MAE 82 - Project

Bessel Equation

The Wave Equation in Polar Coordinate System

Vibration of a Membrane - Circular Drum Head

MAE 82 - Project - Vibration of a Membrane (Circular Drum Head)

Note the following review material is based on the cited references but in some cases it was expanded beyond the brief derivation of the equations, in particular the Bessel equation. You may use the attached notes or refer to the original textbooks which are both available for you.

1. Review the attached reading material / original textbook (DePrima) regarding the Bessel equation
2. Review the attached reading material / original textbook (Goodwine) regarding the wave equation in polar coordinate system and the vibration of a membrane
3. Matlab Assignment
    a. Generate 16 plots (see the next page) of the following term

    $$J_m\left(Z_{m,n} r\right)\cos(m\theta)$$

    For

    $$\begin{aligned} r &: 0 \to 1 \\ \theta &: 0 \to 2\pi \\ m &= 0,1,2,3 \\ n &= 1,2,3 \end{aligned}$$

    b. Repeat 3.a for the following term

    $$J_m\left(Z_{m,n} r\right)\sin(m\theta)$$

4. What is the relationships between the two terms?
5. What do they represent?

Useful Matlab Functions

`besselj` -
https://www.mathworks.com/help/matlab/ref/besselj.html

`cart2pol` –
https://www.mathworks.com/help/matlab/ref/cart2pol.html

`plot::Surface` -
https://www.mathworks.com/help/symbolic/mupad_ref/plot-surface.html

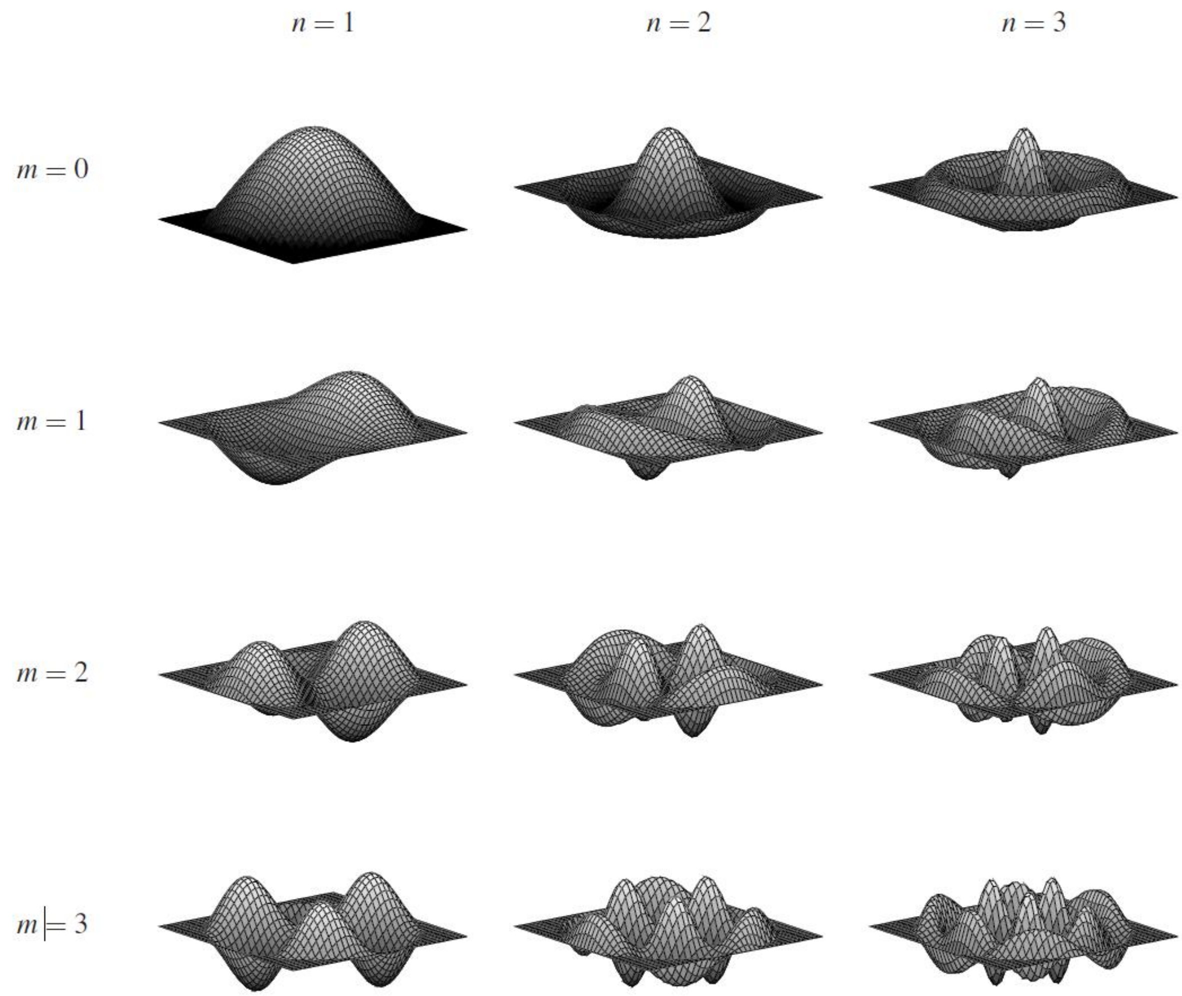


Fig. 11.30 Plots of $J_m\,(z_{m,n}r)\cos m\theta$ for various m and n, which are the modes of vibration for a circular drum head.

Bessel Equation

BESSEL'S EQUATION

SERIES SOLUTION OF LINEAR DIFF EQ.

REGULAR SINGULAR POINT

Ref: BOYCE / DIPRIMA - SECTION 5.7 - BESSEL'S EQ.

The differential equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \quad (1)$$

arises in advanced studies of mathematics, physics and engineering an known as the Bessel's equation of order ν, named after the German mathematician and astronomer Friedrich Wilhelm Bessel (1784 - 1846).

When we solve the diff eq. we shell assume that $\nu \geq 0$

Friedrich Wilhelm Bessel left school at the age of 14 to embark on a career in the import - export business but soon became interested in astronomy and mathematics. He was appointed director of the observatory of Königsberg in 1810 and held this position until his death. His study of planetary pertubations led him is 1824 to make the first systematic analysis of the solution, known as Bessel function of the diff eq. He is also famous for making, in 1838, the first accurate determination of the distance from the earth to a star.

Bessel's equation arises when finding seperable solution to the Laplace's equation ($\nabla^2 \varphi = 0$ OR $\Delta \varphi = 0$) or Helmholtz equation ($\nabla^2 A + k^2 A = 0$) in cylindrical or spherical coordinates
Bessel function are therefore especially important for many problems of wave propagation and static potential

In solving problems in cylindrical coordinates system, one obtain functions of integer order ($\nu = n$); in spherical problems, one obtains half integer order ($\nu = n + \frac{1}{2}$)

LIST OF APPLICATIONS

- Electromagnetic waves in cylindrical waveguide
- Pressure amplitudes of inviscid rotational flows
- Heat conduction in cylindrical object
- Modes of vibration of a thin circular acoustic membrane (such as drum or menbranophone)
- Diffusion problems on a lattice
- Solutions to the radial Schrödinger eq.
- Solving for patterns of acoustical radiation
- Frequency-dependent friction in circular pipelines
- Dynamics of floating bodies
- Angular resolution
- Signal processing (e.g. FM synthesis, Kaiser window, Bessel filter)

- Regular Singular Point (RSP)

$x = 0$ is a RSP of the Bessel's eq.

$$p_0 = \lim_{x \to 0} x \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \frac{1}{x} = 1$$

$$q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{x^2 - \nu^2}{x^2} = -\nu^2$$

- The indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0$$

with the roots $r = \pm \nu$

For the interval $\nu \geq 0$ we will consider 3 cases:

$$\begin{cases} \boxed{\text{CASE 1} \quad \nu = 0} \\ \boxed{\text{CASE 2} \quad \nu = \frac{1}{2}} \\ \boxed{\text{CASE 3} \quad \nu = 1} \end{cases}$$

CASE 1 : Bessel Eq. of order zero ($\nu = 0$)

In case $\nu = 0$ the diff eq (1) reduces to

$$x^2 y'' + x y' + x^2 y = 0 \quad (1)$$

and the roots of the indicial eq. are equal

$$r_1 = r_2 = 0$$

(3)

Substituting $y = \sum_{n=0}^{\infty} a_n x^{r+n} \rightarrow \begin{cases} y' = \sum_{n=0}^{\infty} a_n (r+n)\, x^{r+n-1} \\ y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n-2} \end{cases}$

into the diff eq (2) we obtain

$$x^2 y'' + x y' + x^2 y = \underbrace{x^2 \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n-2}}_{x^2 y''} + \underbrace{x \sum_{n=0}^{\infty} a_n (r+n)\, x^{r+n-1}}_{x y'} + \underbrace{x^2 \sum_{n=0}^{\infty} a_n x^{r+n}}_{x^2 y} = 0 \quad (4)$$

$$\underbrace{\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)\, x^{r+n}}_{x^2y''} + \underbrace{\sum_{n=0}^{\infty} a_n(r+n)\, x^{r+n}}_{xy'} + \underbrace{\sum_{n=0}^{\infty} a_n\, x^{r+n+2}}_{x^2y} = 0$$

$$\sum_{n=0}^{\infty} a_n\left[(r+n)(r+n-1)+(r+n)\right] x^{r+n} + \sum_{n=0}^{\infty} a_n\, x^{r+n+2} = 0$$

↳ Taking out the first & the second terms

$$\underbrace{a_0\left[r(r-1)+r\right]x^r}_{n=0} + \underbrace{a_1\left[(r+1)r+(r+1)\right]x^{r+1}}_{n=1} + \sum_{n=2}^{\infty} a_n\left[(r+n)(r+n-1)+(r+n)\right] x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n\, x^{r+n+2} = 0$$

$k = n+2$

$n = k-2$

$n: 0 \to \infty$

$k: 2 \to \infty$

$$\sum_{k=2}^{\infty} a_{k-2}\, x^{r+k-\cancel{2}+\cancel{2}}$$

$$a_0\left[r(r-1)+r\right]x^r + a_1\left[(r+1)r+(r+1)\right]x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left\{a_n\left[(r+n)(r+n-1)+(r+n)\right]+a_{n-2}\right\}x^{r+n} = 0$$

As we have already noted, the roots of the indicial equation $F(r) = r(r-1)+r=0$ are $r_1=0$; $r_2=0$

$$x^0 \longrightarrow \quad a_0\left[\underset{0}{r}\,(\underset{0}{r}-1)+\underset{0}{r}\right] = 0 \quad \Rightarrow a_0$$

$$x^1 \longrightarrow \quad a_1\left[(r+1)\underset{0}{r}+\underbrace{(r+1)}_{1}\right] = 0 \quad \Rightarrow a_1 = 0$$

$$n:2\to\infty \quad x^n \longrightarrow \quad a_n\left[(r+n)(r+n-1)+(r+n)\right]+a_{n-2} = 0$$

(5)

$$\Rightarrow \quad a_n = -\frac{a_{n-2}(r)}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}}{(r+n)^2} \quad n \geq 2$$

$n=2 \qquad a_2 = -\frac{a_0}{2^2}$

$n=3 \qquad a_3 = -\frac{\cancelto{0}{a_1}}{3^2} = 0$

$n=4 \qquad a_4 = -\frac{a_2}{4^2} = +\frac{a_0}{\underbrace{2^2 4^2}_{2^6}}$

$n=5 \qquad a_5 = -\frac{a_3}{5^2} = 0$

$-a_4$

$n=6 \qquad a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^6 (6)^2} = +\frac{a_0}{2^6 (2\cdot 3)^2}$

In general

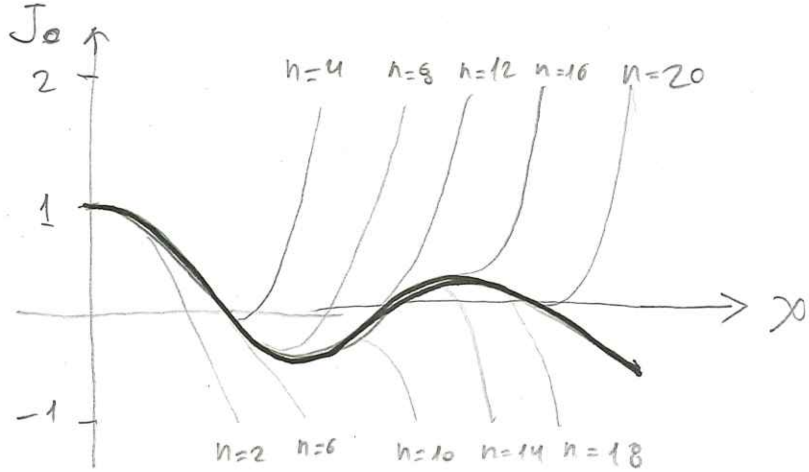
$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2} \qquad m = 1, 2, 3 \quad (6)$$

Hence the first solution y_1 is

$$\boxed{y_1(x) = a_0 \Big(\underbrace{1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}}_{J_0} \Big)} \quad x > 0 \quad (7)$$

J_0 = Bessel function of the first kind of order zero

- The series converges for all x
- J_0 is analytic at $x = 0$



To determine the second solution use

$$y_2 = y_1(x)\ln(x) + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)\, x^n \qquad x>0$$

$$a_n'(r_1) = \left.\frac{da_n}{dr}\right|_{r=r_1}$$

Derivative of a_n with respect to r and evaluted at $r=r_1$

- Since $a_1 = 0 \Rightarrow a_1' = 0$

Based on the recurrence relation

$$a_1 = a_3 = a_5 \cdots = a_{2n+1} = 0 \qquad a_1' = a_3' = a_5' = \cdots = a_{n+1}' = 0$$

- Rewriting Eq 5 by replacing $\boxed{n \rightarrow 2m}$ and running the index $m = 1, 2, 3, 4 \ldots$

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(r+2m)^2}$$

$m = 1$ $$a_2(r) = -\frac{a_0}{(r+2)^2}$$

$m = 2$ $$a_4(r) = -\frac{a_2}{(r+4)^2} = +\frac{a}{(r+2)^2(r+4)^2}$$

and in general

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \ldots (r+2m)^m} \qquad m \geqslant 3 \quad (8)$$

- Thus we need only to compute the derivatives of the even coefficients $a'_{2m}(t)$

- The computation of a'_{2m} can be carried out most conveniently by noting that

$$\boxed{\begin{array}{l} \text{- If} \\ \qquad f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots\cdots(x-\alpha_n)^{\beta_n} \\ \text{- And if } x \text{ is not equal to } \alpha_1, \alpha_2 \cdots, \alpha_n \text{ then} \\ \qquad \dfrac{f'(x)}{f(x)} = \dfrac{\beta_1}{x-\alpha_1} + \dfrac{\beta_2}{x-\alpha_2} + \cdots + \dfrac{\beta_n}{x-\alpha_n} \end{array}}$$

Rewriting (8)

$$a_{2m}(r) = \underbrace{\boxed{(-1)^m a_0}}_{\text{CONST}} (r+2)^{\overset{\beta_1}{-2}} (r+4)^{\overset{\beta_2}{-2}} (r+6)^{\overset{\beta_3}{-2}} \cdots\cdots (r+2m)^{-2}$$

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \overset{\beta_i}{-2}\left(\frac{1}{r+2} + \frac{1}{r+4} + \frac{1}{r+6} + \cdots + \frac{1}{r+2m}\right)$$

seting $r = 0$

$$a'_{2m}(0) = -2\underbrace{\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2m}\right)}_{-\overbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{m}\right)}^{H_m}} a_{2m}(0)$$

from Eq. 6

$$\boxed{a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}}$$

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2} = H_m \frac{(-1)^{m+1} a_0}{2^{2m}(m!)^2}$$

the second solution of the Bessel equation of the order zero is found by setting $a_0 = 1$ and substituting $y_1(0)$ and $a'_{2m}(0) = b_{2m}(0)$

$$y_2(x) = y_1(x)\ln(x) + x^{r_1} \sum_{n=1}^{\infty} \boxed{a'_n(r_1)}\, x^{n}$$

$n \rightarrow 2m$

see previously p. 11 top

$$\boxed{\begin{matrix} y_1 = a_1 J_0 \\ \uparrow \\ a_1 = 0 \end{matrix}} \qquad \boxed{\begin{matrix} r_1 = 0 \\ x^0 = 1 \end{matrix}} \qquad \boxed{H_m \frac{(-1)^{m+1} \cancel{a_0}^{\,1}}{2^{2m}(m!)^2}}$$

$$y_2(x) = J_0 \ln(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m} \qquad 10$$

Insted of y_2 the second solution is usually taken to be a certain linear combination of J_0 and y_2. It is known as the Bessel function of the second kind of order zero and is denoted by Y_0 also known as the Weber function

$$Y_0 = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0 \right] \qquad 11$$

Here γ is a constant known as the Euler-Mascheroni constant; it is defined by the equation

$$\gamma = \lim_{n\to\infty} \left(H_n - \ln(n) \right) \cong 0.5772$$

substituting for $y_2(x)$ in eq. 11 we obtain for $x > 0$

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right]$$

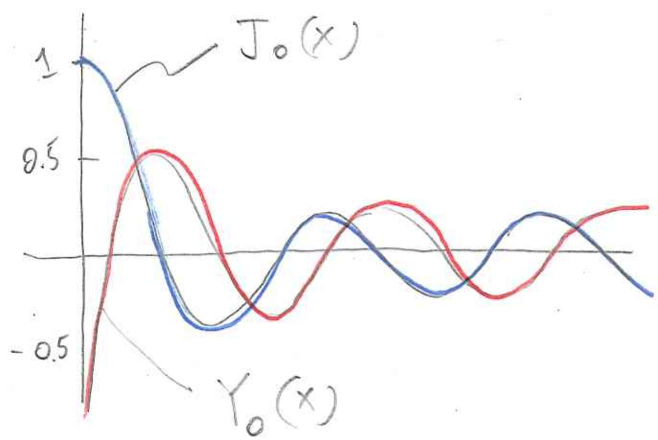
The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1 J_0(x) + c_2 Y_0(x)$$

Notes: 1 Note that as $x \to 0$

$$\begin{cases} J_0(x) \to 1 \\ Y_0(x) \to \frac{2}{\pi} \ln x \to -\infty \end{cases}$$

Thus, if we are interested in solution of the Bessel's eq. of the order zero that are finite at the origin, which is often the case, we must discard Y_0.



② Note that $x \to$ large

$$\left.\begin{matrix} J_0(x) \\ Y_0(x) \end{matrix}\right\} \longrightarrow \text{osillatory}$$

- such a behavior might be anticipated from the original equation (Bessel equation of the order ν)
- If we divide eq. (1) by x^2 we obtain

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$$

For $x \to$ Large $\Rightarrow \left\{ \begin{array}{l} \left(\frac{1}{x}\right) y' \longrightarrow 0 \\ \frac{\nu^2}{x^2} y \longrightarrow 0 \end{array} \right]$ can be neglected

resulting in

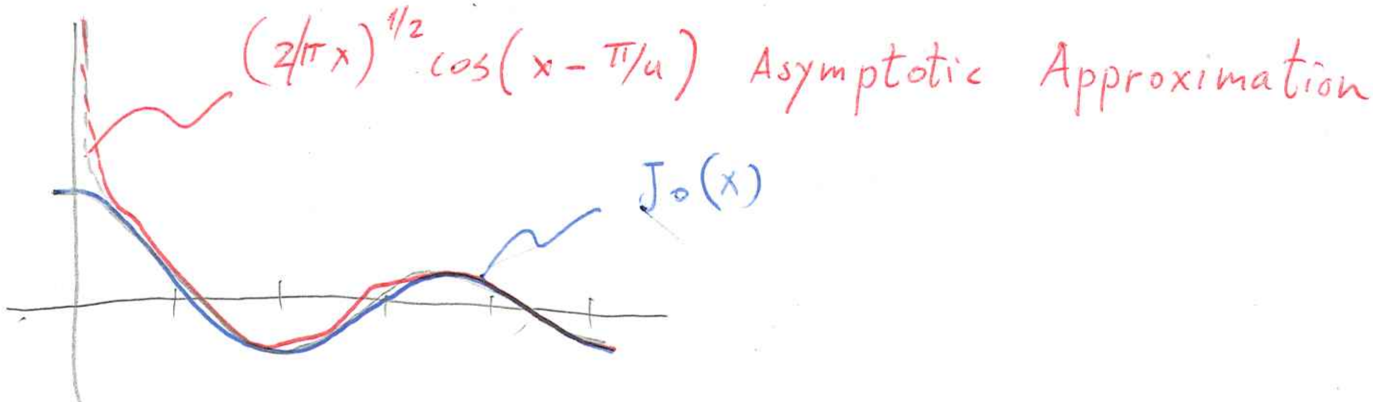
$$y'' + y = 0$$

The solution of this equation are $\sin(x)$ and $\cos(x)$. Thus we might anticipate that the solutions of Bessel's eq. for a large x are similar to linear combinations of sin and cos

This is only partly correct since for a large x the functions J_0 and Y_0 also decay as x increases thus the equation $y'' + y = 0$ does not provide an adequate approximation to the Bessel equation for large x.

It is possible to show that

$$\left.\begin{aligned} J_0(x) &\cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \\ Y_0(x) &\cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) \end{aligned}\right\} \quad \text{as } x \to \infty$$



$\boxed{\text{CASE 2: Bessel Eq of Order } 1/2 \ (\nu = 1/2)}$

This case illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting $\nu = \frac{1}{2}$ in eq. (1) gives

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \quad (16)$$

substitute a series solution of an identical form as (3) into the difl eq. (16)

$$\underbrace{x^2 \sum_{n=0}^{\infty} a_n (r+n)(r+n+1) x^{r+n-2}}_{x^2 y''} + \underbrace{x \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}}_{x y'}$$

$$+ \underbrace{x^2 \sum_{n=0}^{\infty} a_n x^{r+n}}_{x^2 y} - \underbrace{\frac{1}{4} \sum_{n=0}^{\infty} a_n x^{r+n}}_{1/4\, y}$$

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-\frac{1}{4}\right]a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

$$\underbrace{\left[r(r-1)+r-\frac{1}{4}\right]a_0 x^{r}}_{n=0} + \underbrace{\left[\overbrace{(r+1)(r)+(r+1)}^{r^2+2r+1}-\frac{1}{4}\right]a_1 x^{r+1}}_{n=1}$$

$$\sum_{n=2}^{\infty}\left[\underbrace{(r+n)(r+n-1)+(r+n)-\frac{1}{4}}_{\left[(r+n)^2-\frac{1}{4}\right]}\right]a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

$$\left\{\begin{array}{l} n=k-2 \\ k=n+2 \\ \left\{\begin{array}{l} n:\ 0\to\infty \\ k:\ 2\to\infty \end{array}\right. \end{array}\right.$$

$$\sum_{k=2}^{\infty} a_{k-2}\, x^{r+k}$$

$$k \to n$$

$$\left(r^2-\frac{1}{4}\right)a_0 x^{r} + \left((r+1)^2-\frac{1}{4}\right)a_1 x^{r+1} + \sum_{n=2}^{\infty}\left[\left((r+n)^2-\frac{1}{4}\right)a_n + a_{n-2}\right]x^{r+n} = 0 \qquad (17)$$

- The indicial equation

$$r^2 - \frac{1}{4} = 0$$

- The roots are $r_1 = \frac{1}{2}$ $r_2 = -\frac{1}{2}$

- The roots differ by an integer

- Corresponding to the larger root $r_1 = \frac{1}{2}$ the coefficients a_0 and a_1 are

For (a_0)

$$\left(r^2 - \frac{1}{4}\right) a_0 x^r = 0 x^r$$

$$\left(\frac{1}{4} - \frac{1}{4}\right) a_0 = 0 \implies a_0 \text{ arbitrary}$$

For (a_1)

$$\left((r+1)^2 - \frac{1}{4}\right) a_1 x^{r+1} = 0 x^{r+1}$$

$$\left(\left(1\frac{1}{4}\right)^2 - \frac{1}{4}\right) a_1 = 0 \implies a_1 = 0$$

- The recurrence relation is

$$\left((r+n)^2 - \frac{1}{4}\right) a_n + a_{n-2} = 0 \qquad n \geq 2 \qquad (18)$$

$$a_n = -\frac{a_{n-2}}{\left[(r+n)^2 - \frac{1}{4}\right]}$$

- For $r = \frac{1}{2}$

$$a_n = -\frac{a_{n-2}}{\left(\frac{1}{2}+n\right)^2 - \frac{1}{4}} = -\frac{a_{n-2}}{\cancel{\frac{1}{4}} + n + n^2 - \cancel{\frac{1}{4}}}$$

$$a_n = -\frac{a_{n-2}}{n(n+1)} \qquad n \geq 2$$

Since $a_1 = 0 \longrightarrow a_3 = a_5 = a_7 = \dots = a_{2n+1} = 0$
(All the odd elements are equal to zero)

$$a_n = -\frac{a_{n-2}}{n(n+1)} \qquad \underbrace{n = 2, 4, 6, 8 \ldots}_{\text{only the even elements}}$$

Letting $n = 2m$ we obtain

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)} \qquad m = 1, 2, 3, \dots$$

$m=1$ $$a_2 = -\frac{a_0}{2\,3} = -\frac{a_0}{3!}$$

$m=2$ $$a_4 = -\frac{a_2}{4\,5} = +\frac{a_0}{2\,3\,4\,5} = +\frac{a_0}{5!}$$

$m=3$ $$a_6 = -\frac{a_4}{6\,7} = -\frac{a_0}{2\,3\,4\,5\,6\,7} = -\frac{a_0}{7!}$$

$$\vdots$$

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!} \qquad m = 1, 2, 3, 4, \dots$$

- Taking $a_0 = 1$

$$y_1 = \sum_{n=0}^{\infty} a_n \, x^{r+n}$$

$$y_1 = \cancel{a_0}^{\,1} x^{1/2} + \sum_{n=1}^{\infty} a_n \, x^{n+1/2}$$

$$y_1 = x^{1/2} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \cancel{a_0}^{\,1}}{(2m+1)!} \, x^{2m} \right]$$

[inserting $x^{1/2}$ into the sum for $m=0$]

$$y_1 = x^{1/2} + \sum_{\substack{m=1 \\ m=0}}^{\infty} \frac{(-1)^m}{(2m+1)!} \, x^{2m+1/2}$$

$$y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \, x^{2m+1/2}$$

[Extracting $x^{-1/2}$ out of the sum]

$$y_1 = x^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \, x^{2m+1} \qquad (19)$$

Not that the Taylor series of $\sin x$ is

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

Hence one solution of the Bessel equation of order $1/2$ is

$$y_1 = x^{-1/2} \sin x$$

The Bessel function of the first kind of order $1/2$ $J_{1/2}$ is defined as

$$J_{1/2} = \left(\frac{2}{\pi}\right)^{1/2} y_1 = \left(\frac{2}{\pi x}\right)^{1/2} \sin x \qquad x > 0 \qquad (20)$$

$$y_1 = \sqrt{\frac{\pi}{2}}\, J_{1/2}$$

- From equation (17) for $r = -\frac{1}{2}$ the coefficient or x^r and x^{r+1} are both zero regardless of the choice of a_0 and a_1.
- Hence, a_0, and a_1 can be chosen arbitrarily.
- From the recurrence relation (18), we obtain a set of even-numbered coefficients corresponding to a_0 and a set of odd-numbered coefficients corresponding to a_1.
- For $r = -\frac{1}{2}$

$$\left.\begin{aligned} a_{2n} &= \frac{(-1)^n a_0}{(2n)!} \\ a_{2n+1} &= \frac{(-1)^n a_1}{(2n+1)!} \end{aligned}\right\} \quad n = 1, 2, 3, \dots$$

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

$$y_2(x) = a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}} \qquad x > 0$$

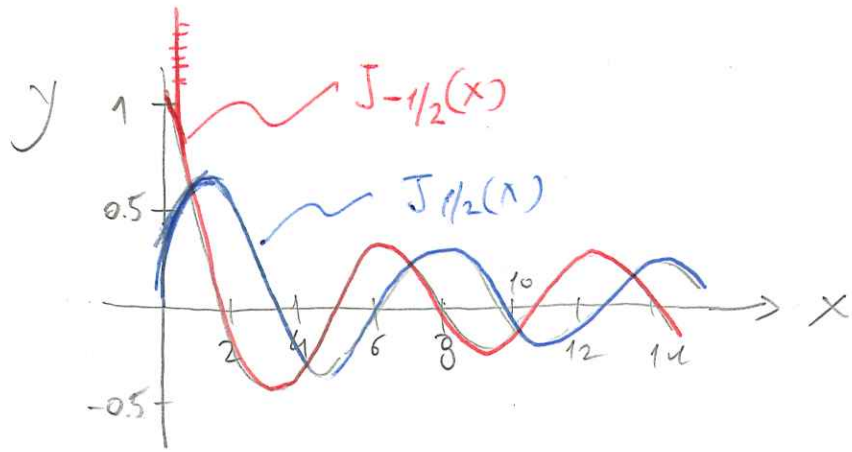
The constant a_1 simply introduces a multiple of $y_1(x)$
The second second solution of the Bessel equation of order $1/2$ is usually taken to be the solution for which $a_0 = \left(\frac{2}{\pi}\right)^{1/2}$ and $a_1 = 0$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2} = \left(\frac{2}{\pi x}\right)^{1/2} \cos x \quad x > 0$$

The general solution of equation (16) is

$$y = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$$

$$\left.\begin{aligned} J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}}\,\sin(x) \\ J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}}\,\cos(x) \end{aligned}\right\}\quad x > 0$$



Compare the graph of $J_{-1/2}$; $J_{1/2}$ with J_0 , Y_0

- There is a phase shift of $\pi/4$

- $J_{-1/2} \longrightarrow J_0$

  $J_{1/2} \longrightarrow Y_0$

- $J_{-1/2}$ and $J_{1/2}$ resemble J_0 and Y_0 respectivly for large x

CASE 3- Bessel Eq of Order 1 ($\nu = 1$)

This case illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term

Setting $\nu = 1$ in equation(1) gives

$$x^2 y'' + x y' + (x^2 - 1) y = 0 \quad (23)$$

substitute a series solution of an identical form as (3) into eq (23)

$$\underbrace{x^2 \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}}_{x^2 y} + \underbrace{x \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}}_{xy'} +$$

$$\underbrace{x^2 \sum_{n=0}^{\infty} a_n x^{r+n}}_{x^2 y} \underset{-y}{- 1 \sum_{n=0}^{\infty} a_n x^{r+n}} = 0$$

$$\boxed{\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n}} + \boxed{\sum_{n=0}^{\infty} a_n (r+n)\, x^{r+n}}$$

$$\boxed{\sum_{n=0}^{\infty} a_n\, x^{\cancel{r}+n+2}} \quad \boxed{- \sum_{n=0}^{\infty} a_n\, x^{r+n}} \quad = 0$$

$k = n+2$

$n = k-2$

$n : 0 \to \infty$

$k : 2 \to \infty$

$$\sum_{k=2}^{\infty} a_{k-2}\, x^{r+k}$$

$$\underbrace{a_0 (r)(r-1)\, x^{r}}_{n=0} + \underbrace{a_1 (r+1)(r)\, x^{r+1}}_{n=1} + \sum_{n=2}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n}$$

$$\underbrace{a_0 (r)\, x^{r}}_{n=0} + \underbrace{a_1 (r+1)\, x^{r+1}}_{n=1} + \sum_{n=2}^{\infty} a_n (r+n)\, x^{r+n}$$

$$+ \sum_{n=2}^{\infty} a_{n-2}\, x^{r+n}$$

$$- \underbrace{a_0\, x^{r}}_{n=0} - \underbrace{a_1\, x^{r+1}}_{n=1} - \sum_{n=2}^{\infty} a_n\, x^{r+n}$$

$$\left[a_0\left(r^2 - \cancel{r} + \cancel{r} - 1\right)\right] x^r +$$

$$\left[a_1\left(r^2 + r + r + 1 - 1\right)\right] x^{r+1} +$$

$$+ \sum_{n=2}^{\infty} a_n \left[(r+n)(r+n-1) + (r+n) - 1\right] x^{r+n} + a_{n-2}\, x^{r+n} = 0$$

$$a_0\left(r^2 - 1\right) x^r + a_1\left((r+1)^2 - 1\right) x^{r+1} +$$

$$\sum_{n=2}^{\infty} \left(a_n\left[(r+n)^2 - \right] + a_{n-1}\right) x^{r+n} = 0 \qquad (2u)$$

The roots of the indicial equation $r^2 - 1 = 0$ are

$$\begin{cases} r_1 = 1 \\ r_2 = -1 \end{cases}$$

The recurrence relation is

$$\left[(r+n)^2-1\right] a_n = -a_{n-2} \qquad n \geq 2 \quad (25)$$

corresponding to the larger root $r_1 = 1$ the recurrence relation becomes

$$a_n = -\frac{a_{n-2}}{(n+2)n} \qquad n = 2, 3, 4, \ldots$$

From equation (24)

First element $\quad a_0(r^2-1)\,x^r = 0\,x^r$

for $r=1$ $\quad a_0(0) = 0 \Rightarrow a_0 = \text{arbitrary}$

Second element $\quad a_1\left((r+1)^2-1\right) x^{r+1} = 0\,x^{r+1}$

for $r=1$ $\quad a_1(2^2-1) = 0 \Rightarrow a_1 = 0$

For $r = 1$ $\qquad a_n = -\dfrac{a_{n-2}}{(n+2)n} \qquad n = 2, 3, 4, 5, \ldots$

$n = 2 \qquad a_2 = -\dfrac{a_0}{4 \cdot 2}$

$n = 3 \qquad a_3 = -\dfrac{a_1}{5 \cdot 3} = \dfrac{0}{5 \cdot 3} = 0$

$n = 4 \qquad a_4 = -\dfrac{a_2}{6 \cdot 4} = +\dfrac{a_0}{6\,4\,4\,2}$

$n = 5 \qquad a_5 = -\dfrac{a_3}{\cancel{7} \cdot 5} = 0$

$n = 6 \qquad a_6 = -\dfrac{a_4}{8\,6} = -\dfrac{a_0}{8\,66\,44\,2} =$

For even values of n, we can write $n = 2m$

$$a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m} = \frac{a_{2m-2}}{2^2(m+1)m} \qquad m = 1, 2, 3, 4, \ldots$$

2

By solving this recurance relation, we obtain

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!\,m!} \qquad m = 1, 2, 3$$

check

$$m = 1 \Rightarrow \quad a_2 = -\frac{a_0}{2^2\,(1+1)!\,1!} = -\frac{a_0}{4 \cdot 2} \quad \checkmark$$

$$m = 2 \Rightarrow \quad a_4 = +\frac{a_0}{\underbrace{2^4}_{2222}\,(3)!\,2!} = +\frac{a_0}{\underbrace{22}_{4}\,\underbrace{22}_{4}\,\underbrace{3\,2}_{6}\,\underbrace{2}_{2}} \quad \checkmark$$

$$m = 3 \Rightarrow \quad a_6 = -\frac{a}{2^6\,4!\,3!} = \frac{a_0}{\underbrace{222}_{8}\,\underbrace{2}_{2}\,\underbrace{22}_{4}\,\underbrace{4}_{4}\cdot\underbrace{3\cdot 2}_{6}\cdot\underbrace{3\cdot 2}_{6}} \quad \checkmark$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{r+n} \overset{n=2m}{=} \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{2^{2m}(m+1)!\,m!} \, x^{1+2m}$$

$$y_1 = a_0 \, x \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!}$$

Seting $= a_0 = \frac{1}{2}$

$$y_1 = \underbrace{\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!}}_{J_1(x)}$$

The series converges absolutely for all x, so the function J_1 is analytic everywhere

$$J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!} \qquad (27)$$

In deremining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in eq. (28) below is rather complicated, but the first few coefficients can be found fairly easily.

$$\boxed{\begin{gathered}\text{According to theorem 5.6.1 (cases 3: If } r_1 - r_2 = N \text{, a positive integer)} \\ y_2(x) = a\, y_1(x) \ln|x| + |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)\, x^n\right)\end{gathered}}$$

For $r_2 = -1$

check !

$$y_2(x) = a\, J_1(x) \ln x + \sum_{n=1}^{\infty} c_n x^{n-1}] \qquad x > 0$$

Compute y_2' and y_2''

$$\begin{cases} y_2'(x) = a\left[J_1' \ln x + J_1 \frac{1}{x}\right] + \sum_{n=1}^{\infty} c_n (n-1) x^{n-2} \\ y_2''(x) = a\left[J_1'' \ln x + J_1' \frac{1}{x} + J_1' \frac{1}{x} - J_1 \frac{1}{x^2}\right] + \sum_{n=1}^{\infty} c_n (n-1)(n-2) x^{n-3} \end{cases}$$

substituting into (23) and making use of the fact that J_1 is a solution of eq (23)

$$x^2 y'' + x y' + (x^2 - 1) y = 0$$

$$x^2 \Big[\underbrace{a\left[J_1'' \ln x + J_1' \tfrac{1}{x} + J_1' \tfrac{1}{x} - J_1 \tfrac{1}{x^2} \right] + x^2 \sum_{n=0}^{\infty} c_n (n+1)(n-2) x^{n-3}}_{y_2''} \Big]$$

$$x \Big[\underbrace{a\left[J' \ln x + J_1 \tfrac{1}{x} \right] + x \sum_{n=0}^{\infty} c_n (n-1) x^{n-2}}_{y_2'} \Big]$$

$$+ (x^2 - 1) \Big(\underbrace{a J_1 \ln(x) + (x^2 - 1) \sum_{n=0}^{\infty} c_n x^{n-1}}_{y_2} \Big) = 0$$

$$\left[x^2 J_1'' + x J_1' + (x^2 - 1) J_1 = 0 \right]$$ J_1 is a solution of eq 23

$$a \ln x \Big[\underbrace{x^2 J_1'' + x J' + (x^2-1) J_1}_{=0} \Big] \to 0$$

$$a\left[J_1' x + J_1' x - \cancel{J_1} \right] + \quad + \sum_{n=1}^{\infty} c_n (n-1)(n-2)\, x^{n-1}$$

$$\cancel{a[J_1]} \quad + \quad + \sum_{n=1}^{\infty} c_n (n-1)\, x^{n-1}$$

$$+ \quad + \sum_{n=1}^{\infty} c_n \left(x^{n+1} - x^{n-1} \right)$$

$$2ax J_1' + \sum_{n=0}^{\infty} \left[c_n (n-1)(n-2) + c_n (n-1) - c_n \right] x^{n-1} + \sum_{n=1}^{\infty} c_n\, x^{n+1}$$

We obtain

$$\boxed{\boxed{2ax J_1'(x)} + \boxed{\boxed{\sum_{n=0}^{\infty}\left((n-1)(n-2)c_n+(n-1)c_n-c_n\right)x^{n-1}} + \boxed{\sum_{n=0}^{\infty} c_n x^{n+1}}} = 0}$$

where $c_0 = 1$. "Substituting for $J_1(x)$ from equation (27)

$$\underbrace{\left((-1)(-2)\,1 + (-1)\,1 - 1\right)}_{n=0\,\to\,0} x^{-1} + \underbrace{\left((1-1)(1-2)c_1 + (1-1)c_1 - c_1\right)}_{n=1\,\to\,-c_1} x^{0} + \underbrace{\left[(2-1)(2-2)c_2 + (2-1)c_2 - c_2\right]}_{n=2\,\to\,0} x^{1}$$

$$\sum_{n=3}^{\infty}\underbrace{\left((n-1)(n-2)c_n+(n-1)c_n-c_n\right)}\,x^{n-1}$$

$k = n-1$

$n = k+1$

$n: 3 \to \infty$

$k = 2 \to \infty$

$$0\cancel{c_0}x^{-1} - c_1 x^0 + 0\cancel{c_2}x^1 + \sum_{k=2}^{\infty}\left[(k)(k-1)c_{k+1} + k c_{k+1} - c_{k+1}\right]x^{n-1}$$

$$\underbrace{(\overset{c_0}{1})\, x^1}_{n=0} + \sum_{n=1}^{\infty} c_n\, x^{n+1}$$

$k = n + 1$

$n = k - 1$

$n : 1 \to \infty$

$k = 2 \to \infty$

$$x + \sum_{k=2}^{\infty} c_{k-1}\, x^k$$

$k \to n$

$$-c_1 + x + \sum_{n=2}^{\infty} \Big[\underbrace{n(n-1)\, c_{n+1} + n\, c_{n+1} - c_{n+1}}_{(n^2-1)\, c_{n+1}} \Big] x^n + c_{n-1}\, x^n =$$

$$= -c_1 + x + \sum_{n=2}^{\infty} \left[(n^2 - 1)\, c_{n+1} + c_{n-1} \right] x^n$$

$$\boxed{\begin{aligned} 2ax J_1' &= \cancel{2}ax \overbrace{\left[\frac{1}{\cancel{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!} \right]}^{J'} \\ &= ax \left[\underbrace{\overbrace{\frac{(-1)^0 x^0}{2^0 (1)!\,0!}}^{1}}_{m=0} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\,m!} \right] \\ &\ a \left[x + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m}(m+1)!\,m!} \right] \end{aligned}}$$

(30)

$$\boxed{\underbrace{-c_1 + -x}_{-c_1 x^0 + (0c_2 + c_0)x^1} + \sum_{n=2}^{\infty} \left[(n^2-1) c_{n+1} + c_{n-1} \right] x^n = -a \left[x + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m}(m+1)!\,m!} \right]}$$

$$0x^0 - ax^1 - a \sum_{m=1}^{\infty} \text{— " —}$$

From Eq. 30

$$x^0: \quad -c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$x^1: \quad 0\,c_2 + c_0 = -a \quad \Rightarrow \quad c_0 = -a = 1 \longrightarrow \overbrace{\begin{matrix} c_0 = 1 \\ a = -1 \end{matrix}}^{\text{Assume}}$$

Since $c_1 = 0 \Rightarrow c_3 = c_5 = c_7 = \cdots = 0$

Corresponding to the odd power of x writing $n = 2m+1$ on the left side of eq. 30

$$\underbrace{\left((2m+1)^2 - 1\right) c_{2m+2} + c_{2m}}_{\substack{\text{multipliars of } x^n \text{ in eq. 30} \\ n \to 2m+1}} = \frac{(-1)^m (2m+1)}{2^{2m} (m+1)!\, m!} \qquad m = 1, 2, 3, \qquad (31)$$

When we set $m=1$ in eq. 31, we obtain

$$(3^2-1)\,c_4 + c_2 = \frac{(-1)\,3}{2^2\,2!\,1!}$$

- Notice that c_2 can be selected arbitrarily, and then this equation determine c_4

- Notice that in equation (30) for the coefficient of x, c_2 appeared multiplied by 0, and that equation was used to determine a. That c_2 is arbitrary is not surprising, since c_2 is the coefficient of x in the expression $x^{-1}\left(1+\sum_{n=1}^{\infty} c_n x^n\right)$

- Consequently, c_2 simply generates multiple of J_1, and y_2 is determined only up to an additive multiple of J_1

- In accordance with the usual practice, we choose

$$c_2 = 1/2^2$$

- Then we obtain

$$c_4 = \left(\frac{(-1)3}{2^2\, 2!\,1!} - \underbrace{\frac{1}{2^2}}_{c_2}\right)\underbrace{\frac{1}{3^2-1}}_{\frac{1}{2^3}}$$

$$= \left(\frac{(-1)3}{2^3} - \frac{1}{2^2}\right)\frac{1}{2^3} =$$

$$= \frac{(-1)}{2^3 2^2}\left(\frac{3}{2}+1\right) = \frac{(-1)}{2^4 2}\left(\frac{3}{2}+1\right) = \frac{-1}{2^4 2!}\left(\left(1+\frac{1}{2}\right)+1\right)$$

Given $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$; $H_0 = 0$

$$C_4 = \frac{(-1)}{2^4 \, 2!} \left[H_2 + H_1 \right]$$

It is possible to show that the solution of the recurrence relation (31) is

$$C_{2m} = \frac{(-1)^{m+1} (H_m + H_{m-1}}{2^{2m} \, m! \, (m-1)!} \qquad m = 1, 2, \ldots$$

Thus

$$y_2 = -J_1 \ln(x) + \frac{1}{x} \left(1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} \, m! \, (m-1)!} x^{2m} \right) \qquad x > 0$$

- The second solution of equation (23), the Bessel function of the second kind of order one, Y_1 is usually taken to be a certain linear combination of J_1 and y_2. Y_1 is defined as

$$Y_1(x) = \frac{2}{\pi}\left(-y_2(x) + (\gamma - \ln 2)\, J_1(x)\right) \quad (33)$$

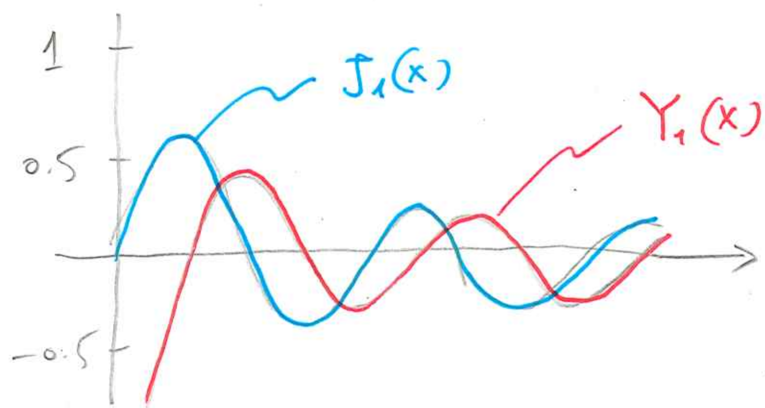
where γ is defined in eq. (12).

- The general solution of eq. 23 for $x > 0$ is

$$y = C_1 J_1(x) + C_2 Y_1(x)$$

- Notice that J_1 is analytic at $x = 0$ and the second solution Y_1 becomes undounded in the same manner as $1/x$ as $x \to 0$

- The graphs of J_1, Y_1



BESSEL'S EQ. - SUMMARY

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

$$J_\nu(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu}\, n!\, (n+\nu)!} x^{2n+\nu}$$

$$Y_\nu(x) = \frac{J_\nu(t) \cos(\nu\pi) - J_{-\nu}(t)}{\sin(\nu\pi)}$$

The Wave Equation in Polar Coordinate System

Vibration of a Membrane - Circular Drum Head

THE WAVE EQUATION

Ref: BILL GOODWINE, ENGINEERING DIFFERANTIAL EQUATION – THEORY AND APPLICATIONS
SECTION 11.5 VIBRATION MEMBRANE

– THE WAVE EQ.

The wave equation is an important second order linear partial differential equation for the description of waves – as they occur in classical physics – such as mechanical waves (e.g. water waves, sound waves, seismic waves) or light waves. It arises in the fields like accustics, electromagnetics and fluid dynamics

The wave equation is hyperbolic partial differential equation. It typically concerns
α – Fixed constant
t – time variable

$x_1, x_2, \dots x_n$ – Spacial variable
$u = u(x_1, x_2, \dots x_n)$ – Scalar function whose

values coild model the mechanical displacement of the wave. The wave equation of u is

$$(3D) \qquad \frac{\partial^2 u}{\partial t^2} = \alpha^2 \underbrace{\nabla^2}_{\text{spacial laplacian}} u = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

$$(2D) \qquad \frac{\partial^2 u}{\partial t^2} = \alpha^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$$(1D) \qquad \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

THE TWO DIMENTIONAL WAVE EQUATION IN POLAR COORDINATES

The two dimentional wave equation is given by

$$\boxed{\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$$

$$\frac{1}{\alpha^2} u_{tt} = u_{xx} + u_{yy}$$

- For a circular membrane like a drum, because the boundary condition will hold at a fixed radius, it is much more convenient to solve it in a polar coordinate system

- The relationship between polar and cartesian coordinates is given by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

and the inverse transformation is given by

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

- We need to relate derivatives with respect to the variable x and y to derivatives with respect to the variables r and θ

-

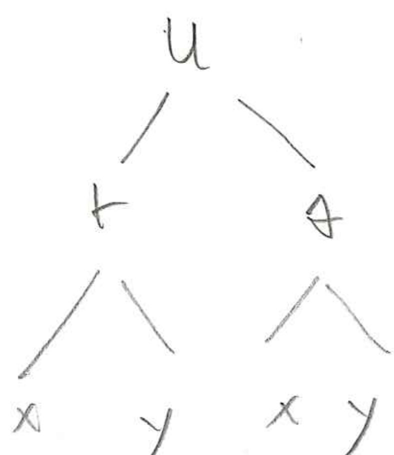
- Becouse we know the expressions for the change of coordinates, we can write the functions as

$$u(r,\theta,t) = u(r(x,y),\theta(x,y),t)$$

- By the chain rule

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y} \end{cases}$$

$$\begin{cases} u_x = u_r r_x + u_\theta \theta_x \\ u_y = u_r r_y + u_\theta \theta_y \end{cases}$$



- Differentiate with respect to x again and use the product rule

$$u_{xx} = u_r r_{xx} + \boxed{(u_r)_x} r_x + u_\theta \theta_{xx} + \boxed{(u_\theta)_x} \theta_x$$

$$u_{xx} = u_r r_{xx} + \boxed{(u_{rr} r_x + u_{r\theta} \theta_x)} r_x + u_\theta \theta_{xx} + \boxed{(u_{\theta r} r_x + u_{\theta\theta} \theta_x)} \theta_x$$

$$= u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta} \theta_x^2$$

- In a similar fashion sub x with y $x \to y$

$$u_{yy} = u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta} \theta_y^2$$

$$u_{xx} + u_{yy} = \overbrace{u_r r_{xx} + u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_\theta \theta_{xx} + u_{\theta\theta} \theta_x^2}^{u_{xx}} +$$
$$\underbrace{u_r r_{yy} + u_{rr} r_y^2 + 2u_{r\theta} r_y \theta_y + u_\theta \theta_{yy} + u_{\theta\theta} \theta_y^2}_{u_{yy}}$$

$$= u_r \boxed{(r_{xx} + r_{yy})} + u_{rr} \boxed{(r_x^2 + r_y^2)} + 2u_{r\theta} \boxed{(r_x \theta_x + r_y \theta_y)}$$
$$+ u_\theta \boxed{(\theta_{xx} + \theta_{yy})} + u_{\theta\theta} \boxed{(\theta_x^2 + \theta_y^2)}$$

- Differentiating the relationship $x^2 + y^2 = r^2$ with respect to x

$$(x^2 + y^2)_x = (r^2)_x$$
$$2x = 2r r_x \longrightarrow r_x = \frac{x}{r}$$

and with respect to y

$$(x^2 + y^2)_y = (r^2)_y$$
$$2y = 2r r_y \longrightarrow r_y = \frac{y}{r}$$

Differentiating $(r_x)_x$ and $(r_y)_y$

$$(r_x)_x = r_{xx} = \left(\frac{x}{r}\right)_x = \frac{r - x\, r_x}{r^2}$$

Pluging the expression for r_x

$$r_{xx} = \frac{r - x\,\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

in a similar fashion differentiating $(r_y)_y$

$$r_{yy} = \frac{x^2}{r^3}$$

so that

$$\boxed{r_{xx} + r_{yy}} = \frac{\overbrace{y^2 + x^2}^{r^2}}{r^3} = \frac{1}{r}$$

$$\boxed{r_x^2 + r_y^2} = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \frac{\overbrace{x^2+y^2}^{r^2}}{r^2} = 1$$

- Differentiate $\tan\theta = \frac{y}{x}$ with respect to (x) and then (y)

$$(\tan\theta)_x = \left(\frac{y}{x}\right)_x \longrightarrow (\sec^2\theta)\,\theta_x = \frac{x\cdot 0 - y\,1}{x^2} = -\frac{y}{x^2}$$

$$\theta_x = -\frac{y\cos^2\theta}{x^2}$$

$$(\tan\theta)_y = \left(\frac{y}{x}\right)_y \longrightarrow (\sec^2\theta)\,\theta_y = \frac{x\,1 - y\cdot 0}{x^2} = \frac{1}{x}$$

$$\theta_y = \frac{\overbrace{\cos^2\theta}^{\frac{x^2}{r^2}}}{x} = \frac{x}{r^2}$$

- Differentiate $(\theta_x)_x$ and $(\theta_y)_y$

$$(\theta_x)_x = \theta_{xx} = \left(-\frac{y}{r^2}\right)_x = \left(-\frac{y}{x^2+y^2}\right)_x = -\left(\frac{(x^2+y^2)\,0 - y\,2x}{r^4}\right)$$

$$\theta_{xx} = \frac{2xy}{r^4}$$

$$(\theta_y)_y = \theta_{yy} = \left(\frac{x}{r^2}\right)_y = \left(\frac{x}{x^2+y^2}\right)_y = \frac{(x^2+y^2)\,0 - x\,2y}{r^4}$$

$$\theta_{yy} = \frac{-2xy}{r^4}$$

So that $\boxed{\theta_{xx} + \theta_{yy}} = \frac{2xy}{r^2} + \frac{-2xy}{r^2} = 0$

$$\boxed{(\theta_x)^2 + (\theta_y)^2} = \left(-\frac{y}{r^2}\right)^2 + \left(\frac{x}{r^2}\right)^2 = \frac{\overbrace{x^2+y^2}^{r^2}}{r^4} = \frac{1}{r^2}$$

$$\boxed{r_x \theta_x + r_y \theta_y} = \underbrace{\frac{x}{r}}_{r_x}\left(-\frac{y}{r^2}\right) + \underbrace{\frac{y}{r}}_{r_y}\left(\frac{x}{r^2}\right) = \frac{-xy}{r^3} + \frac{xy}{r^3} = 0$$

We finally obtain

$$u_{xx} + u_{yy} = u_r \overbrace{(r_{xx} + r_{yy})}^{\frac{1}{r}} + u_{rr}\overbrace{(r_x^2 + r_y^2)}^{1} + 2u_{r\theta}\overbrace{(r_x\theta_x + r_y\theta_y)}^{0}$$
$$+ u_\theta \underbrace{(\theta_{xx} + \theta_{yy})}_{0} + u_{\theta\theta}\underbrace{(\theta_x^2 + \theta_y^2)}_{\frac{1}{r^2}}$$

$$u_{xx} + u_{yy} = \frac{1}{r} u_r + u_{rr} + \frac{1}{r^2} u_{\theta\theta}$$

Substituting these expressions into the right hand side of the wave equation gives the wave equation in polar coordinates

$$\frac{1}{\alpha^2} u_{tt} = u_{xx} + u_{yy}$$

$$\frac{1}{\alpha^2} u_{tt} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$\boxed{\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}}$$

Boundary and Intiat condition BC, IC

- For a circular drum the BC is $u(r, \theta, t) = 0$

and the intial conditions

$$\begin{cases} u(r, \theta, 0) = f(r, \theta) \\ \frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta) \end{cases}$$

Assume a solution of the form

$$u(r, \theta, t) = R(r)\,\Theta(\theta)\,T(t)$$

and substitution into the wave equation in polar coordinates gives

$$\left\{\begin{aligned} u_r &= R'\Theta T \\ u_{rr} &= R''\Theta T \\ u_{\theta\theta} &= R\Theta'' T \\ u_{tt} &= R\Theta T'' \end{aligned}\right.$$

$$R''\Theta T + \frac{1}{r} R'\Theta T + \frac{1}{r^2} R\Theta'' T = \frac{1}{\alpha^2} R\Theta T''$$

Dividing by $R\Theta T$ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\alpha^2}\frac{T''}{T}$$

- Note: that the right side of the equation only depends on t and the left side depens on r and θ, and all three veriables are independent, therefore both sides must be constant

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

where λ is yet to be determined constant. Hence

$$\begin{cases} T'' + \alpha^2\lambda T = 0 \\ \dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} = -\lambda \end{cases}$$

Multiplying by r^2 and rearranging gives

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = -\frac{\Theta''}{\Theta}$$

The left hand side of this equation only depends or r and the right side only depends on Θ and the variable R, Θ are independent, therefore these also must be equal to a constant, which is not necessarily the same as λ. Calling this constant γ, we have

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + r^2\lambda = -\frac{\Theta''}{\Theta} = \gamma$$

Hence

$$\begin{cases} \Theta'' + \gamma\,\Theta = 0 \\ r^2 R'' + rR' + (r^2\lambda - \gamma)R = 0 \end{cases}$$

If we determine the solution to equation

$$\begin{cases} T'' + \alpha^2 \lambda T = 0 \\ \Theta'' + \gamma \Theta = 0 \\ r^2 R'' + r R' + \left(r^2 \lambda - \gamma\right) R = 0 \end{cases}$$

we have the solution to the wave equation

$$\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

where

$$u(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t)$$

SOLUTION FOR $\Theta(\theta)$

$$\Theta''(\theta) + \gamma\,\Theta(\theta) = 0$$

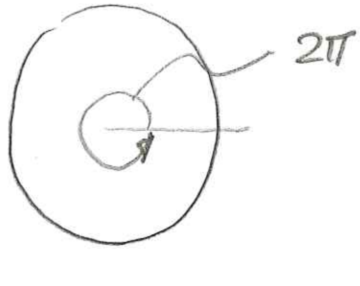
Although it appears that we only have one boundary condition given by

radius of the drum $u(\hat{r}, \theta, t) = 0$

there is also the fact that the solution for $\Theta(\theta)$ must be periodic that is

$$\Theta(\theta) = \Theta(\theta + 2\pi)$$

given the circular geometry of the drum



Thus γ must be positive and the solution is

$$\Theta(\theta) = c_1 \sin\sqrt{\gamma}\,\theta + c_2 \cos\sqrt{\gamma}\,\theta$$

In order for $\Theta(\theta) = \Theta(\theta + 2\pi)$, $\sqrt{\gamma}$ must be an integer, or

$$\sqrt{\tau} = m = 0, 1, 2, \ldots$$

$$\Theta_m(\theta) = c_1 \sin(m\theta) + c_2 \cos(m\theta) \qquad m = 1, 2, 3, \ldots$$

SOLUTION FOR $R(r)$

$$r^2 R''(r) + r R'(r) + (r^2\lambda - \gamma) R(r) = 0$$

Because $\sqrt{\gamma} = m \rightarrow \gamma = m^2$

The equation becomes

$$r^2 R''(r) + r R'(r) + (r^2\lambda - m^2) R(r) = 0$$

In the special case wher m is an integer which is the present case, the general solution may be written as

$$R(r) = C_1 J_m(\sqrt{\lambda}\, r) + C_2 Y_m(\sqrt{\lambda}\, r)$$

where

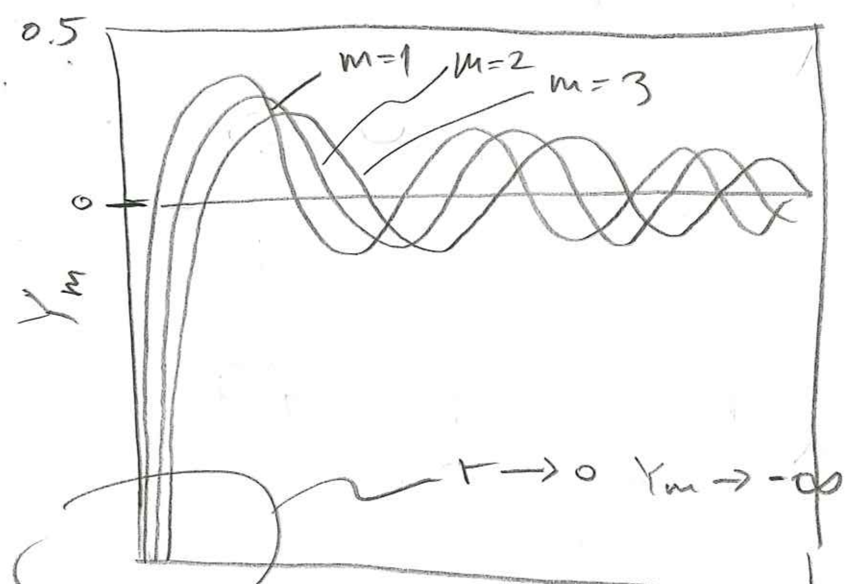
$$J_m(\sqrt{\lambda}\, r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-m)!} \left(\frac{\sqrt{\lambda}\, r}{2}\right)^{2n-k}$$

and

$$Y_m(\sqrt{\lambda}\, r) = \ln(\sqrt{\lambda}\, r)\, J_m(\sqrt{\lambda}\, r)$$
$$- \frac{1}{2} \sum_{n=0}^{m-1} \frac{(m-n-1)!}{n!} \left(\frac{\sqrt{\lambda}\, r}{2}\right)^{2n-m}$$
$$- \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) + \left(1+\frac{1}{2}+\cdots+\frac{1}{n+m}\right)\right]}{n!\,(n+m)!} \left(\frac{\sqrt{\lambda}\, r}{2}\right)^{2n+m}$$

- The function J_m and Y_m are the bessel function of the first and the second kind respectivly, or order m.

- Note that as $r \to 0$ $Y_m(r) \to -\infty$



and hence we assume to motion of the center of the drum is bounded; then we must have $c_2 = 0$

Hence

$$R(r) = c_1 J_m(\sqrt{\lambda}\, r)$$

and becouse the boundary condition

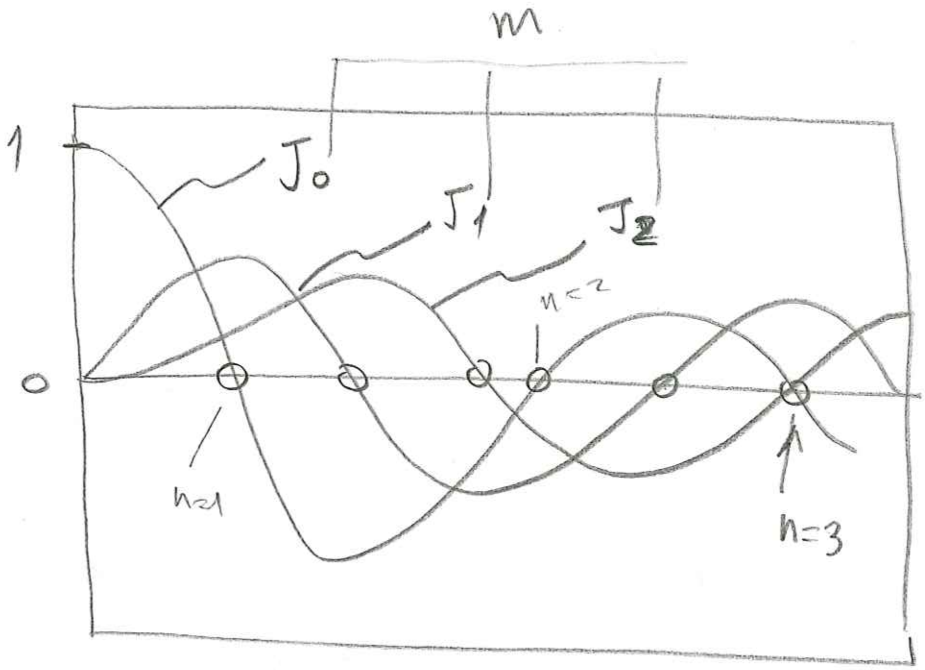
$$u(\hat{r}, \theta, t) = 0$$

requires that $R(\hat{r}) = 0$

at the boundary of the drum

either $c_1 = 0$, which would give the trivial solution or

$$J_m(\sqrt{\lambda}\, \hat{r}) = 0$$



It is apparent that Bessel function of the first kind of verious order are equal to zero for multiple values of r and they are tabulated in Table 5.2 P. 188 [see ref]

ORDER	1st Zero	2nd Zero	3ed Zero	. . . . .
0	2.40483	5.52008	8.65373	
1	3.83171	7.01559	10.1735	
2	5.13562	8.41724	11.6198	
3	6.38016	9.76102	13.0152	

Table of zeros of the Bessel function of the first kind

Let $Z_{m,n}$ denote the nth zero of the bessel function of the first kind of order m

zero

$$Z_{m,n}$$

mth order nth zero

$$\sqrt{\lambda}\,\hat{r} = Z_{m,n}$$

$$\lambda = \left(\frac{Z_{m,n}}{\hat{r}}\right)^2 \qquad n = 1, 2, 3, \ldots$$

Radius of the drum

and

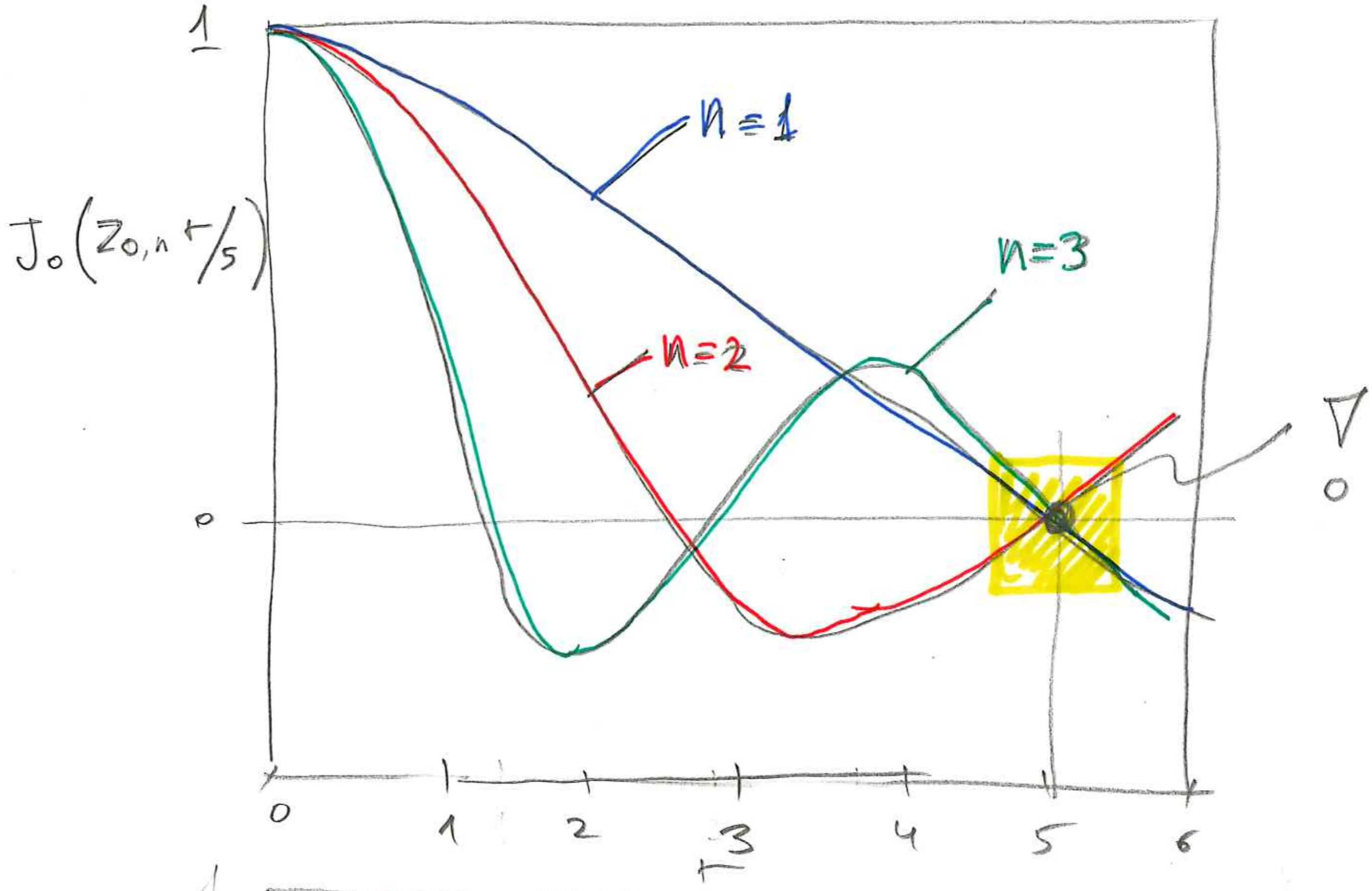
$$R(r) = C_1 J_m\left(\frac{Z_{m,n}\, r}{\hat{r}}\right)$$

The way to intuitively think of the role of $Z_{m,n}$ is that it scale r in such a way that it will go through zero at the radius of the drum

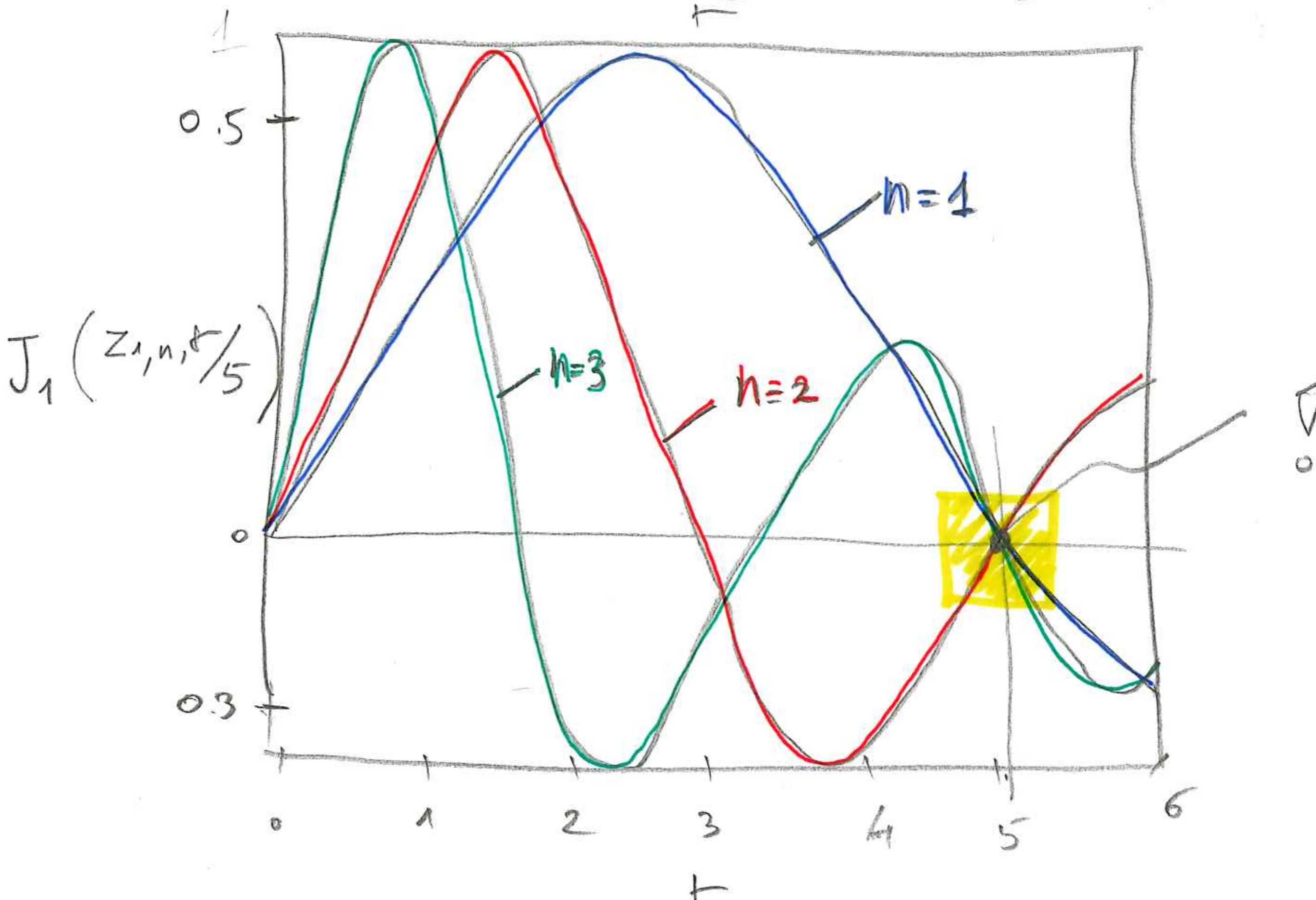
- The following figures illustrate the Bessel functions of the first kind of order zero and one with $\hat{r} = 5$ and $z_{m,n}$ equal to the first three zeros for each one
- The feature to observe is that scaling the argument be be

$$z_{m,n} \frac{r}{\hat{r}}$$

makes all the functions go to zero at $\hat{r} = 5$, which is what is necessary to match the boundary condition at the radius of the drum



1
$J_0(z_{0,n}\, r/5)$
n=1
n=2
n=3
0
0
1
2
3
4
5
6
r



1
0.5
$J_1(z_{1,n}\, r/5)$
n=1
n=2
n=3
0
0.3
0
1
2
3
4
5
6
r

SOLUTION FOR $T(t)$

$$T''(t) + \alpha^2 \lambda T(t) = 0$$

plugging the detenition of λ

$$\lambda = \left(\frac{z_{m,n}}{r^2}\right)^2$$

into the diff. eq.

$$T''(t) + \left(\frac{\alpha z_{m,n}}{\hat{r}}\right)^2 T(t) = 0$$

and hance to solution is

$$T(t) = d_1 \cos\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right) + d_2 \sin\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right)$$

THE ENTIRE SOLUTION

For a fixed integer m and n

$$u(r, \theta, t) = R(r)\,\Theta(\theta)\,T(t)$$

$$= \underbrace{J_m\left(\sqrt{\lambda}\, r\right)}_{R(r)} \underbrace{\left(a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)\right)}_{\Theta(\theta)} \underbrace{\left(d_1 \cos\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right) + d_2 \sin\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right)\right)}_{T(t)}$$

Any linear combination of these solutions is also a solution. Summing over both m and n and combining some of the constants gives

$$u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) \left[\left(a_{m,n} \cos(m\theta) + b_{m,n} \sin m\theta\right) \cos\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right) + \left(c_{m,n} \cos(m\theta) + d_{m,n}(\sin m\theta)\right) \sin\left(\frac{\alpha z_{m,n} t}{\hat{r}}\right)\right]$$

- This solution satisfies the wave equation in polar coordinates as well as the boundary condition

- The intial condition given in

$$\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \dfrac{\partial u}{\partial t}(r,\theta,0) = u_t(r,\theta,0) = g(r,\theta) \end{cases}$$

still need to be satisfied.

- Substituting $t=0$ into the solution gives

$$u(r,\theta,0) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J_m\left(\frac{z_{m,n} r}{\hat{r}}\right)\left(a_{m,n}\cos(m\theta) + b_{m,n}\sin(m\theta)\right) = f(r,\theta)$$

- To determine the coeffients, we make use of orthogonality of the sin and cosine functions as well as the following fact

$$\int_0^{\hat{r}} r\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) J_m\left(\frac{z_{m,\hat{n}} r}{\hat{r}}\right) dr = \begin{cases} 0 & n \neq \hat{n} \\ \frac{1}{2}\left[\frac{dJ_k}{dr}(z_{m,n})\right]^2 ; & n = \hat{n} \end{cases}$$

- observe that the integral is weighted by r and also the Bessel functions in the integral is that a different $z_{m,n}$ appears in the argument to the function. The two terms in the integral would be two of the curve in the previous graph

- To determine the coeffients $a_{m,n}$ that setisfy

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{z_{m,n}\, r}{\hat{r}}\right)\left(a_{m,n}\cos(m\theta)+b_{m,n}\sin(m\theta)\right) = f(r,\theta)$$

multiply both sides of the equation by $\cos(\hat{m}\theta)$ and integrate from 0 to 2π gives

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_0^{2\pi} J_m\left(\frac{z_{m,n}\, r}{\hat{r}}\right)\left(a_{m,n}\cos(m\theta)+b_{m,n}\sin(m\theta)\right)\cos(\hat{m}\theta)\,d\theta$$

$$= \int_0^{2\pi} \cos \hat{m}\theta\, f(r,\theta)\,d\theta$$

- Becouse of the orthogonality of the sine and the cosine functions, every term in the series indexed by m is zero except for when $\hat{m} = m$ and hance

$$\sum_{n=1}^{\infty} J_m\left(\frac{Z_{m,n} r}{\hat{r}}\right) \int_0^{2\pi} a_{m,n} \cos^2(m\theta)\, d\theta$$

$$= \int_0^{2\pi} f(r,\theta) \cos(m\theta)\, d\theta$$

or

$$\sum_{n=1}^{\infty} a_{m,n} J_m\left(\frac{Z_{m,n} r}{\hat{r}}\right) = \frac{\int_0^{2\pi} f(r,\theta)\cos(m\theta)\, d\theta}{\int_0^{2\pi} a_{m,n}\cos^2(m\theta)\, d\theta}$$

Now multiplying by $r\, J_m\left(\frac{Z_{m,\hat{n}} r}{\hat{r}}\right)$ and integrating from 0 to $\hat{r}$ gives

$$\sum_{n=1}^{\infty} \int_0^{\hat{r}} a_{m,n}\, r\, J_m\left(\frac{J_{m,n} r}{\hat{r}}\right) J_m\left(\frac{J_{m,\hat{n}} r}{\hat{r}}\right) dr$$

$$= \int_0^{\hat{r}} r\, J_m\left(\frac{Z_{m,\hat{n}} r}{\hat{r}}\right) \frac{\int_0^{2\pi} f(r,\theta)\cos(m\theta)\, d\theta}{\int_0^{2\pi} a_{m,n}\cos^2(m\theta)\, d\theta}$$

or

$$a_{m,n} = \frac{\int_0^{\hat{r}}\int_0^{2\pi} r\, f(r,\theta)\cos(m\theta)\, J_m\left(\frac{Z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi}\cos^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{Z_{m,n} r}{\hat{r}}\right) dr\right)}$$

An analogous computation gives

$$b_{m,n} = \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, f(r,\theta) \sin(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \sin^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$

To determine $c_{m,n}$ and $d_{m,n}$ differentiate $u(r, \theta, t)$ with respet to time, substitute $t=0$, and follow the same procedure, which gives

$$c_{m,n} = \frac{\hat{r} \int_0^{\hat{r}} \int_0^{2\pi} r\, g(r,\theta) \cos(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\alpha z_{m,n}\left(\int_0^{2\pi} \cos^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$

$$d_{m,n} = \frac{\hat{r}}{\alpha z_{m,n}} \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, g(r,\theta) \sin(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \cos^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$

and with that, we have solved the two dimensional wave equation in a polar coordinates

Summary of the solution to the Wave Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$$

$$BC \rightarrow \left\{ u(\hat{r}, \theta, t) = 0 \right.$$

$$IC \rightarrow \begin{cases} u(r, \theta, 0) = f(r, \theta) \\ \dfrac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta) \end{cases}$$

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{Z_{m,n} r}{\hat{r}}\right)\left[\left(a_{mn}\cos(m\theta) + b_{m,n}\sin(m\theta)\right)\cos\left(\frac{\alpha Z_{m,n} t}{\hat{r}}\right) + \left(c_{m,n}\cos(m\theta) + d_{m,n}\sin(m\theta)\right)\sin\left(\frac{\alpha Z_{m,n} t}{\hat{r}}\right)\right]$$

$$a_{m,n} = \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, f(r,\theta) \cos(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \cos^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$

$$b_{m,n} = \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, f(r,\theta) \sin(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \sin^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n}}{\hat{r}}\right) dr\right)}$$

$$c_{m,n} = \frac{\hat{r}}{\alpha\, z_{m,n}} \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, g(r,\theta) \cos(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \cos^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$

$$d_{m,n} = \frac{\hat{r}}{\alpha\, z_{m,n}} \frac{\int_0^{\hat{r}} \int_0^{2\pi} r\, g(r,\theta) \sin(m\theta)\, J_m\left(\frac{z_{m,n} r}{\hat{r}}\right) d\theta\, dr}{\left(\int_0^{2\pi} \sin^2(m\theta)\, d\theta\right)\left(\int_0^{\hat{r}} J_m^2\left(\frac{z_{m,n} r}{\hat{r}}\right) dr\right)}$$