$

$$= s^2 \mathcal{L}(y) - s\,y(0) - y'(0)$$

$$\mathcal{L}(y^{(n)}) = s^n \mathcal{L}(y) - s^{n-1} y(0) - \dots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

Solving IVP's

- transfor DE into s-domain
- solve equation algebraicly
- invert back to original domain

$y'' - y' - 2y = 0 \quad y(0) = 1 \quad y'(0) = 0$

$\mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) = 0$ Take LT + seperate terms

$\left(s^2 \mathcal{L}(y) - s y(0) - y'(0)\right) - \left(s\mathcal{L}(y) - y(0)\right) - 2\mathcal{L}(y) = 0$

$(s^2 - s - 2)\, Y(s) + (1-s)\, y(0) - y'(0) = 0$

$y(0) = 1, \quad y'(0) = 0$

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)}$$

Now convert back. Goal is to manipulate $Y(s)$ until its in a recognizable form to use table.

Partial fractions

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)}$$

$s - 1 = a(s+1) + b(s-2)$ must hold for all s

let $s = 2 \rightarrow a = \frac{1}{3}$

$s = -1 \rightarrow b = \frac{2}{3}$

$$Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$$\boxed{y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}}$$

$y'' + y = \sin 2t \qquad y(0) = 2 \quad y'(0) = 1$

$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{2}{s^2+4} \qquad \frac{b}{s^2+b^2}$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$= \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(as+b)(s^2+4) + (cs+d)(s^2+1)}{(s^2+1)(s^2+4)}$$

Numerator

$2s^3 + s^2 + 8s + 6 = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$

$a + c = 2 \qquad b + d = 1$

$4a + c = 8 \qquad 4b + d = 6$

$a = 2 \quad c = 0 \quad b = \frac{5}{3} \quad d = -\frac{2}{3}$

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

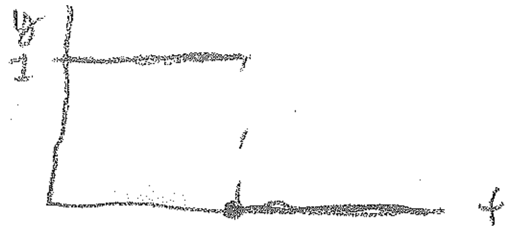
$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Step functions

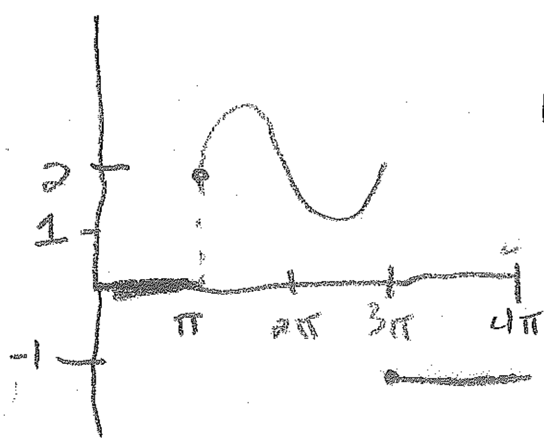
$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

$u_c(t)$

- steps to 1
- steps at c.
- Allows us to deal with piecewise continuous functions



$1 - u_c(t)$



$$y(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 2\sin(t-\pi) & \pi \leq t \leq 3\pi \\ -1 & 3\pi \leq t < 4\pi \end{cases}$$

$$y(t) = 2u_\pi(t)\sin(t-\pi) - 2u_{3\pi}(t)\sin(t-\pi) - u_{3\pi}(t)$$

$$\mathcal{L}(u_c(t)) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$\boxed{\mathcal{L}\,u_c(t) = \frac{e^{-cs}}{s} \quad s > 0}$$

From previous ex. we know

if $y = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$ $\quad y = u_c(t) f(t-c)$

Theorem

$$\mathcal{L}\left(u_c(t)\,y(t-c)\right) = e^{-cs}\,\mathcal{L}(y(t)) = e^{-cs}\,Y(s)$$

also

$$u_c(t)\,y(t-c) = \mathcal{L}^{-1}\left(e^{-cs}\,Y(s)\right)$$

To shift a fn we multiply laplace by e^{-cs}

$$\mathcal{L}\left(e^{ct}g(t)\right) = Y(s-c)$$

$$e^{ct}\,y(t) = \mathcal{L}^{-1}\,Y(s-c)$$

substituting $(s-c)$ in laplace is equivalent to multiplying original func. by e^{ct}

$$y'' + 4y = g(t) \qquad y(0)=0 \quad y'(0)=0$$

$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{t-5}{5} & 5 \le t < 10 \\ 1 & t \ge 10 \end{cases}$$

rewrite

$$g(t) = \frac{u_5(t-5) - u_{10}(t)(t-10)}{5}$$

DE

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \mathcal{L}(g(t))$$

$$(s^2+4)\,Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

$$Y(s) = \frac{e^{-5s} - e^{-10s}}{s} H(s) \qquad H(s) = \frac{1}{s^2(s^2+4)}$$

use theorem

$$y = \frac{u_5(t)\,h(t-5) - u_{10}(t)\,h(t-10)}{s}$$

$$H(s) = \frac{a}{s^2} + \frac{b}{s^2+4} \qquad a(s^2+4) + bs^2 = 1$$

$$= \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}$$

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$y(t) = u_5(t)\left[\frac{1}{4}(t-5) - \frac{1}{8}\sin(2(t-5))\right] - u_{10}(t)\left[\frac{1}{4}(t-10) - \frac{1}{8}\sin(2(t-10))\right]$$

Impulse functions

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

- $g(t)$ occurs in very short interval
- $g(t) = 0$ outside τ

$$= \int_{-\infty}^{\infty} g(t)\,dt$$

Unit impulse

$$\delta(t) = 0 \quad t \neq 0$$
$$= 1 \quad t = 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

↑ dirac delta function

if impulse occurs at t_0

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

$$\mathcal{L}\left(\delta(t-t_0)\right) = e^{-st_0}$$

$$2y'' + y' + 2y = \delta(t-5) \qquad y(0)=0, \quad y'(0)=0$$

$$(2s^2+s+2)\,Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2(s^2+\frac{1}{2}s+1)} \qquad \frac{b}{(s-a)^2+b^2}$$

$$Y(s) = \frac{e^{-5s}}{2}\left(\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right) \leftarrow H(s)$$

$$\mathcal{L}^{-1}(H(s)) = \frac{4}{\sqrt{15}} e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4}$$

$$Y(s) = \frac{1}{2} e^{-5s} H(s) \quad \text{using theorem}$$

$$\boxed{y(t) = \frac{2}{\sqrt{15}} e^{-\frac{(t-5)}{4}} \sin \frac{\sqrt{15}}{4}(t-5)}$$