Modeling Arms Races

by Michael Olinick

NICOLAS ASFOURI/AFP/Getty Images/Newscom

Weapons and ammunition recovered during military operations against Taliban militants in South Waziristan in October 2009

The last hundred years have seen numerous dangerous, destabilizing, and expensive arms races. The outbreak of World War I climaxed a rapid buildup of armaments among rival European powers. There was a similar mutual accumulation of conventional arms just prior to World War II. The United States and the Soviet Union engaged in a costly nuclear arms race during the forty years of the Cold War. Stockpiling of ever-more deadly weapons is common today in many parts of the world, including the Middle East, the Indian subcontinent, and the Korean peninsula.

British meteorologist and educator Lewis F. Richardson (1881–1953) developed several mathematical models to analyze the dynamics of arms races, the evolution over time of the process of interaction between countries in their acquisition of weapons. Arms race models generally assume that each nation adjusts its accumulation of weapons in some manner dependent on the size of its own stockpile and the armament levels of the other nations.

Richardson's primary model of a two country arms race is based on *mutual fear*: A nation is spurred to increase its arms stockpile at a rate proportional to the level of armament expenditures of its rival. Richardson's model takes into account internal constraints within a nation that slow down arms buildups: The more a nation is spending on arms, the harder it is to make greater increases, because it becomes increasingly difficult to divert society's resources from basic needs such as food and housing to weapons. Richardson also built into his model other factors driving or slowing down an arms race that are independent of levels of arms expenditures.

The mathematical structure of this model is a linked system of two first-orde linear differential equations. If x and y represent the amount of wealth being spent on arms by two nations at time t, then the model has the form

$$\frac{dx}{dt} = ay - mx + r$$
$$\frac{dy}{dt} = bx - ny + s$$

where a, b, m, and n are positive constants while r and s are constants which can be positive or negative. The constants a and b measure mutual fear; the constants m and n represent proportionality factors for the "internal brakes" to further arms increases. Positive values for r and s correspond to underlying factors of ill will or distrust that would persist even if arms expenditures dropped to zero. Negative values for r and s indicate a contribution based on goodwill.

The dynamic behavior of this system of differential equations depends on the relative sizes of ab and mn together with the signs of r and s. Although the model is a relatively simple one, it allows us to consider several different long-term outcomes. It's possible that two nations might move simultaneously toward mutual disarmament, with x and y each approaching zero. A vicious cycle of unbounded increases in x and y is another possible scenario. A third eventuality is that the arms expenditures asymptotically approach a stable point (x^*, y^*) regardless of the initial level of arms expenditures. In other cases, the eventual outcome depends on the starting point. Figure 1 shows one possible situation with four different initial

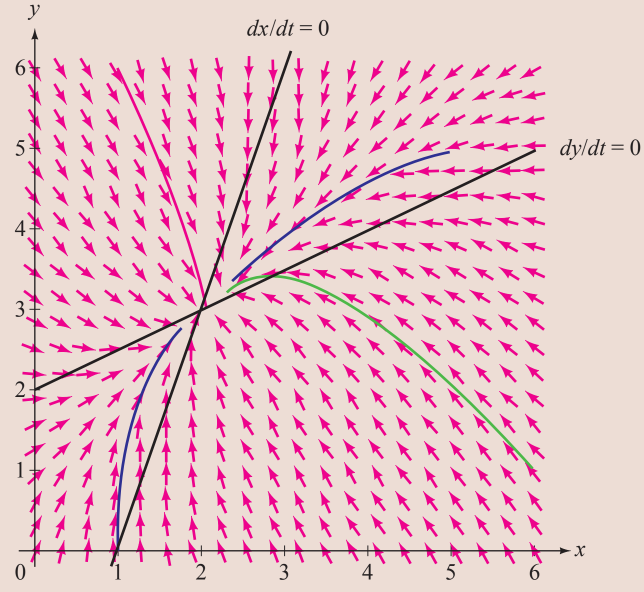

FIGURE 1 Expenditures approaching a stable point

levels, each of which leads to a "stable outcome," the intersection of the nullclines $dx/dt = 0$ and $dy/dt = 0$.

Although "real world" arms races seldom match exactly with Richardson's model, his pioneering work has led to many fruitful applications of differential equation models to problems in international relations and political science. As two leading researchers in the field note in [**3**], "The Richardson arms race model constitutes one of the most important models of arms race phenomena and, at the same time, one of the most influentia formal models in all of the international relations literature."

Arms races are not limited to the interaction of nation states. They can take place between a government and a paramilitary terrorist group within its borders as, for example, the Tamil Tigers in Sri Lanka, the Shining Path in Peru, or the Taliban in Afghanistan. Arms phenomena have also been observed between rival urban gangs and between law enforcement agencies and organized crime.

The "arms" need not even be weapons. Colleges have engaged in "amenities arms races," often spending millions of dollars on more luxurious dormitories, state- of-the-art athletic facilities, epicurean dining options, and the like, to be more competitive in attracting student applications. Biologists have identified the possibility of evolutionary arms races between and within species as an adaptation in one lineage may change the selection pressure on another lineage, giving rise to a counter-adaptation. Most generally, the assumptions represented in a Richardson-type model also characterize many competitions in which each side perceives a need to stay ahead of the other in some mutually important measure.

Related Problems

1. (a) By substituting the proposed solutions into the differential equations, show that the solution of the particular Richardson arms model

$$\frac{dx}{dt} = y - 3x + 3$$

$$\frac{dy}{dt} = 2x - 4y + 8$$

with initial condition $x(0) = 12, y(0) = 15$ is

$$x(t) = \frac{32}{3}e^{-2t} - \frac{2}{3}e^{-5t} + 2$$
$$y(t) = \frac{32}{3}e^{-2t} + \frac{4}{3}e^{-5t} + 3$$

What is the long-term behavior of this arms race?

(b) For the Richardson arms race model (a) with arbitrary initial conditions $x(0) = A, y(0) = B$, show that the solution is given by

$$x(t) = Ce^{-5t} + De^{-2t} + 2 \qquad \text{where} \qquad C = (A - B + 1)/3$$
$$y(t) = -2Ce^{-5t} + De^{-2t} + 3 \qquad \qquad D = (2A + B - 7)/3$$

Show that this result implies that the qualitative long-term behavior of such an arms race is the same ($x(t) \to 2, y(t) \to 3$), no matter what the initial values of x and y are.

2. The qualitative long-term behavior of a Richardson arms race model can, in some cases, depend on the initial conditions. Consider, for example, the system

$$\frac{dx}{dt} = 3y - 2x - 10$$
$$\frac{dy}{dt} = 4x - 3y - 10$$

For each of the given initial conditions below, verify that the proposed solution works and discuss the long-term behavior:

(a) $x(0) = 1, y(0) = 1 : x(t) = 10 - 9e^{t}, y(t) = 10 - 9e^{t}$
(b) $x(0) = 1, y(0) = 22 : x(t) = 10 - 9e^{-6t}, y(t) = 10 + 12e^{-6t}$
(c) $x(0) = 1, y(0) = 29 : x(t) = -12e^{-6t} + 3e^{t} + 10, y(t) = 16e^{-6t} + 3e^{t} + 10$
(d) $x(0) = 10, y(0) = 10 : x(t) = 10, y(t) = 10$ for all t

3. (a) As a possible alternative to the Richardson model, consider a *stock adjustment model* for an arms race. The assumption here is that each country sets a desired level of arms expenditures for itself and then changes its weapons stock proportionally to the gap between its current level and the desired one. Show that this assumption can be represented by the system of differential equations

$$\frac{dx}{dt} = a(x^* - x)$$
$$\frac{dx}{dt} = b(y^* - y)$$

where x^* and y^* are desired constant levels and a, b are positive constants. How will x and y evolve over time under such a model?

(b) Generalize the stock adjustment model of (a) to a more realistic one where the desired level for each country depends on the levels of both countries. In particular, suppose x^* has the form $x^* = c + dy$ where c and d are positive constants and that y^* has a similar format. Show that, under these assumptions, the stock adjustment model is equivalent to a Richardson model.

4. Extend the Richardson model to three nations, deriving a system of linear differential equations if the three are mutually fearful: each one is spurred to arm by the expenditures of the other two. How might the equations change if two of the nations are close allies not threatened by the arms buildup of each other, but fearful of the armaments of the third. Investigate the long-term behavior of such arms races.

5. In the real world, an unbounded runaway arms race is impossible since there is an absolute limit to the amount any country can spend on weapons; *e.g.* gross national product minus some amount for survival. Modify the Richardson model to incorporate this idea and analyze the dynamics of an arms race governed by these new differential equations.

References

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ABOUT THE AUTHOR

Courtesy of Michael Olinick

After earning a BA in mathematics and philosophy at the University of Michigan and an MA and PhD from the University of Wisconsin (Madison), **Michael Olinick** moved from the Midwest to New England where he joined the Middlebury College faculty in 1970 and now serves as Professor of Mathematics. Dr. Olinick has held visiting positions at University College Nairobi, University of California at Berkeley, Wesleyan University, and Lancaster University in Great Britain. He is the author or co-author of a number of books on single and multivariable calculus, mathematical modeling, probability, topology, and principles and practice of mathematics. He is currently developing a new textbook on mathematical models in the humanities, social, and life sciences.