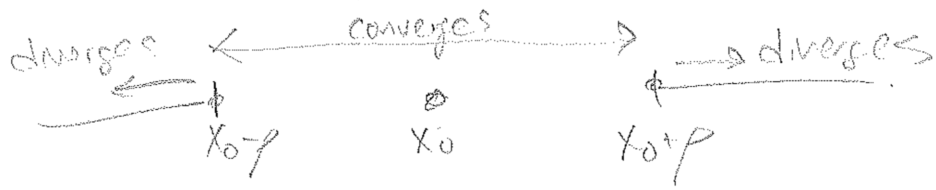
Ch. 5 Series Solutions

Power Series about x_0 $\quad y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

This series will converge at x_0 and in the interval of convergence

$$|x - x_0| < \rho \quad \uparrow \text{radius of convergence} \qquad \rho > 0$$



If a function y has a Taylor series about $x = x_0$ with radius of convergence $\rho > 0$ it is analytic

- This mean y is continuous and infinitely differentiable on the interval of convergence

Goal is to solve

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \text{around } x_0$$

with y as a power series $\quad y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Divide by $P(x)$

$$y'' + p(x)y' + q(x)y = 0$$

Ordinary Point $\quad P(x_0) \neq 0$

Singular Point $\quad P(x_0) = 0$

$p(x)$, $q(x)$ become unbanded.

Existance - We need $p(x)$, $q(x)$ to be continuous on interval $|x-x_0| < \rho$ like we would for any type of sol.

For power series we also need $p + q$ to be analytic. pg 266 proof.

- The radius of convergence is defined by the distance between the ordinary point x_0 and the nearest singular point.

Ordinary Point ($x_0 = 0$)	Singular Point
$y = \sum_{n=0}^{\infty} a_n x^n$	$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$
$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$	$y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}$
$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$	$y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$

- Find if X_0 is ord. or sing. point
- choose appropriate solution
- plug into Diff. Eq.

- note starting index of derivatives!

Example - Ordinary Point

$$y'' - (1+x)y = 0$$

$$y'' - xy - y = 0$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)C_n x^{n-2}}_{y''} - \underbrace{x\sum_{n=0}^{\infty} C_n x^n}_{xy} - \underbrace{\sum_{n=0}^{\infty} C_n x^n}_{y} = 0$$

$$\sum_{n=2} n(n-1)C_n x^{n-2} - \sum_{n=0} C_n x^{n+1} - \sum_{n=0} C_n x^n = 0$$

need to match <u>exponents</u> and <u>starting index</u>

Exp. first... need x^n in term 1, 2.

$$\sum_{n=2} n(n-1)C_n x^{n-2} \rightarrow n = n+2 \rightarrow \sum_{n=0} (n+2)(n+1)C_{n+2} x^n$$

$$\sum_{n=0} C_n x^{n+1} \rightarrow n = n-1 \rightarrow \sum_{n=1} C_{n-1} x^n$$

$$\sum_{n=0} (n+2)(n+1)C_{n+2} x^n - \sum_{n=1} C_{n-1} x^n - \sum_{n=0} C_n x^n = 0$$

Now match index to $n=1$

$\rightarrow$ Take out terms

$$\left(2c_2 + \sum_{n=1}(n+2)(n+1)c_{n+2}x^n\right) - \sum_{n=1} c_{n-1}x^n - \left(c_0 + \sum_{n=1} c_n x^n\right) = 0$$

$$(2c_2 - c_0) + \sum_{n=1}^{\infty}\left[(n+2)(n+1)c_{n+2} - c_{n-1} - c_n\right]x^n = 0$$

$$2c_2 - c_0 = 0 \rightarrow \boxed{c_2 = \frac{c_0}{2}}$$

$$(n+2)(n+1)c_{n+2} - c_{n-1} - c_n = 0$$

Recurrence relation

$$\boxed{c_{n+2} = \frac{c_n + c_{n-1}}{(n+2)(n+1)}}$$

$n = 1, 2, 3, \ldots$

$c_2 = \frac{1}{2}c_0$

$c_3 = \frac{1}{6}(c_0 + c_1)$ $n=1$

$c_4 = \frac{c_2 + c_1}{3 \cdot 4}$ $n=2$

$c_5 = \frac{c_3 + c_2}{4 \cdot 5}$ $n=3$

All coefficients are some multiple of c_1, c_0

$y = c_0 y_1 + c_1 y_2$

We can find each sol: y_1, y_2 seperately since each is a sol on its own.

let $c_0 \neq 0 \quad c_1 = 0$

$$c_2 = \frac{1}{2} c_0$$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{6}$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{c_0}{24}$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{30}$$

$c_0 = 0 \quad c_1 \neq 0$

$$c_2 = \frac{1}{2} c_0 = 0$$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{6}$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{12}$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{120}$$

$$y = c_0 y_1 + c_1 y_2$$

$$c_0 y_1 = c_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \dots \right]$$

$$c_1 y_2 = c_1 \left[x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \dots \right]$$

Example – Singular Point

$$2x^2y'' - xy' + (1+x)y = 0$$

$$y'' - \left(\frac{1}{2x}\right)y' + \left(\frac{1+x}{2x^2}\right)y = 0$$

unbounded at $x = 0$

$$2x^2y'' - xy' + (1+x)y = \sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n}$$

$$- \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0} a_n x^{r+n} + \underbrace{\sum_{n=0} a_n x^{r+n+1}}_{\sum_{n=1} a_{n-1} x^{r+n}}$$

$$L[y] = a_0[2r(r-1) - r + 1]x^r$$

$$+ \sum_{n=1}^{\infty} \left[[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}\right]x^{r+n} = 0$$

Indicial equation $\rightarrow 2r(r-1) - r + 1 = 0$

$$\boxed{r_1 = 1 \quad r_2 = \frac{1}{2}}$$

r_1, r_2 are called exponents at the singularity

Recurrence relation

$$[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0$$

$$a_n = \frac{-a_{n-1}}{2(r+n)^2 - 3(r+n) + 1}$$

$$\boxed{a_n = \frac{-a_{n-1}}{[(r+n)-1][2(r+n)-1]}} \qquad n = 1, 2, 3 \ldots$$

Now we use each root to get a Recurrance relation. Basically we find a solution y_1 for r_1 then y_2 for r_2

<u>$r_1 = 1$</u> $\qquad a_n = \frac{-a_{n-1}}{(2n+1)n} \qquad n = 1, 2, 3 \ldots$

$$a_1 = \frac{-a_0}{3}$$

$$a_2 = \frac{-a_1}{5 \cdot 2} = +\frac{a_0}{30}$$

$$a_3 = \frac{-a_2}{7 \cdot 3} = \frac{-a_0}{630}$$

x^{r+1}

$$a_0 y_1 = a_0\left[x - \frac{1}{3}x^2 + \frac{1}{30}x^3 - \frac{1}{630}x^4 \ldots\right]$$

$$y_1 = x - \frac{1}{3}x^2 + \frac{1}{30}x^3 - \frac{1}{630}x^4 + \ldots$$

$$r_2 = \frac{1}{2} \qquad a_n = \frac{-a_{n-1}}{n(2n-1)} \qquad n = 1, 2, 3 \ldots$$

$$a_1 = -\frac{a_0}{1}$$

$$a_2 = \frac{-a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)} = \frac{a_0}{6}$$

$$a_3 = \frac{-a_2}{3 \cdot 5} = \frac{-a_0}{90}$$

$$y_2 = x^{\frac{1}{2}} - x^{3/2} + \frac{1}{6} x^{5/2} - \frac{1}{90} x^{7/2}$$

$$y = c_1 y_1 + c_2 y_2$$

Example Singular Point

$$xy'' + y = 0$$

$$\underbrace{\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-1}}_{xy''} + \underbrace{\sum_{n=0}^{\infty} a_n x^{r+n}}_{y} = 0$$

multiply by x + shift index in second term

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n} + \sum_{n=1} a_{n-1} x^{r+n} = 0$$

$$r(r-1)a_0 x^r + \sum_{n=1} (r+n)(r+n-1)a_n x^{r+n} + \sum_{n=1} a_{n-1} x^{r+n} = 0$$

$$a_0 [r(r-1)] x^r + \sum_{n=1} [(r+n)(r+n-1)a_n + a_{n-1}] x^{r+n} = 0$$

Indicial Eq. $r(r-1) = 0 \rightarrow \boxed{r = 1, 0}$

Recurrence $a_n = \dfrac{-a_{n-1}}{(r+n)(r+n-1)}$ $n = 1, 2, 3 \ldots$

for $r = 1$ $a_n = \dfrac{-a_{n-1}}{(n+1)n}$

$$a_1 = \frac{-a_0}{2}$$

$$a_2 = \frac{-a_1}{6} = \frac{a_0}{12}$$

$$a_3 = \frac{-a_2}{12} = \frac{-a_0}{144}$$

$$y = \sum_{n=0}^{\infty} a_n x^{r+n} \qquad r = 1$$

$$a_0 y_1 = a_0\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 \cdots\right.$$