Chapter 3 Second Order DE

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) = \underbrace{g(t) - p(t)\frac{dy}{dt} - q(t)y}_{\text{If Linear}}$$

$y(t_0) = y_0$

$y'(t_0) = y'_0$

$$y'' + p(t)y' + q(t)y = \underset{\substack{\uparrow \\ \text{non homogeneous} \\ \text{term}}}{g(t)}$$

Const. coefficients

$ay'' + by' + cy = 0 \rightarrow$ seek sol. of form $y = e^{rt}$

$$\underbrace{(ar^2 + br + c)}_{\text{must equal zero}}\underset{\text{never zero}}{e^{rt}} = 0$$

$\underline{ar^2 + br + c = 0}$
characteristic eq.

Second order eq. gives 2 roots $\rightarrow$ 2 solutions

$$Y_1 = e^{r_1 t}, \quad Y_2 = e^{r_2 t} \rightarrow \boxed{y = C_1 y_1 + C_2 y_2}$$

general solution

Linear operator, L

$$L[y] = y'' + p(t)y' + q(t)y$$

Theorems

Existance + Uniqueness – pg 146

$y'' + p(t)y' + q(t)y = g(t) \quad y(t_0) = y_0 \quad y'(t_0) = y_0'$

if p, q, g is continuous on interval I containing to
There is exactly one solution

Super position pg 147

If y_1, y_2 are two sol. of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

$y = C_1 y_1 + C_2 y_2$ is also a solution

$L[C_1 y_1 + C_2 y_2] = C_1 L[y_1] + C_2 L[y_2]$ pg 148
Result of linearity of DE!

Wronksian

for two solutions $y_1 + y_2$
we know $y = C_1 y_1(t) + C_2 y_2(t)$ is solution.
We can always find C_1, C_2 if the Wronksian Determinant of y_1, y_2 is not zero at to

$$W = y_1 y_2' - y_1' y_2 = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $W(t_0) \neq 0$ (some point) then $y = c_1 y_1 + c_2 y_2$ contains every solution.

$y = c_1 y_1 + c_2 y_2$ — general solution

y_1 & y_2 form fundamental set of solutions

Complex Roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \underline{b^2-4ac < 0}$$

$$r_{1,2} = \lambda \pm i\mu \qquad y_1 = e^{r_1 t} \quad y_2 = e^{r_2 t}$$

We can use Taylor Series to show

$$e^{it} = \cos t + i \sin t$$

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos\mu t + i \sin\mu t)$$

Since IVP has only real coefficient we use Theorem 3.2.6 which states the real + imaginary parts of a sol. are also solutions

$$\begin{cases} y_1 = e^{\lambda t} \cos\mu t + i \sin\mu t \\ y_2 = e^{\lambda t} \cos\mu t - i \sin\mu t. \end{cases}$$

$$y = e^{\lambda t}(c_1 \cos\mu t + c_2 \sin\mu t)$$

Repeated Roots $\quad r_1 = r_2$

$$y = c_1 e^{rt} + c_2 t e^{rt}$$

Nonhomogeneous Equations

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

Theorem 3.5.1

if Y_1, Y_2 sol. to nonhomogeneous eq.

$$Y_1 - Y_2 = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homogeneous solution}}$$

General solution to non homogeneous eq.

$$y = \phi(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{homogeneous}} + Y_p(t)$$

↑ particular Solution.

To find Particular sol.

Method of Undetermined Coefficients

- choose solution of the same form as non homogeneous term.
  - polynomial
  - exponential.
  - cos, sin.

- If Y_p same form as homogeneous solution multiply by t. until there is no duplication

$L[y]$	$[Y_P]$
$5t^2$	$At^2 + Bt + C$
$\cos 3t$	$A\cos 3t + B\sin 3t$
$8e^{5t}$	Ae^{5t}
$t^2 \sin 2t$	$(At^2 + Bt + C)(\cos 2t + \sin 2t)$
$t + e^{5t}$	$\underbrace{(At + B)}_{g_1(t)} + \underbrace{Ce^{5t}}_{g_2(t)}$

For sum you can solve as two seperate problems!

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$$y'' - 2y' + y = te^t + 4$$

Homogeneous equation $\rightarrow$ $r^2 - 2r + 1 = 0$
$r_1 = r_2 = 1$

$$y_h(t) = c_1 e^t + c_2 t e^t$$

Consider $g_1(t) = te^t$, we would guess $(At+b)e^t$
but te^t is a homo solution

- set $Y_1 = At^2e^t + Bt^3e^t$ (multiplied by t^2)

- Plug into nonhomogeneous eq. and solve for coefficients

$$[At^2e^t + Bt^3e^t]'' - 2[At^2e^t + Bt^3e^t]' + At^2e^t + Bt^3e^t = te^t$$

collect like terms and set coeff $= 0$

$$Y_1 = \frac{t^3e^t}{6}$$

for $g_2(t) = 4$

- set $Y_2 = C$
- plug in

$$0 - 0 + C = 4$$

$$C = 4$$

$$Y_2 = 4$$

$$y = c_1 e^t + c_2 t e^t + \frac{t^3 e^t}{6} + 4$$

Variation of Parameters

- general method
- g(t) can have any form

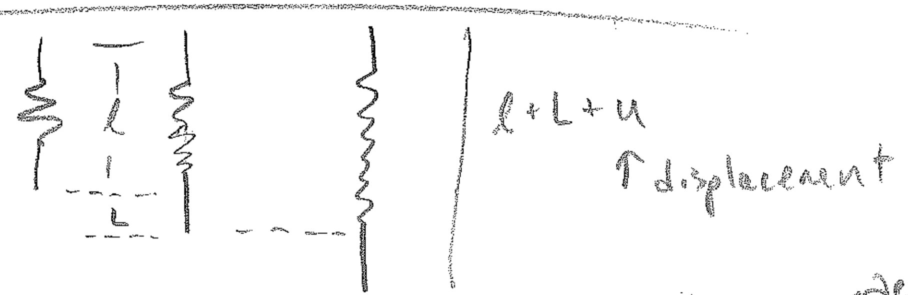
$$\boxed{\begin{gathered} Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds \\ y = c_1 y_1(t) + c_2 y_2(t) + Y(t) \end{gathered}}$$

- don't forget bounds
- Integrals may be difficult to evaluate.
- equation must be in correct form to use above eq.

$$5y'' + ty' + 7 = t^2 \rightarrow g(t) = \frac{t^2}{5}$$

$$g(t) \neq t^2 \; !$$

Mechanical Vibrations



$$m u''(t) + \gamma u'(t) + k u(t) = F(t)$$

derived in Discussion 1

↑ Forcing function

Free vibration $F(t) = 0$

<u>Undamped</u>. $\gamma = 0 \rightarrow m u'' + k u = 0$

$$m r^2 + k = 0 \rightarrow r = \pm i\sqrt{\frac{k}{m}}$$

$$u = A\cos\omega_0 t + B\sin\omega_0 t \quad \text{where } \omega_0^2 = \frac{k}{m}$$

Lets write in different form

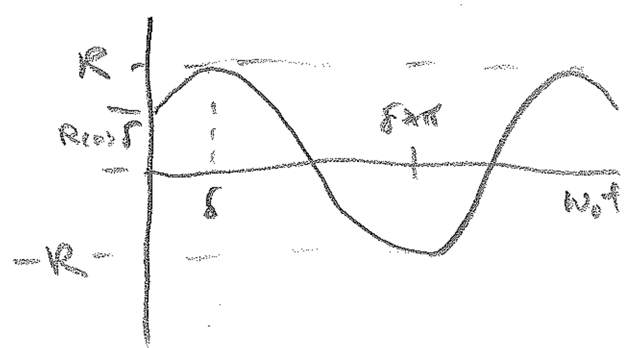
$$u = R\cos(\omega_0 t - \delta)$$

$$= R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t$$

$A = R\cos\delta$ $\qquad R = \sqrt{A^2 + B^2}$

$B = R\sin\delta$ $\qquad \tan\delta = B/A$

amplitude (R) phase (δ)



Period $T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{1/2}$

natural frequency (ω_0)

Damped

$$mu'' + \gamma u' + ku = 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

$\gamma^2 - 4km > 0 \rightarrow u = Ae^{r_1 t} + Be^{r_2 t}$ roots are negative

$\gamma^2 - 4km = 0 \quad u = (A + Bt)e^{-\gamma t/2m}$

$\gamma^2 - 4km < 0 \quad u = e^{-\gamma t}(A\cos\mu t + B\cos\mu t)$ negative real part

$$\mu = \frac{(4km - \gamma^2)^{1/2}}{2m} > 0$$

Negative exponentials

$u \rightarrow 0, \quad t \rightarrow \infty$

note! γ value doesn't matter

γ^2 relative to $4km$ matters

quasi frequency - μ

- Motion not periodic but oscillates at μ.

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}$$

Damping reduces frequency!

Forced Vibrations

$$mu''(t) + \gamma u' + ku = F(t) = F_0 \cos \omega t$$

$$u = C_1 u_1(t) + C_2 u_2(t) + A\cos \omega t + B \sin \omega t$$

$$= \underset{\substack{\uparrow \\ \text{transient}}}{u_c(t)} + \underset{\substack{\uparrow \\ \text{steady}}}{U(t)}$$

We know transient homogeneous solution dies out but it allows us to satisfy initial conditions

- convenient to express $U(t)$ as single term

$$U(t) = R\cos(\omega t - \delta)$$

$$R = \frac{F_0}{\Delta} \qquad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta} \qquad \sin\delta = \frac{\gamma\omega}{\Delta}$$

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \omega_0^2 = \frac{k}{m} \qquad \Gamma = \frac{\gamma^2}{mk}$$

gives

$$\frac{Rk}{F_0} = \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2}\right]^{1/2}}$$

$\omega \to 0 \quad R = \frac{F_0}{k}$ low frequency

$\omega \to \infty \quad R \to 0$ high frequency

max in between?

differentiate and set = 0

$$\omega_{max}^2 = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)$$

$\omega_{max} < 0$

$$R_{max} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - \frac{\gamma^2}{4mk}}}$$

Resonance Frequency