Project # 9 Solution

Problem # 1

(a) Richardson Model

① $\dfrac{dx}{dt} = y - 3x + 3 \qquad x(0) = 12$

② $\dfrac{dy}{dt} = 2x - 4y + 8 \qquad y(0) = 15$

$x(t) = \frac{32}{3} e^{-2t} - \frac{2}{3} e^{-5t} + 2 \qquad \Big| \qquad \frac{dx}{dt} = -\frac{64}{3} e^{-2t} + \frac{10}{3} e^{-5t}$

$y(t) = \frac{32}{3} e^{-2t} + \frac{4}{3} e^{-5t} + 3 \qquad \Big| \qquad \frac{dy}{dt} = -\frac{64}{3} e^{-2t} - \frac{20}{3} e^{-5t}$

① $\left(-\frac{64}{3} e^{-2t} + \frac{10}{3} e^{-5t}\right) = \left(\frac{32}{3} e^{-2t} + \frac{4}{3} e^{-5t} + 3\right) - 3\left[\frac{32}{3} e^{-2t} - \frac{2}{3} e^{-5t} + 2\right] + 3$

Collect terms

$e^{-2t}: \; -\frac{64}{3} = \frac{32}{3} - \frac{96}{3}$ ✓

$e^{-5t}: \; \frac{10}{3} = \frac{4}{3} + \frac{6}{3}$ ✓

$C: \; 0 = 3 - 6 + 3$ ✓

IC

$x(0) = 12 = \frac{32}{3} - \frac{2}{3} + \frac{6}{3}$

$= \frac{36}{3} = 12$ ✓

② $-\frac{64}{3} e^{-2t} - \frac{20}{3} e^{-5t} = 2\left[\frac{32}{3} e^{-2t} - \frac{2}{3} e^{-5t} + 2\right] - 4\left[\frac{32}{3} e^{-2t} + \frac{4}{3} e^{-5t} + 3\right] + 8$

$e^{-2t}: \; -\frac{64}{3} = \frac{64}{3} - \frac{128}{3}$ ✓

$e^{-5t}: \; -\frac{20}{3} = -\frac{4}{3} - \frac{16}{3}$ ✓

$C: \; 0 = 4 - 12 + 8$ ✓

IC $y(0) = 15 = \frac{32}{3} + \frac{4}{3} + \frac{9}{3}$

$= \frac{45}{3} = 15$ ✓

Long Term Behavior

$$x(t) = \frac{32}{3}e^{-2t} - \frac{2}{3}e^{-5t} + 2$$

$$y(t) = \frac{32}{3}e^{-2t} + \frac{4}{3}e^{-5t} + 3$$

0 as $t \to \infty$

The first two terms go to zero as $t \to \infty$ due to the negative exponents so $x(t) \to 2$
$y(t) \to 3$

The IC is $\overset{x}{12}$ and $\overset{y}{15}$ so both countries decrease their supplies exponentially. Looking at the derivatives, Country y decreases much faster due to the negative sign on the second term.

(b) $x(0) = A \quad y(0) = B$

$$\bar{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\bar{x} + \begin{bmatrix} 3 \\ 8 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(A - rI)\xi = 0$$

$$\det(A - rI) = 0$$

$$\begin{vmatrix} -3-r & 1 \\ 2 & -4-r \end{vmatrix} = (-3-r)(-4-r) - 2 = 0$$

$$= 12 + 7r + r^2 - 2 = 0$$

$$r^2 + 7 + 10 = 0$$

$$(r+2)(r+5) = 0$$

$$\boxed{\begin{aligned} r_1 &= -2 \\ r_2 &= -5 \end{aligned}}$$

$\underline{r_1 = -2}$

$(A + 2I)\xi^{(1)} = 0$

$$\begin{pmatrix} -3+2 & 1 \\ 2 & -4+2 \end{pmatrix} \bar{\xi}^{(1)} = \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \bar{\xi}^{(1)} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$-\xi_1^{(1)} + \xi_2^{(1)} = 0 \Rightarrow \xi_1^{(1)} = \xi_2^{(1)}$$

$$\boxed{\xi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } r_1 = -2}$$

$\underline{r_2 = -5}$

$(A + 5I)\xi^{(2)} = 0$

$$\begin{pmatrix} -3+5 & 1 \\ 2 & -4+5 \end{pmatrix} \xi^{(2)} = \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \xi^{(2)} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$2\xi_1^{(2)} + 1\xi_2^{(2)} = 0 \rightarrow \xi_1^{(2)} = -\frac{1}{2}\xi_2^{(2)}$$

$$\boxed{\xi^{(2)} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{for } r_2 = -5}$$

$$X_h = D\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + C\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$$

$X_p = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ → plug in

$$0 = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{cases} 0 = -3b_1 + b_2 + 3 \\ 0 = 2b_1 - 4b_2 + 8 \end{cases} \rightarrow \begin{array}{l} b_2 = 3b_1 - 3 = 3(b_1 - 1) \\ b_1 = 2b_2 - 4 = 2(b_2 - 2) \end{array}$$

$$b_1 = 2(3(b_1 - 1) - 2)$$
$$= 6b_1 - 6 - 4$$
$$= 6b_1 - 10$$
$$5b_1 = 10 \rightarrow b_1 = 2$$
$$b_2 = 3(2-1) = 3$$

$$\boxed{\begin{array}{l} b_1 = 2 \\ b_2 = 3 \end{array}}$$

$$X_p = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$X = X_h + X_p$

$$\boxed{X = D \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + C \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\boxed{x(t) = -Ce^{-5t} + De^{-2t} + 2}$$

$$x(0) = A = C + D + 2$$

$$\boxed{y(t) = -2Ce^{-5t} + De^{-2t} + 3}$$

$$y(0) = B = -2C + D + 3$$

$$A = C + D + 2 \qquad C = A - D - 2$$

$$B = -2C + D + 3$$

$$B = -2(A - D - 2) + D + 3$$

$$= -2A + 2D + 4 + D + 3$$

$$B = -2A + 3D + 7$$

$$\boxed{D = \frac{2A + B - 7}{3}} \quad \checkmark$$

$$C = A - \left(\frac{2A + B - 7}{3}\right) - 2$$

$$\boxed{C = \frac{A - B + 1}{3}} \quad \checkmark$$

Problem #2 $\quad \frac{dx}{dt} = 3y - 2x - 10$

$\frac{dy}{dt} = 4x - 3y - 10$

(a) $x(t) = 10 - 9e^{t} \qquad x(0) = 1 \qquad x(0) = 10 - 9 = 1$ ✓

$y(t) = 10 - 9e^{t} \qquad y(0) = 1 \qquad y(0) = 10 - 9 = 1$ ✓

$$\frac{dx}{dt} = -9e^{t} = 3(10 - 9e^{t}) - 2(10 - 9e^{t}) - 10$$
$$= 30 - 27e^{t} - 20 + 18e^{t} - 10$$
$$= -9e^{t} \checkmark$$

$$\frac{dy}{dt} = -9e^{t} = 4(10 - 9e^{t}) - 3(10 - 9e^{t}) - 10$$
$$= 40 - 36e^{t} - 30 + 27e^{t} - 10$$
$$= -9e^{t} \checkmark$$

as $t \to \infty$ both countries will eventually reduce their arms to zero (since negative guns isn't possible)

(b) $x(t) = 10 - 9e^{-6t} \qquad x(0) = 1 = 10 - 9 = 1$ ✓

$y(t) = 10 + 12e^{-6t} \qquad y(0) = 22 = 10 + 12 = 22$ ✓

$$\frac{dx}{dt} = -54e^{-6t} = 3(10 + 12e^{-6t}) - 2(10 - 9e^{-6t}) - 10$$
$$= \cancel{30} + 36e^{-6t} - \cancel{20} + 18e^{-6t} - \cancel{10}$$
$$= -54e^{-6t} \checkmark$$

$$\frac{dy}{dt} = -72e^{-6t} = 4(10 - 9e^{-6t}) - 3(10 + 12e^{-6t}) - 10$$
$$= \cancel{40} - 36e^{-6t} - \cancel{30} - 36e^{-6t} - \cancel{10}$$
$$= -72e^{-6t} \checkmark$$

as $t \to \infty$ both countries will have 10 guns however y will reduce from 22 to 10 and x will increase from 1 to 10

ⓒ $x(t) = -12e^{-6t} + 3e^{t} + 10 \qquad x(0) = 1 = -12 + 3 + 10 = 1$ ✓

$y(t) = 16e^{-6t} + 3e^{t} + 10 \qquad y(0) = 29 = 16 + 3 + 10 = 29$ ✓

$$\frac{dx}{dt} = 72e^{-6t} + 3e^{t} = 3(16e^{-6t} + 3e^{t} + 10) - 2(-12e^{-6t} + 3e^{t} + 10) - 10$$
$$= 48e^{-6t} + 9e^{t} + \cancel{30} - 24e^{-6t} - 6e^{t} \cancel{-20} \cancel{-10}$$
$$= 72e^{-6t} + 3e^{t} \quad ✓$$

$$\frac{dy}{dt} = -96e^{-6t} + 3e^{t} = 4(-12e^{-6t} + 3e^{t} + 10) - 3(16e^{-6t} + 3e^{t} + 10) - 10$$
$$= -48e^{-6t} + 12e^{t} + \cancel{40} - 48e^{-6t} - 9e^{t} \cancel{-30} \cancel{-10}$$
$$= -96e^{-6t} + 3e^{t} \quad ✓$$

as $t \to \infty$ both countries grow indefinately (e^{t}) term

ⓓ $x(t) = 10 \qquad x(0) = 10$ ✓

$y(t) = 10 \qquad y(0) = 10$ ✓

as $t \to \infty$ both stay at 10

$$\frac{dx}{dt} = 30 - 20 - 10 = 0 \quad ✓$$

$$\frac{dy}{dt} = 40 - 30 - 10 = 0 \quad ✓$$

problem #3

(a) $\frac{dx}{dt} = a(x^* - x)$

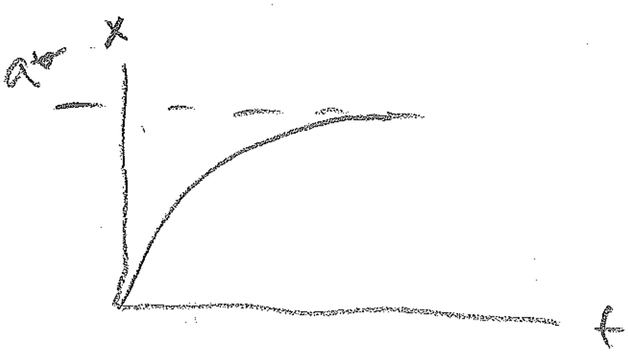
$\frac{dy}{dt} = b(y^* - y)$

as $t \to \infty$ both countries would increase until they reach the desired level exponentially

$\frac{dx}{dt} = -ax + \underset{\uparrow \text{const}}{ax^*}$

$x = ax^* e^{-at}$

$y = by^* e^{-bt}$



(b) $\frac{dx}{dt} = a(c + dy - x) = (ad)y - ax + (ac)$

$\frac{dy}{dt} = b(e + fx - y) = (bf)x - by + (be)$

Problem #4

$$\frac{dx}{dt} = ay + bz - cx + q$$

$$\frac{dy}{dt} = dx + ez - fy + r$$

$$\frac{dz}{dt} = gx + hy - jz + s$$

If x + y are allies we drop the y + x terms from x + y respectively

$$\frac{dx}{dt} = bz - cx + q$$

$$\frac{dy}{dt} = ez - fy + r$$

$$\frac{dz}{dt} = gx + hy - jz + s$$

The long term behavior depends on the values of the constants.

problem #5

using what we learned from population control in project 1, we need to add a coeff. that limits the growth to a (carrying capacity K)

$$\frac{dx}{dt} = (ay - mx + r)\left(1 - \frac{x}{K_x}\right)$$

$$\frac{dy}{dt} = (bx - ny + s)\left(1 - \frac{y}{K_y}\right)$$