Chapter 2 First Order DE

$$\frac{dy}{dt} = f(t,y) \quad \text{or} \quad \frac{dy}{dt} + p(t)y = g(t)$$

If we can't seperate variable and integrate we can use an <u>integrating factor</u> $\mu(t)$

<u>Procedure</u>

- multiply through by $\mu(t)$
- Product rule for derivaties

$$\boxed{\frac{d}{dt}\left[\mu(t)\,y\right] = \mu(t)\frac{dy}{dt} + \underbrace{\frac{d\mu(t)}{dt}\,y}_{\text{set equal to second term}}}$$

First term

Example

$$\frac{dy}{dt} + ay = g(t)$$

$$\underbrace{\mu(t)\frac{dy}{dt}}_{\text{First Term}} + \underbrace{a\,\mu(t)\,y}_{} = g(t)\,y$$

$$a\,\mu(t)\,y = \frac{d\mu(t)}{dt}\,y$$

Seperate variables . . .

can be a

If a is a function of t, $a = p(t)$

$$\ln|\mu(t)| = \int p(t)\,dt + c$$

$$\boxed{\mu(t) = e^{\int p(t)\,dt}}$$

If you use these equations for integrating factor make sure the initial form of the DE is correct!!

Finally we can solve

$$\boxed{\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\,g(t)}$$

by direct integration.

<u>Example</u>: pg 40 problem 15

put eq. in correct form...

$$t\,y' + 2y = t^2 - t + 1 \quad \rightarrow \quad \underline{y' + \frac{2}{t}y = t - 1 + \frac{1}{t}}$$

$$\int \frac{2}{t}\,dt$$

ung

$$\frac{d}{dt}\left[t^2 y\right] = t^3 - t^2 + t$$

$$y = \frac{\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C}{t^2}$$

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

Apply IC $\quad y(1) = \frac{1}{2} \quad t > 0$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

$$C = \frac{1}{12}$$

$\frac{1}{t}$

Seperable Equations

$$\frac{dy}{dx} = f(x,y)$$

What if non-linear

$$M(x,y) + \underbrace{N(x,y)\frac{dy}{dx}}_{\text{non linear}} = 0$$

If M is fn x
N is fn y $\rightarrow$ $\underbrace{M(x) + N(y)\frac{dy}{dx} = 0}_{\text{Seperable}}$

$$M(x)\,dx + N(y)\,dy = 0$$

Example prob 3 pg 48

$$y' + \underbrace{y^2}_{\text{Non-linear!}}\sin x = 0$$

$$\frac{1}{y^2}dy = -\sin x\,dx$$

Modeling w/ FODE

Example problem 18 pg 63

Box w/ internal temp $u(t)$.

$$\frac{du}{dt} = -k\,[u - T(t)]$$

const. $\left[\frac{h}{\rho c_p V}\right]$ ambient Temp.

$$T(t) = T_0 + T_1 \cos \omega t$$

$$\frac{du}{dt} + ku = kT(t) \qquad \text{Linear}$$

Integrating factor: e^{kt}

$$\frac{d}{dt}\left[e^{kt}u\right] = k\,(T_0 + T_1 \cos\omega t)\,e^{kt}$$

$$e^{kt}u = T_0 e^{kt} + kT_1 \underbrace{\int \cos\omega t\, e^{kt}\,dt}_{\text{Integration by parts}} + C$$

$$u(t) = T_0 + kT_1 \frac{(k\cos\omega t + \omega \sin\omega t)}{k^2 + \omega^2} + Ce^{-kt}$$

Exact Equations

$$M(x,y) + N(x,y)y' = 0$$

General
Nonlinear FO ODE
Homogeneos

This equation is exact if + only if

$$\boxed{M_y(x,y) = N_x(x,y)}$$

Then a function ψ exist where

$$\boxed{\begin{aligned} \psi_x(x,y) &= M(x,y) \\ \psi_y(x,y) &= N(x,y) \end{aligned}}$$

$\psi(x,y) = C$

Example problem 1 pg 101

$$(2x+3) + (2y-2)y' = 0$$

$$M = 2x+3 \quad , \quad N = 2y-2$$

$$M_y = N_x = 0 \rightarrow \text{exact}$$

$$\psi_x = M = 2x+3$$

will be fn of only y

$$\psi = x^2 + 3x + h(y)$$

If equation is not exact an integrating factor can be used.

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

Just as hard to solve

Assume μ is function of one variable

$$\mu = \mu(x) \rightarrow \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

$$\mu = \mu(y) \longrightarrow \frac{d\mu}{dM} = \frac{N_x - M_y}{M}\mu$$

Example : prob. 25 pg 102.

$$(3x^2y + 2xy + y^2) + (x^2 + y^2)y' = 0$$

$$\frac{M_y - N_x}{N} = 3 \quad \text{fn only of } x \rightarrow \mu = e^{3x}$$

$$\psi_x = \mu M = e^{3x}(3x^2y + 2xy + y^2)$$

$$\psi = \left(x^2y + y^3/3\right)e^{3x} + h(y)$$

$$\psi_y = (x^2 + y^2)e^{3x} + h'(y) = \mu N = e^{3x}(x^2 + y^2)$$

$$h'(y) = 0$$

$$\boxed{\psi = (3x^2y + y^3)e^{3x} = c}$$