MAE 82 - Project

Bessel Equation
The Wave Equation in Polar Coordinate System
Vibration of a Membrane - Circular Drum Head
MAE 82 - Project - Vibration of a Membrane (Circular Drum Head)

Note the following review material is based on the cited references but in some cases it was expanded beyond the brief derivation of the equations, in particular the Bessel equation. You may use the attached notes or refer to the original textbooks which are both available for you.

1. Review the attached reading material / original textbook (DePrima) regarding the Bessel equation
2. Review the attached reading material / original textbook (Goodwine) regarding the wave equation in polar coordinate system and the vibration of a membrane
3. Matlab Assignment
   a. Generate 16 plots (see the next page) of the following term

   \[ J_m(Z_{m,n}r)\cos(m\theta) \]

   For
   \[ r: 0 \rightarrow 1 \]
   \[ \theta: 0 \rightarrow 2\pi \]
   \[ m = 0,1,2,3 \]
   \[ n = 1,2,3 \]

   b. Repeat 3.a for the following term

   \[ J_m(Z_{m,n}r)\sin(m\theta) \]

4. What is the relationships between the two terms?
5. What do they represent?

Useful Matlab Functions

bessely -

cart2pol -
https://www.mathworks.com/help/matlab/ref/cart2pol.html

plot::Surface -
Fig. 11.30 Plots of $J_m(z_{m,n,r}) \cos m\theta$ for various $m$ and $n$, which are the modes of vibration for a circular drum head.
Bessel Equation
BESSEL'S EQUATION

SERIES SOLUTION OF LINEAR DIFF EQ.

REGULAR SINGULAR POINT

Ref: BOYCE / DIPRIMA - SECTION 5.7 - BESSEL'S EQ.

The differential equation

\[ x^2 y'' + xy' + (x^2 - \nu^2) y = 0 \quad (1) \]

arises in advanced studies of mathematics, physics and engineering as known as the Bessel's equation of order \( \nu \), named after the German mathematician and astronomer Friedrich Wilhelm Bessel (1784-1846). When we solve the diff eq. we shall assume that \( \nu \geq 0 \).

Friedrich Wilhelm Bessel left school at the age of 14 to embark on a career in the import-export business but soon
became interested in astronomy and mathematics. He was appointed director of the observatory of Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him is 1824 to make the first systematic analysis of the solution, known as Bessel function of the diff eq. He is also famous for making, in 1838, the first accurate determination of the distance from the earth to a star.

Bessel's equation arises when finding separable solution to the Laplace's equation (\( \nabla^2 \phi = 0 \) or \( \Delta \phi = 0 \)) or Helmholtz equation (\( \nabla^2 A + k^2 A = 0 \)) in cylindrical or spherical coordinates. Bessel function are therefore especially important for many problems of wave propagation and static potential.
In solving problems in cylindrical coordinates system, one obtains functions of integer order \((\nu = n)\); in spherical problems, one obtains half integer order \((\nu = n + \frac{1}{2})\).

**LIST OF APPLICATIONS**

- Electromagnetic waves in cylindrical waveguide
- Pressure amplitudes of inviscid rotational flows
- Heat conduction in cylindrical object
- Modes of vibration of a thin circular acoustic membrane (such as drum or membranophone)
- Diffusion problems on a lattice
- Solutions to the radial Schrödinger eq.
- Solving for patterns of acoustical radiation
- Frequency-dependent friction in circular pipelines
- Dynamics of floating bodies
- Angular resolution
- Signal processing (e.g. FM synthesis, Kaiser window, Bessel filter)
- Regular Singular Point (RSP)

$x = 0$ is a RSP of the Bessel's eq.

$$p_0 = \lim_{x \to 0} \frac{Q(x)}{P(x)} = \lim_{x \to 0} \frac{x}{x} = 1$$

$$q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} \frac{x^2 - y^2}{x^2} = -y^2$$

- The indicial equation

$$t(t-1) + p_0 t + q_0 = t(t-1) + t - y^2 = t^2 - y^2 = 0$$

with the roots $t = \pm y$

For the interval $y \geq 0$ we will consider 3 cases:

- \text{CASE 1} \quad y = 0
- \text{CASE 2} \quad y = \frac{1}{2}
- \text{CASE 3} \quad y = 1
CASE 1: Bessel Eq. of Order Zero (\( \nu = 0 \))

In case \( \nu = 0 \) the diff eq(1) reduces to

\[
x^2 y'' + xy' + x^2 y = 0 \quad (1)
\]

and the roots of the indicial eq. are equal

\[
\tau_1 = \tau_2 = 0
\]

Substituting

\[
y = \sum_{n=0}^{\infty} a_n x^{\tau + n} \rightarrow \begin{cases} 
  y' = \sum_{n=0}^{\infty} a_n (\tau + n) x^{\tau + n - 1} \\
  y'' = \sum_{n=0}^{\infty} a_n (\tau + n)(\tau + n - 1) x^{\tau + n - 2}
\end{cases}
\]

into the diff eq (2) we obtain

\[
x^2 y'' + xy' + x^2 y = \underbrace{x^2 \sum_{n=0}^{\infty} a_n (\tau + n)(\tau + n - 1) x^{\tau + n - 2}}_{x^2 y''} + \underbrace{x \sum_{n=0}^{\infty} a_n (\tau + n) x^{\tau + n - 1}}_{xy'} + \underbrace{x^2 \sum_{n=0}^{\infty} a_n x^{\tau + n}}_{x^2 y} = 0
\]

\[
y^2 y'' - 5
\]
\[ a_0 \left[ r(r-1) + r \right] x^r + a_1 \left[ (r+1)(r+2) \right] x^{r+1} + \sum_{n=2}^{\infty} \left\{ a_n \left[ (r+n)(r+n-1) + (r+n) \right] + a_{n-2} \right\} x^{r+n} = 0 \]

As we have already noted, the roots of the indicial equation \( \tau(t) = r(r-1) + t = 0 \) are \( r_1 = 0 \); \( r_2 = 0 \)

\[ X^0 \rightarrow a_0 \left[ r(r-1) + r \right] = 0 \quad \Rightarrow \quad a_0 = 0 \]

\[ X^1 \rightarrow a_1 \left[ (r+1)(r+2) \right] = 0 \quad \Rightarrow \quad a_1 = 0 \]

\[ \lim_{n \to \infty} X^n \rightarrow a_n \left[ (r+n)(r+n-1) + (r+n) \right] + a_{n-2} = 0 \]

\[ = \quad a_n = \frac{a_{n-2}(r)}{(r+n)(r+n-1) + (r+n)} = \frac{a_{n-2}}{(r+n)^2} \quad n \geq 2 \]
\[ n=2 \quad a_2 = -\frac{a_0}{2^2} \]
\[ n=3 \quad a_3 = -\frac{a_4}{3^2} = 0 \]
\[ n=4 \quad a_4 = -\frac{a_2}{4^2} = +\frac{a_0}{2^4 4^2} \]
\[ n=5 \quad a_5 = -\frac{a_3}{5^2} = 0 \]
\[ n=6 \quad a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^6 (6)^2} = +\frac{a_0}{2^6 (2 \cdot 3)^2} \]

In general

\[ a_{2m} = \frac{(-1)^m a_0}{2^{2m} (m!)^2} \quad m = 1, 2, 3 \quad (6) \]
Hence the first solution $y_1$ is

$$y_1(x) = x \left( \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^m (2m)!} \right)$$

$J_0$ is a Bessel function of the first kind of order zero.

$J_0$ converges for all $x$.

$J_0$ is analytic at $x=0$.

The series converges for all $x$. 

$x = 0$
Based on the recurrence relation

\[ a_n = 0, \quad n = 0, 1, 2, \ldots \]

Since \( a_k = 0 \) for \( k > 0 \), and evaluated at \( n = 0 \),

\[ a_n (t) = \frac{\partial}{\partial n} \left. 0_n \right|_{n=0} = 0 \]

\[ y_2 = y_2 (x) \ln (x) + \sum_{n=1}^{\infty} a_n (x) x^n \]

To determine the second solution use
Rewriting Eq 5 by replacing \( n \rightarrow 2m \) and running the index \( m = 1, 2, 3, 4 \ldots \)

\[
A_{2m}(r) = - \frac{A_{2m-2}(r)}{(r+2m)^2}
\]

\( m = 1 \)

\[
a_2(r) = - \frac{a_0}{(r+2)^2}
\]

\( m = 2 \)

\[
a_2(r) = - \frac{A_2}{(r+4)^2} = \frac{a}{(r+2)^2(r+4)^2}
\]

and in general

\[
a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \ldots (r+2m)^m}
\]

\( m \geq 3 \) (8)
Thus we need only to compute the derivatives of the even coefficients $a'_{2m}(x)$.

The computation of $a'_{2m}$ can be carried out most conveniently by noting that

- If

$$f(x) = (x - \alpha_1)^{\beta_1} (x - \alpha_2)^{\beta_2} (x - \alpha_3)^{\beta_3} \cdots (x - \alpha_n)^{\beta_n}$$

- And if $x$ is not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$ then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x - \alpha_1} + \frac{\beta_2}{x - \alpha_2} + \cdots + \frac{\beta_n}{x - \alpha_n}$$
\[ A_{2m}(r) = (-1)^m \frac{\alpha_0}{r^{m+2}} \left( \frac{1}{r} + \frac{1}{r+2} + \frac{1}{r+6} + \cdots \right) \]

\[ A_{2m}(0) = \begin{cases} 0 & r = 0 \\ \frac{1}{2} + \frac{1}{u} + \frac{1}{6} + \cdots + \frac{1}{2m} & r \neq 0 \end{cases} \]

From Eq. 6

\[ A_{2m}(r) = (-1)^m \frac{\alpha_0}{2^{2m}} \left( \frac{1}{m+2} \right)^2 \]
The second solution of the Bessel equation of the order zero is found by setting $a_0 = 1$ and substituting $y_1(0)$ and $a_{2m}(0) = \beta_{2m}(0)$.
\[ y_2(x) = J_0 \ln(x) + \sum_{m=1}^{\infty} \frac{(1)^{m+1} H_m}{2^{2m} (m!)^2} x^{2m} \]

Instead of \( y_2 \) the second solution is usually taken to be a certain linear combination of \( J_0 \) and \( y_2 \).

It is known as the Bessel function of the second kind of order zero and is denoted by \( Y_0 \), also known as the Weber function.

\[ Y_0 = \frac{2}{\pi} \left[ y_2(x) + (x - \ln 2) J_0 \right] \]

Here \( \gamma \) is a constant known as the Euler–Mascheroni constant; it is defined by the equation

\[ \gamma = \lim_{n \to \infty} \left( H_n - \ln(n) \right) \approx 0.5772 \]
substituting for \( y_2(x) \) in eq. 11 we obtain for \( x > 0 \)

\[
Y_0(x) = \frac{2}{\pi} \left[ \left( \gamma + \ln \frac{x}{2} \right) J_0(x) + \sum_{m=1}^{\infty} \frac{(x)^{m+1} 4m}{2^{2m} (m!)^2} x^{2m} \right]
\]

The general solution of the Bessel equation of order zero for \( x > 0 \) is

\[
y = c_1 J_0(x) + c_2 Y_0(x)
\]

Notes: (1) Note that as \( x \to 0 \)

\[
\begin{cases} 
J_0(x) \to 1 \\
Y_0(x) \to \frac{2}{\pi} \ln x \to -\infty
\end{cases}
\]
Thus, if we are interested in solutions of the Bessel's eq. of the order zero that are finite at the origin, which is often the case, we must discard $Y_0$.

\[ J_0(x) \]
\[ Y_0(x) \]

2. Note that $x \to \text{large}$

\[ J_0(x) \to \text{oscillatory} \]
\[ Y_0(x) \to \text{oscillatory} \]
such a behavior might be anticipated from the original equation (Bessel equation of the order ν)

If we divide eq. (1) by $x^2$ we obtain

$$y'' + \frac{1}{x} y' + \left(1 - \frac{ν^2}{x^2}\right)y = 0$$

For $x \to \text{large} \Rightarrow \left\{ \begin{array}{l}
\left(\frac{1}{x}\right)y' \to 0 \\
\frac{y^2}{x^2} \to 0
\end{array} \right. \text{can be neglected}

resulting in

$$y'' + y = 0$$

The solution of this equation are $\sin(x)$ and $\cos(x)$. Thus we might anticipate that the solutions of Bessel's eq. for a large $x$ are similar to linear combinations of $\sin$ and $\cos$.
This is only partly correct since for a large $x$ the functions $J_0$ and $Y_0$ also decay as $x$ increases thus the equation $y'' + y = 0$ does not provide an adequate approximation to the Bessel equation for large $x$.

It is possible to show that

\[ J_0(x) \approx \left( \frac{2}{\pi x} \right)^{1/2} \cos \left( x - \frac{\pi}{u} \right) \]\n
\[ Y_0(x) \approx \left( \frac{2}{\pi x} \right)^{1/2} \sin \left( x - \frac{\pi}{u} \right) \] as $x \to \infty$

\( (2/\pi x)^{1/2} \cos(x - \pi/u) \) Asymptotic Approximation
CASE 2: Bessel Eq of Order \( \nu = \frac{1}{2} \)

This case illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting \( \nu = \frac{1}{2} \) in eq. (\( 6 \)) gives

\[ x^2 y'' + y' + \left( 2 - \frac{1}{4} \right) y = 0 \tag{10} \]

Substitute a series solution of an identical form as (3)

\[ y(x) = \sum_{n=0}^{\infty} A_n x^{n+\nu} \]

Substitute into the diff eq (10)
\[
\sum_{n=0}^{\infty} \left[ \frac{(r+n)(r+n-1) + (r+n) - \frac{1}{u}}{n} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0
\]

\[
\left[ \frac{r(r-1) + r - \frac{1}{u}}{n} \right] a_0 x^r + \left[ \frac{(r+1)(r) + (r+1) - \frac{1}{u}}{n} \right] a_1 x^{r+1} = 0
\]

\[
\sum_{n=2}^{\infty} \left[ \frac{(r+n)(r+n-1) + (r+n) - \frac{1}{u}}{(r+n)^2 - \frac{1}{u}} \right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0
\]

\[
\begin{align*}
\{ & n = k-2 \\
& \{ n : k \to \infty \\
& \{ k : 2 \to \infty \\
& \sum_{k=2}^{\infty} a_{k-2} x^{r+k} \\
& k \to n
\end{align*}
\]

\[
(r^2 - \frac{1}{4}) a_0 x^r + \left(r+1, \frac{1}{4}\right) a_1 x^{r+1} + \sum_{n=2}^{\infty} \left[ \frac{(r+n)^2 - \frac{1}{u}}{n} \right] a_n x^{r+n} = 0
\]
The indicial equation

\[ r^2 - \frac{1}{u} = 0 \]

The roots are \( r_1 = \frac{1}{2}, \ r_2 = -\frac{1}{2} \)

The roots differ by an integer

Corresponding to the larger root \( r_1 = \frac{1}{2} \) the coefficients \( a_0 \) and \( a_1 \) are

\[ (r^2 - \frac{1}{u})a_0 x^r = 0 \times x^r \]
\[ \frac{1}{u} - \frac{1}{u} a_0 = 0 \Rightarrow a_0 \text{ arbitrary} \]

For \( a_1 \)

\[ \frac{(r+1)^2 - \frac{1}{u}}{u} a_1 x^{r+1} = 0 \times x^{r+1} \]
\[ \frac{(r + 1)^2 - \frac{1}{u}}{u} a_1 = 0 \Rightarrow a_1 = 0 \]
The recurrence relation is

\[(F + h)^2 - \frac{1}{u} \] \(a_n + a_{n-2} = 0 \quad n \geq 2\) (18)

\[a_n = - \frac{a_{n-2}}{(F + h)^2 - \frac{1}{u}}\]

For \(F + h = \frac{1}{2}\)

\[a_n = - \frac{a_{n-2}}{\left(\frac{1}{2} + h\right)^2 - \frac{1}{u}} = - \frac{a_{n-2}}{\frac{1}{4} + h + h^2 - \frac{1}{u}}\]

\[a_n = - \frac{a_{n-2}}{h(n+1)} \quad h \geq 2\]

Since \(a_1 = 0 \rightarrow a_3 = a_5 = a_7 = \cdots = 0, 2n+1 = 0\)

(All the odd elements are equal to zero)

\[a_n = - \frac{a_{n-2}}{h(n+1)} \quad n = 2, 4, 6, 8, \ldots \]

only the even elements
By letting \( n = 2m \) we obtain

\[
L = \frac{a^{2m-2}}{2m(2m+1)}
\]

\[Q_2 = -2..\]

\[
Q_0 = -\frac{a_0}{2^{3/2}} = \frac{a_0}{3}
\]

\[
Q_2 = -2
\]

\[
Q_4 = -\frac{a_0}{2^{4/2}} = -\frac{a_0}{2}
\]

\[
\cdots
\]

\[
Q_{2m} = \frac{(-1)^m a_0}{2^{2m+1}}
\]

\[
L = \frac{a_0}{5!}
\]

\[
\cdots
\]

\[
0
\]

\[
-h_0 = -2
\]

\[
0
\]

\[
0
\]
Taking $A_0 = 1$

\[ y_1 = \sum_{n=0}^{\infty} a_n x^{r+n} \]

\[ y_1 = A_0 x^{1/2} + \sum_{n=1}^{\infty} a_n x^{n+1/2} \]

\[ y_1 = x^{1/2} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m A_0^{1/2}}{(2m+1)!} x^{2m} \right] \]

[inserting $x^{1/2}$ into the sum]

\[ y_1 = x^{1/2} + \sum_{m=1}^{\infty} \frac{(-1)^m A_0^{1/2}}{(2m+1)!} x^{2m+1/2} \]

[extracting $x^{-1/2}$ out of the sum]

\[ y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m A_0^{1/2}}{(2m+1)!} x^{2m+1} \]

\[ y_1 = x^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m A_0^{1/2}}{(2m+1)!} x^{2m+1} \]

(19)
Not that the Taylor series of \( \sin x \) is

\[
\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}
\]

Hence one solution of the Bessel equation of order \( \frac{1}{2} \) is

\[
y_1 = x^{-\frac{1}{2}} \sin x
\]

The Bessel function of the first kind of order \( \frac{1}{2} \) \( J_{\frac{1}{2}} \) is defined as

\[
J_{\frac{1}{2}} = \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} y_1 = \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} \sin x \quad x > 0
\]

\[
y_1' = \sqrt{\frac{\pi}{2}} J_{\frac{1}{2}}
\]
From equation (17) for \( t = -\frac{1}{2} \), the coefficients or

\[ x^n \] and \( x^{-n} \) and both zero regardless of the choice of \( a_0 \) and \( a_1 \), can be chosen arbitrarily.

Hence, \( a_0 \) and \( a_1 \) can be taken as

From the recurrence relation (18), we obtain a set of even-numbered coefficients corresponding to \( a_0 \) and an odd-numbered coefficients corresponding to \( a_1 \).

\[
\begin{align*}
0_{2n} &= \frac{(-1)^n a_0}{(2n)!} \\
0_{2n+1} &= \frac{(-1)^n a_1}{(2n+1)!}
\end{align*}
\]

\( n = 1, 2, \ldots \)
\[ y_2(x) = x^{-1/2} \left[ a_0 \sum_{n=0}^{\infty} \frac{(1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(1)^n x^{2n+1}}{(2n+1)!} \right] \]

\[ y_2(x) = a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \quad x > 0 \]

The constant \( a_1 \) simply introduces a multiple of \( y_1(x) \).

The second second solution of the Bessel equation of order \( 1/2 \) is usually taken to be the solution for which \( a_0 = \left( \frac{2}{\pi x} \right)^{1/2} \) and \( a_1 = 0 \). It is denoted by \( J^{-1/2} \). Then

\[ J^{-1/2} = \left( \frac{2}{\pi x} \right)^{1/2} \cos x > 0 \]

The general solution of equation (16) is

\[ y = C_1 J_{1/2}(x) + C_2 J^{-1/2}(x) \]
\[ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \]

\[ J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \]

\[ x \geq 0 \]

Compare the graph of \( J_{-1/2} \) and \( J_{1/2} \) with \( J_0 \) and \( Y_0 \)

- There is a phase shift of \( \pi/4 \)

- \( J_{-1/2} \rightarrow J_0 \)
- \( J_{1/2} \rightarrow Y_0 \)

- \( J_{-1/2} \) and \( J_{1/2} \) resemble \( J_0 \) and \( Y_0 \) respectively for large \( x \)
CASE 3 - Bessel Eq of Order 1 ( \( \nu = 1 \) )

This case illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term.

Setting \( \nu = 1 \) in equation (3) gives

\[
x^2 y'' + x y' + (y^2 - 1) y = 0 \quad (23)
\]

Substitute a series solution of an identical form as (3) into eq (23)

\[
\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2} + x \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1} + x^2 y
\]

\[
-1 \sum_{n=0}^{\infty} a_n x^{r+n} = 0
\]

\[
- y - 30
\]
\[
\sum_{n=2}^{\infty} a_n \left[ (r+n)(r+n-1) + (r+n) - 1 \right] x^{r+n} + a_{n-2} x^{r+n} = 0
\]  
\[
a_0 (r^2 - r + r' - 1) x^r + a_1 (r^2 + r + r + 1 - 1) x^{r+1} + 
\sum_{n=2}^{\infty} a_n \left[ (r+n)(r+n-1) + (r+n) - 1 \right] x^{r+n} + a_{n-2} x^{r+n} = 0 
\]  
\[
a_0 (r^2 - 1) x^r + a_1 (r+1)^2 - 1) x^{r+1} + 
\sum_{n=2}^{\infty} \left( a_n \left[ (r+n)^2 - \right] + a_{n-1} \right) x^{r+n} = 0 
\]  
(2u)

The roots of the indicial equation \( r^2 - 1 = 0 \) are

\[
\begin{cases} 
    r_1 = 1 \\
    r_2 = -1 
\end{cases}
\]
The recurrence relation is

\[(r^2+n)^2 - 1\] \(a_n = -a_{n-2}\) \(n \geq 2\) (25)

corresponding to the larger root \(r_1 = 1\) the recurrence relation becomes

\[a_n = -\frac{a_{n-2}}{(n+2)n}\] \(n = 2, 3, 4, \ldots\)

From equation (24)

First element \(a_0(r^2 - 1) x^5 = 0 x^5\)

for \(r = 1\) \(a_0(0) = 0 \Rightarrow a_0 = \text{arbitrary}\)

Second element \(a_1((r+1)^2 - 1) x^{3+1} = 0 x^{r+1}\)

for \(r = 1\) \(a_1(2^2 - 1) = 0 \Rightarrow a_1 = 0\)
For \( t = 1 \)
\[ a_n = -\frac{a_{n-2}}{(n+2)n} \quad n = 2, 3, 4, 5, \ldots \]

\( h = 2 \)
\[ a_2 = -\frac{a_0}{4.2} \]

\( h = 3 \)
\[ a_3 = -\frac{a_1}{5.3} = \frac{0}{5.3} = 0 \]

\( h = 4 \)
\[ a_4 = -\frac{a_2}{6.4} = \frac{a_0}{6.4} \]

\( h = 5 \)
\[ a_5 = -\frac{a_3}{7.5} = 0 \]

\( h = 6 \)
\[ a_6 = -\frac{a_4}{8.6} = -\frac{a_0}{8.6} \]

For even values of \( h \), we can write \( h = 2m \)

\[ a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m)} = \frac{a_{2m-2}}{2^2(m+1)m} \quad m = 1, 2, 3, 4, \ldots \]
By solving this recurrence relation, we obtain

\[ a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!m!} \quad m = 1, 2, 3 \]

check

\[ m = 1 \implies a_2 = -\frac{a_0}{2^2 (1+1)!1!} = -\frac{a_0}{4 \cdot 2} \quad \checkmark \]

\[ m = 2 \implies a_4 = \frac{a_0}{2^4 (3)!2!} = \frac{a_0}{2 \cdot 2 \cdot 6 \cdot 2} \quad \checkmark \]

\[ m = 3 \implies a_6 = -\frac{a_0}{2^6 4!3!} = \frac{a_0}{2 \cdot 2 \cdot 2 \cdot 4 \cdot 3 \cdot 2} \quad \checkmark \]

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\[ y_1 = \sum_{n=0}^{\infty} a_n x^{1+n} = \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{2^m (m+1)!} x^{1+2m} \]

\[ y_1 = a_0 \times \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m+1)!} \]

\[ y_1 = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m+1)!} \]

\[ J_1(x) \]

The series converges absolutely for all \( x \), so the function \( J_1 \) is analytic everywhere.

\[ J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m+1)!} \]

(27)
In determining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in eqn (28) below is rather complicated, but the first few coefficients can be found fairly easily.

According to theorem 5.6.1 (case 3: If $\Gamma_1 - \Gamma_2 = n_1$, a positive integer)

\[ y_2(x) = ay_1(x) \ln |x| + |x|^{-1} \left( 1 + \sum_{n=1}^{\infty} C_n (n-2) x^n \right) \]

For $\Gamma_2 = -1$

\[
\begin{align*}
    y_2(x) &= a J_1(x) \ln x \quad + \sum_{n=1}^{\infty} C_n x^{n-1} \\
    y'_2(x) &= a \left[ J'_1 \ln x + J_1 \frac{1}{x} \right] + \sum_{n=1}^{\infty} C_n (n-1) x^{n-2} \\
    y''_2(x) &= a \left[ J''_1 \ln x + J'_1 \frac{1}{x} + J_1 \frac{1}{x^2} - J_2 \frac{1}{x^2} \right] + \sum_{n=1}^{\infty} C_n (n-1)(n-2) x^{n-3}
\end{align*}
\]
Substituting into \((23)\) and making use of the fact that \(J_1\) is a solution of eq \((23)\)

\[
x^2 y'' + x y' + (x^2 - 1) y = 0
\]

\[
\begin{align*}
\left[ a \left[ J'' \ln x + J'_1 \frac{1}{x} + J_1 \frac{1}{x} - J_1 \frac{1}{x^2} \right] \right] + x^2 \sum_{n=0}^{\infty} c_n (n+1)(n-2) x^{n-3} \\
\left[ a \left[ J'_1 \ln x + J_1 \frac{1}{x} \right] \right] + x \sum_{n=0}^{\infty} c_n (n-1) x^{n-2}
\end{align*}
\]

\[
\begin{align*}
+ (x^2 - 1) \left[ a J_1 \ln(x) + \left( x^2 - 1 \right) \sum_{n=0}^{\infty} c_n x^{n-1} \right] = 0
\end{align*}
\]

\[
\begin{align*}
\left[ x^2 J''_1 + x J'_1 + (x^2 - 1) J_1 = 0 \right] \quad J_1 \text{ is a solution of eq } 23
\end{align*}
\]
\[ a \ln x \left[ \frac{x^2 J''_1 + x J'_1 + (x^2 - 1) J_1}{= 0} \right] \rightarrow 0 \]

\[ a \left[ J'_1 x + J'_1 \frac{x - J'_1}{J_1} \right] + \sum_{n=1}^{\infty} C_n (n-1)(h-2) x^{n-1} \]

\[ a \left[ J_1 \right] + \sum_{n=1}^{\infty} C_n (n-1) x^{n-1} \]

\[ 2a x J'_1 + \sum_{n=0}^{\infty} \left[ C_n (h-1)(h-2) + C_n (n-1) - C_n \right] x^{n-1} + \sum_{n=1}^{\infty} C_n x^{n+1} \]
We obtain

\[ 2a \times J'_1(x) + \sum_{n=0}^{\infty} \left( (n-1)(n-2)C_n + (n-1)C_n - C_n \right) x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0 \]

where \( c_0 = 1 \). "Substituting for \( J_1(x) \) from equation (27)

\[
\begin{align*}
\sum_{n=0}^{\infty} &\left( (n-1)(n-2)C_n + (n-1)C_n - C_n \right) x^{n-1} \\
&+ \left( (1-1)(1-2)C_1 + (1-1)C_1 \right) x^0 \\
&+ \left[ (2-1)(2-2)C_2 + (2-1)C_2 \right] x^1 \\
&\text{with} \quad n = 0 \rightarrow 0 \\
&\text{and} \quad n = 1 \rightarrow C_1 \\
&\text{and} \quad n = 2 \rightarrow 0
\end{align*}
\]

\[ \sum_{n=3}^{\infty} \left( (n-1)(n-2)C_n + (n-1)C_n - C_n \right) x^{n-1} \]

- \( k = n-1 \)
- \( n = k+1 \)
- \( h : 3 \rightarrow \infty \)
- \( k = 2 \rightarrow \infty \)

\[ 0 \int_0^1 x^{-1} C_1 x^0 + 0 \int_0^1 x^4 + \sum_{k=2}^{\infty} \left[ (k)(k-1)C_{k+2} + kC_{k+2} - C_{k+1} \right] x^{n-2} \]
\[
\begin{align*}
\sum_{k=0}^{\infty} c_k x^k &= \sum_{n=0}^{\infty} c_n x^{n+1} \\
\sum_{k=2}^{\infty} c_{k-1} x^k = 0 &\quad \text{for } n \geq 2 \\
\sum_{n=2}^{\infty} \left[ \left( n^2 - 1 \right) c_{n+1} + c_{n-1} \right] x^n &= \left( \frac{1}{2} \right) \left( \frac{1}{x} \right) \\
= -c_1 + x + \sum_{n=2}^{\infty} \left( n^2 - 1 \right) c_{n+1} x^n + c_{n-1} x^n
\end{align*}
\]
\[ 2ax J'_1 = 2ax \left[ \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m+1)! m!} \right] \]

\[ = ax \left[ \frac{(-1)^0 x^0}{2^0 (1)! 0!} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m+1)! m!} \right] \]

\[ \theta \left[ x + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+1}}{2^m (m+1)! m!} \right] \]

\[ -c_1 + x - \sum_{n=2}^{\infty} \left[ (h^2 - 1) C_{n+1} + c_{n-1} \right] x^n = -a \left[ x + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+1}}{2^m (m+1)! m!} \right] \]

\[ -c_1 x^0 + (0 c_2 + c_0) x^1 \]

\[ -42 \quad 0 x^0 - ax^1 - a \sum_{m=1}^{\infty} \]
From Eq. 30

\[ X^0: \quad -c_1 = 0 \quad \Rightarrow \quad c_1 = 0 \]

\[ X^1: \quad 0c_2 + c_0 = -a \quad \Rightarrow \quad c_0 = -a = 1 \quad \rightarrow \]

Assume \( c_0 = 1 \)
\( a = -1 \)

Since \( c_1 = 0 \) \( \Rightarrow \) \( c_3 = c_5 = c_7 = \ldots = 0 \)

Corresponding to the odd power of \( x \) writing \( n = 2m + 1 \) on the left side of Eq. 30

\[
\left( (2m+1)^2 - 1 \right) c_{2m+2} + c_{2m} = \frac{(-1)^m (2m+1)}{2^{2m} (m+1)! \ m!}, \quad m=1, 2, 3, \ldots
\]

multipliers of \( x^n \) in Eq. 30

\( n \rightarrow 2m+1 \)
when we set \( m=1 \) in eq. 31, we obtain

\[
(3^2 - 1) \, c_4 + c_2 = \frac{(-1)^3}{2^2 \, 2! \, 1!}
\]

- Notice that \( c_2 \) can be selected arbitrarily, and then this equation determine \( c_4 \)

- Notice that in equation (30) for the coefficient of \( x \), \( c_2 \) appeared multiplied by \( 0 \), and that equation was used to determine \( a \). That \( c_2 \) is arbitrary is not surprising, since \( c_2 \) is the coefficient of \( x \) in

the expression

\[
x^{-1} \left( 1 + \sum_{n=1}^{\infty} c_n x^n \right)
\]
Consequently, $C_2$ simply generates multiple of $J_1$, and $y_2$ is determined only up to an additive multiple of $J_1$.

In accordance with the usual practice, we choose

$$C_2 = \frac{1}{2^2}$$

Then we obtain

$$C_4 = \left(\frac{-1)^3}{2^2 2! 1!} - \frac{1}{2^2}\right) \frac{1}{3^2 - 1}$$

$$= \left(\frac{(-1)^3}{2^3} - \frac{1}{2^2}\right) \frac{1}{2^3} =$$

$$= \frac{(-1)^3}{2^3 2^2} \left(\frac{3}{2} + 1\right) = \frac{(-1)^3}{2^3 2^2} \left(\frac{3}{2} + 1\right) = -\frac{1}{2^4 2^1} \left(1 - \frac{1}{2}\right) + 1$$

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Given
\[ H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \quad ; \quad H_0 = 0 \]

\[ C_n = \frac{(-1)}{2^{n-1}} \left[ H_n + H_1 \right] \]

It is possible to show that the solution of the recurrence relation (31) is

\[ C_{2m} = \frac{(-1)^{m+1} \left( H_m + H_{m-1} \right)}{2^{2m}\cdot m! \cdot (m-1)!} \quad m = 1, 2, \ldots \]

Thus

\[ y_2 = -J_1 \ln(x) + \frac{1}{x} \left( 1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} \cdot m! \cdot (m-1)!} \cdot x^{2m} \right) \quad x > 0 \]
The Second Solution of equation (23), the Bessel function of the second kind of order one, $Y_1(x)$ is usually taken to be a certain linear combination of $J_1(x)$ and $Y_2(x)$. $Y_1(x)$ is defined as

$$Y_1(x) = \frac{2}{\pi} \left( y_2(x) + (\tfrac{\mu}{\nu} - \ln 2) J_1(x) \right) \quad (33)$$

Where $y_2(x)$ is defined in eq. (12).

The general solution of eq. 23 for $x > 0$ is

$$y = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x)$$
Notice that $J_1$ is analytic at $x = 0$ and the second solution $Y_1$ becomes unbounded in the same manner as $1/x \cos x \to 0$.

The graphs of $f_1$, $Y_1$, $J_1(x)$, and $Y_1(x)$ are shown in the diagram.
Bessel's Eq. - Summary

\[ x^2 y'' + x y' + (x^2 - u^2) y = 0 \]

\[ y = C_1 J_u(x) + C_2 Y_u(x) \]

\[ J_u(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+u} n! (n+u)!} x^{2n+u} \]

\[ Y_u(x) = \frac{J_u(x) \cos(\nu \pi) - J_{-\nu}(x)}{\sin(\nu \pi)} \]
The Wave Equation in Polar Coordinate System
Vibration of a Membrane - Circular Drum Head
THE WAVE EQUATION

Ref: BILL GOODWIN, ENGINEERING DIFFERENTIAL EQUATION - THEORY AND APPLICATIONS
SECT. 1.5 VIBRATION MEMBRANE

THE WAVE EQ.

The wave equation is an important second order linear partial differential equation for the description of waves - as they occur in classical physics - such as mechanical waves (e.g. water waves, sound waves, seismic waves) or light waves. It arises in the fields like acoustics, electromagnetics and fluid dynamics.

The wave equation is hyperbolic partial differential equation. It typically concerns
\[ x - \text{fixed constant} \]
\[ t - \text{time variable} \]
\[ x_1, x_2, \ldots, x_n - \text{spacial variable} \]
\[ U = U(x_1, x_2, \ldots, x_n) - \text{scalar function whose} \]
values could model the mechanical displacement of the wave. The wave equation of \( u \) is

\[
\frac{\partial^2 u}{\partial t^2} = \alpha^2 \nabla^2 u = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]
\]

\[
\text{↑ spatial laplacian}
\]

\[
\frac{\partial^2 u}{\partial t^2} = \alpha^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]
\]

\[
\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}
\]

---

**THE TWO DIMENSIONAL WAVE EQUATION**

**IN POLAR COORDINATES**

The two dimensional wave equation is given by

\[
\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\]

\[
\frac{1}{\alpha^2} \ u_{tt} = u_{xx} + u_{yy}
\]
- For a circular membrane like a drum, because the boundary condition will hold at a fixed radius, it is much more convenient to solve it in a polar coordinate system.

- The relationship between polar and cartesian coordinates is given by

\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta
\end{align*}
\]

and the inverse transformation is given by

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2} \\
    \theta &= \tan^{-1} \left( \frac{y}{x} \right)
\end{align*}
\]

- We need to relate derivatives with respect to the variable \( x \) and \( y \) to derivatives with respect to the variables \( r \) and \( \theta \).
Because we know the expressions for the change of coordinates, we can write the functions as

\[ u(r, \theta, t) = u(r(x, y), \theta(x, y), t) \]

By the chain rule

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y}
\end{align*}
\]

\[
\begin{align*}
U_x &= u_r x_x + u_\theta A_x \\
U_y &= u_r y_y + u_\theta A_y
\end{align*}
\]
- Differentiate with respect to $x$ again and use the product rule

\[ U_{xx} = U_{rr} R_{xx} + (U_r)_x \Gamma_x + U_\theta A_{xx} + (U_\theta)_x A_x \]

\[ \hat{U}_{xx} = U_{rr} R_{xx} + (U_{rr} \Gamma_x + U_\theta \theta_x) \Gamma_x + U_\theta A_{xx} + (U_\theta \Gamma_x + U_\theta \theta_x) A_x \]

\[ = U_{rr} R_{xx} + U_{rr} \Gamma_x^2 + 2 U_\theta \Gamma_x A_x + U_\theta A_{xx} + U_\theta \theta_x A_x^2 \]

- In a similar fashion sub $x$ with $y \ x \rightarrow y$

\[ U_{yy} = U_{rr} \Gamma_{yy} + U_{rr} \Gamma_y^2 + 2 U_\theta \Gamma_y A_y + U_\theta A_{yy} + U_\theta \theta_y A_y^2 \]
\[ U_{xx} + U_{yy} = U_r \left( \Gamma_{xx} + \Gamma_{yy} \right) + U_{rr} \left( \Gamma_x^2 + \Gamma_y^2 \right) + 2U_{rg} \Gamma_x \Gamma_y + U_\theta \Theta_{xx} + U_{g\theta} \Gamma_x^2 + U_{g\theta} \Gamma_y^2 \]

\[ U_{yy} \]

\[ = U_r \left( \Gamma_{xx} + \Gamma_{yy} \right) + U_{rr} \left( \Gamma_x^2 + \Gamma_y^2 \right) + 2U_{rg} \Gamma_x \Gamma_y + U_\theta \Theta_{xx} + U_{g\theta} \Gamma_x^2 + U_{g\theta} \Gamma_y^2 \]

- Differentiating the relationship \( x^2 + y^2 = r^2 \)

with respect to \( x \)

\[ (x^2 + y^2)_x = (r^2)_x \]

\[ 2x = 2rx \quad \rightarrow \quad \Gamma_x = \frac{x}{r} \]

and with respect to \( y \)

\[ (x^2 + y^2)_y = (r^2)_y \]

\[ 2y = 2ry \quad \rightarrow \quad \Gamma_y = \frac{y}{r} \]
Differentiating $(f(x))^x$ and $(f(y))^y$

Plugging the expression for $x$

Differentiating $(f(x))^x$:

$$x^x = e^{x \ln x}$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{x \ln x}$$

$$\frac{d}{dx} x^x = x^x (\ln x + 1)$$

So that:

$$\frac{d}{dx} x^x = x^x \ln x + x^x$$

Differentiating $(f(y))^y$:

$$y^y = e^{y \ln y}$$

$$\frac{d}{dy} y^y = \frac{d}{dy} e^{y \ln y}$$

$$\frac{d}{dy} y^y = y^y (\ln y + 1)$$

Thus:

$$\frac{d}{dy} y^y = y^y \ln y + y^y$$

In a similar fashion, differentiating $(f(y))^y$:

$$\frac{d}{dy} y^y = \frac{y^y}{y}$$

So that:

$$\frac{d}{dy} y^y = y^{y-1}$$
\[ r^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = 1 \]

Differentiate \( \tan \theta = -\frac{y}{x} \) with respect to \( x \) and \( y \):

\[ (\tan \theta)_x = \left(\frac{y}{x}\right)_x \rightarrow (\sec^2 \theta)_x = \frac{x \cdot 0 - y \cdot 1}{x^2} = -\frac{y}{x^2} \]

\[ a_x = -\frac{y \cos^2 \theta}{x^2} \]

\[ (\tan \theta)_y = \left(\frac{y}{x}\right)_y \rightarrow (\sec^2 \theta)_y = \frac{x \cdot 1 - y \cdot 0}{x^2} = \frac{1}{x} \]

\[ a_y = \frac{\cos^2 \theta}{x} = \frac{x}{r^2} \]
Differentiate \((A_x)\) and \((A_y)\)

\[
A_{xx} = \frac{2}{r} \quad A_{xy} = \frac{2xy}{r^2}
\]

\[
A_{yy} = -\frac{2y}{r^2}
\]

So that

\[
\theta_{xx} + \theta_{yy} = 2\left(\frac{y}{r^2} - \frac{y}{r^2}\right) = 0
\]
\[(4x)^2 + (4y)^2 = (\frac{y}{r^2})^2 + (\frac{x}{r^2})^2 = \frac{x^2 + y^2}{r^4} = \frac{1}{r^2}\]

\[r_x \theta_x + r_y \theta_y = \frac{x}{r} \left( -\frac{y}{r^2} \right) + \frac{y}{r} \left( \frac{x}{r^2} \right) = \frac{-xy}{r^3} + \frac{xy}{r^3} = 0\]

We finally obtain

\[u_{xx} + u_{yy} = u_{rr} \left( \frac{1}{r^2} \right) + u_{rr} \left( \frac{r^2 - 1}{r^4} \right) + 2u_r \left( \frac{r}{r^2} \right) + u_\theta \left( \frac{4x}{r^2} \right) + u_\theta \left( \frac{4y}{r^2} \right) + u_{\theta \theta} \left( \frac{x^2}{r^4} + \frac{y^2}{r^4} \right)\]

\[u_{xx} + u_{yy} = \frac{1}{r} u_r + u_{rr} + \frac{1}{r^2} u_{\theta \theta}\]
Substituting these expressions into the right hand side of the wave equation gives the wave equation in polar coordinates

\[
\frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}
\]

**Boundary and Initial condition BC, IC**

- for a circular drum the BC is

\[
\begin{align*}
\frac{\partial u}{\partial r}(r,\theta, t) &= 0 \\
\end{align*}
\]

and the initial conditions

\[
\begin{align*}
u(r,\theta, 0) &= f(r,\theta) \\
\frac{\partial u}{\partial t}(r,\theta, 0) &= g(r,\theta)
\end{align*}
\]
Assume a solution of the form

\[ u(\tau, \theta, t) = R(\tau) \Theta(\theta) T(t) \]

and substitution into the wave equation in polar coordinates gives

\[
\begin{align*}
U_\tau &= R' \Theta T \\
U_{\tau\tau} &= R'' \Theta T \\
U_{\theta\theta} &= R \Theta'' T \\
U_{tt} &= R \Theta T''
\end{align*}
\]

\[ R'' \Theta T + \frac{1}{\tau} R' \Theta T + \frac{1}{r^2} R \Theta'' T = \frac{1}{v^2} R \Theta T'' \]

Dividing by \( R \Theta T \) gives
\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = \frac{1}{\alpha^2} \frac{T''}{T}
\]

- Note that the right side of the equation only depends on \(T\) and the left side depends on \(r\) and \(\Theta\), and all three variables are independent, therefore both sides must be constant.

\[
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = \frac{1}{\alpha^2} \frac{T''}{T} = -\lambda
\]

where \(\lambda\) is yet to be determined constant. Hence

\[
\begin{cases}
T'' + \alpha^2 \lambda T = 0 \\
\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \Theta'' = -\lambda
\end{cases}
\]
Multiplying by \( t^2 \) and rearranging gives

\[
\frac{R'}{R} + t^2 \frac{R''}{R} = -\frac{c^2}{c} \frac{R'}{R} + \frac{c^2}{c} \frac{R''}{R} = -c^2 \frac{R'}{R} + \frac{c^2}{c} \frac{R''}{R}
\]

The left-hand side of this equation only depends on \( \theta \) and the variable \( R \), hence the right side is independent, which is necessarily the same as \( \frac{R'}{R} + \frac{R''}{R} \).

Calling this constant \( -c^2 \), we have

\[
\frac{R'}{R} + \frac{R''}{R} = \frac{c^2}{c} = c^2
\]

Hence

\[
r^2 \frac{R'}{R} + \frac{R''}{R} + (t^2 \alpha - \alpha) R = 0
\]
If we determine the solution to equation

\[
\begin{align*}
T'' + \lambda^2 \xi T &= 0 \\
\Theta'' + \tau \Theta &= 0 \\
\tau^2 \Phi'' + \tau \Phi' + \left(\frac{\tau^2 \xi}{\tau} \right) \Phi &= 0
\end{align*}
\]

we have the solution to the wave equation

\[
\frac{1}{\lambda^2} \frac{d^2 u}{dt^2} = \frac{\partial^2 u}{\tau^2} + \frac{1}{\tau} \frac{du}{d\tau} + \frac{1}{\tau^2} \frac{d^2 u}{d\xi^2}
\]

where

\[
U(t, \xi, \tau) = R(t) \Theta(\xi) T(\tau)
\]
**Solution for \( \Theta (\theta) \)**

\[ \Theta''(\theta) + \lambda \Theta(\theta) = 0 \]

Although it appears that we only have one boundary condition given by

radius of the drum \( U(\hat{r}, \theta, t) = 0 \)

there is also the fact that the solution for \( \Theta(\theta) \) must be periodic, that is

\[ \Theta(\theta) = \Theta(\theta + 2\pi) \]

given the circular geometry of the drum.

Thus \( \lambda \) must be positive and the solution is

\[ \Theta(\theta) = C_1 \sin \sqrt{\lambda} \theta + C_2 \cos \sqrt{\lambda} \theta \]

In order for \( \Theta(\theta) = \Theta(\theta + 2\pi) \), \( \sqrt{\lambda} \) must be an integer, or

-16-
\[ V_F = m = 0, 1, 2, \ldots \]

\[ A_m(\theta) = C_1 \sin(m \theta) + C_2 \cos(m \theta) \quad m = 1, 2, 3, \ldots \]
Solution for $R(t)$

$$r^2 R''(t) + r R'(t) + (r^2 - \omega^2) R(t) = 0$$

Because $\sqrt{r} = m \Rightarrow \omega = m^2$

The equation becomes

$$r^2 R''(t) + r R'(t) + (r^2 - m^2) R(t) = 0$$

In the special case where $m$ is an integer which is the present case, the general solution may be written as

$$R(t) = C_1 J_m(\sqrt{r} t) + C_2 Y_m(\sqrt{r} t)$$

Where

$$J_m(\sqrt{r} t) = \sum_{h=0}^{\infty} \frac{(-1)^h}{h! (h-m)!} \left( \frac{\sqrt{r} t}{2} \right)^{2h-m}$$
and

\[ Y_m(\sqrt{\alpha}r) = \ln(\sqrt{\alpha}r) J_m(\sqrt{\alpha}r) \]

\[- \frac{1}{2} \sum_{n=0}^{m-1} \frac{(m-n-1)!}{n!} \left( \frac{\sqrt{\alpha}r}{2} \right)^{2n-m} \]

\[- \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left( \frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+m} }{n! (n+m)!} \right) \left( \frac{\sqrt{\alpha}r}{2} \right)^{2n+m} \]

- The function \( J_m \) and \( Y_m \) are the Bessel function of the first and the second kind respectively, or order \( m \).

- Note that as \( r \to 0 \), \( Y_m(r) \to -\infty \).
and hence we assume to motion of the center of the drum is bounded, then we must have \( c_2 = 0 \)

Hence

\[
R(t) = c_1 \text{J}_m \left( \sqrt{\alpha} t \right)
\]

and because the boundary condition

\[
U \left( \hat{r}, 4, t \right) = 0
\]

requires that \( R(\hat{r}) = 0 \)

at the boundary of

either \( c_1 = 0 \), which would give the trivial solution or

\[
\text{J}_m \left( \sqrt{\alpha} \hat{r} \right) = 0
\]
It is apparent that Bessel function of the first kind of various order are equal to zero for multiple values of $r$ and they are tabulated in Table 5.2 p. 188 [see ref].

<table>
<thead>
<tr>
<th>ORDER</th>
<th>1st Zero</th>
<th>2nd Zero</th>
<th>3rd Zero</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.40483</td>
<td>5.52008</td>
<td>8.65373</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>3.83171</td>
<td>7.01559</td>
<td>10.1735</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>5.13562</td>
<td>8.41724</td>
<td>11.6198</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>6.38016</td>
<td>9.76102</td>
<td>13.0152</td>
<td>...</td>
</tr>
</tbody>
</table>

Table of zeros of the Bessel function of the first kind
Let $Z_m,n$ denote the $n$th zero of the Bessel function of the first kind of order $m$.

\[ Z_m, n \]

with order $m$, $n$th zero

\[ \sqrt{\lambda} = Z_m, n \]

\[ \lambda = \left( \frac{Z_m,n}{R} \right)^2 \quad n = 1, 2, 3, \ldots \]

and

\[ R(t) = C_1 J_m\left( \frac{Z_m,n R}{\lambda} \right) \]

The way to intuitively think of the role of $Z_m,n$ is that it scales $\lambda$ in such a way that it will go through zero at the radius of the drum.
The following figures illustrate the Bessel functions of the first kind of order zero and one with \( \hat{r} = 5 \) and \( Z_{m,n} \) equal to the first three zeros for each one.

The feature to observe is that scaling the argument be

\[ Z_{m,n} \frac{r}{\hat{r}} \]

makes all the functions go to zero at \( \hat{r} = 5 \), which is what is necessary to match the boundary condition at the radius of the drum.
SOLUTION FOR $I(t)$

$$I''(t) + \lambda^2 \lambda I(t) = 0$$

plugging the definition of $\lambda$

$$\lambda = \left( \frac{Z_{m,n}}{r^2} \right)^2$$

into the diff. eq.

$$I''(t) + \left( \frac{\lambda Z_{m,n}}{F} \right)^2 I(t) = 0$$

and hence the solution is

$$I(t) = d_1 \cos \left( \frac{\lambda Z_{m,n} t}{F} \right) + d_2 \sin \left( \frac{\lambda Z_{m,n} t}{F} \right)$$
THE ENTIRE SOLUTION

For a fixed integer \( m \) and \( n \)

\[
U(r, \theta, t) = R(r) \Theta(\theta) T(t)
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\partial}{\partial \theta} \left( a_{mn} \cos(\theta) + b_{mn} \sin(\theta) \right) \right) \left( \frac{\partial}{\partial \theta} \left( \frac{Z_{mn}}{r} \right) \right) \left( \frac{\partial^2}{\partial t^2} \right) \left( \frac{r}{Z_{mn}} \right)
\]

Any linear combination of these solutions is also a solution.

Summing over both \( m \) and \( n \) and combining some of the constants gives

\[
U(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left( a_{mn} \cos(\theta) + b_{mn} \sin(\theta) \right) \left( \frac{Z_{mn}}{r} \right) \left( \frac{\partial^2}{\partial t^2} \right) \left( \frac{r}{Z_{mn}} \right)
\]
- This solution satisfies the wave equation in polar coordinates as well as the boundary condition.

- The initial condition given in

\[
\begin{align*}
\{ \\
U(r, \theta, 0) &= f(r, \theta) \\
\frac{\partial U}{\partial t}(r, \theta, 0) &= U_t(r, \theta, 0) = g(r, \theta)
\end{align*}
\]

still need to be satisfied.

- Substituting \( t=0 \) into the solution gives

\[
U(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) \left( a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta) \right) = f(r, \theta)
\]

- To determine the coefficients, we make use of orthogonality of the sin and cosine functions as well as the following fact:

\[
\int_0^\infty J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) J_m \left( \frac{Z_{m,n} \hat{r}}{r} \right) dr = \begin{cases} 
0 & n \neq \hat{n} \\
\frac{1}{2} \left[ \frac{d}{dr} \left( Z_{m,n} \right) \right]^2 & n = \hat{n}
\end{cases}
\]

observe that the integral is weighted by \( r \) and also the Bessel functions in the integral is
that a different $z_{m,n}$ appears in the argument to the function. The two terms in the integral would be two of the curve in the previous graph.

- To determine the coefficients $a_{m,n}$ that satisfy

$$
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{z_{m,n}}{r}\right)(a_{m,n} \cos(m\theta) + b_{m,n} \sin(m\theta)) = f(r, \theta)
$$

multiply both sides of the equation by $\cos(\hat{m}\Theta)$ and integrate from 0 to $2\pi \hat{m}$ gives

$$
\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_0^{2\pi} J_m\left(\frac{z_{m,n}}{r}\right)(a_{m,n} \cos(m\Theta) + b_{m,n} \sin(m\Theta)) \cos(\hat{m}\Theta) d\Theta
$$

$$
= \int_0^{2\pi} \cos(\hat{m}\Theta) f(r, \Theta) d\Theta
$$

- Because of the orthogonality of the sine and the cosine functions, every term in the series indexed by $m$ is zero except for when $\hat{m} = m$ and hence

-28-
\[ \sum_{n=1}^{\infty} J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \int_0^{2\pi} a_m,n \cos^2(m\theta) \, d\theta \]

\[ = \int_0^{2\pi} F(r, \theta) \cos(m\theta) \, d\theta \]

So

\[ \sum_{n=1}^{\infty} a_m,n J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) = \frac{\int_0^{2\pi} f(r, \theta) \cos(m\theta) \, d\theta}{\int_0^{2\pi} a_m,n \cos^2(m\theta) \, d\theta} \]

Now multiplying by \( r \cdot J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \) and integrating from 0 to \( \hat{r} \) gives

\[ \sum_{n=1}^{\infty} \int_0^{\hat{r}} r \cdot a_m,n \cdot J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \cdot J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \, dr \]

\[ = \int_0^{\hat{r}} r \cdot J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \cdot \frac{\int_0^{2\pi} f(r, \theta) \cos(m\theta) \, d\theta}{\int_0^{2\pi} a_m,n \cos^2(m\theta) \, d\theta} \, dr \]

or

\[ a_m,n = \frac{\int_0^{2\pi} \int_0^{\hat{r}} r \cdot f(r, \theta) \cos(m\theta) \cdot J_m \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \, d\theta \, dr}{\left( \int_0^{2\pi} \cos^2(m\theta) \, d\theta \right) \left( \int_0^{\hat{r}} J_m^2 \left( \frac{Zm,n \hat{r}}{\hat{r}} \right) \, dr \right)} \]
An analogous computation gives

$$b_{m,n} = \frac{\int_0^{2\pi} \int_0^R f(r, \theta) \sin(m \theta) \ J_m \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \, d\theta}{\left( \int_0^{2\pi} \sin^2(m \theta) \, d\theta \right) \left( \int_0^R J_m^2 \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \right)}$$

To determine $c_{m,n}$ and $d_{m,n}$ differentiate $u(r, \theta, t)$ with respect to time, substitute $t = 0$, and follow the same procedure, which gives

$$c_{m,n} = \frac{\int_0^{2\pi} \int_0^R r g(r, \theta) \cos(m \theta) \ J_m \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \, d\theta}{\int_0^{2\pi} \cos^2(m \theta) \, d\theta} \left( \int_0^R J_m^2 \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \right)$$

$$d_{m,n} = \frac{\int_0^{2\pi} \int_0^R r g(r, \theta) \sin(m \theta) \ J_m \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \, d\theta}{\int_0^{2\pi} \cos^2(m \theta) \, d\theta} \left( \int_0^R J_m^2 \left( \frac{Z_{mn} r}{\hat{r}} \right) \, dr \right)$$

and with that, we have solved the two dimensional wave equation in a polar coordinates
Summary of the solution to the Wave Equation in Polar Coordinates

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2}
\]

**BC** \( \rightarrow \) \[
\begin{align*}
u(r, \theta, t) &= 0 \\
u(r, \theta, 0) &= f(r, \theta) \\
\frac{\partial u}{\partial t}(r, \theta, 0) &= g(r, \theta)
\end{align*}
\]

**UC** \( \rightarrow \) \[
\begin{align*}
U(r, \theta, t) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left( \frac{Z_{mn} r}{\hat{r}} \right) \left[ (a_{mn} \cos(m \theta) + b_{mn} \sin(m \theta)) \cos \left( \frac{\alpha Z_{mn} n t}{\hat{r}} \right) \right] \\
&\quad + (c_{mn} \cos(m \theta) + d_{mn} \sin(m \theta)) \sin \left( \frac{\alpha Z_{mn} n t}{\hat{r}} \right) \\
&\quad - 3r
\end{align*}
\]
\begin{align*}
 a_{m,n} &= \frac{\int_{0}^{2\pi} \int_{0}^{\hat{r}} f(r, \theta) \cos(m\theta) \ J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) \ d\theta \ dr}{\left( \int_{0}^{2\pi} \cos^2(m\theta) \ d\theta \right) \left( \int_{0}^{\hat{r}} J_m \left( \frac{Z_{m,n} \hat{r}}{r} \right) \ dr \right)} \\
 b_{m,n} &= \frac{\int_{0}^{2\pi} \int_{0}^{\hat{r}} r f(r, \theta) \sin(m\theta) \ J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) \ d\theta \ dr}{\left( \int_{0}^{2\pi} \sin^2(m\theta) \ d\theta \right) \left( \int_{0}^{\hat{r}} J_m \left( \frac{Z_{m,n} \hat{r}}{r} \right) \ dr \right)} \\
 c_{m,n} &= \frac{\int_{0}^{2\pi} \int_{0}^{\hat{r}} g(r, \theta) \cos(m\theta) \ J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) \ d\theta \ dr}{\int_{0}^{2\pi} \cos^2(m\theta) \ d\theta \left( \int_{0}^{\hat{r}} J_m \left( \frac{Z_{m,n} \hat{r}}{r} \right) \ dr \right)} \\
 d_{m,n} &= \frac{\int_{0}^{2\pi} \int_{0}^{\hat{r}} r g(r, \theta) \sin(m\theta) \ J_m \left( \frac{Z_{m,n} r}{\hat{r}} \right) \ d\theta \ dr}{\int_{0}^{2\pi} \sin^2(m\theta) \ d\theta \left( \int_{0}^{\hat{r}} J_m \left( \frac{Z_{m,n} \hat{r}}{r} \right) \ dr \right)}
\end{align*}