PROJECT
COLUMN BUCKLING
HIGH ORDER DIFF EQ.


**PROJECT - COLUMN BUCKLING**

**MATH BACKGROUND**

1. **TWO EQUATION WORTH KNOWING**

\[
\begin{align*}
\left\{ y'' + k^2 y &= 0 \quad (1) \right. \\
\left. y'' - k^2 y &= 0 \quad (2) \right. \\
\end{align*}
\]

$k$ is real

The two differential equations (1), (2) where $k$ is real, are important in applied mathematics.

\[\text{Eq 1}\]

The auxiliary equation

\[m^2 + k^2 = 0\]

\[m^2 = -k^2\]

roots \[m = \pm k \sqrt{-1} = \pm k i\]

The general solution of the DE is

\[y = c_1 \cos(kx) + c_2 \sin(kx) \quad (3)\]
The auxiliary equation

\[ m^2 - k^2 = 0 \]
\[ m^2 = k^2 \]
\[ m = \pm k \]

The general solution of the DE is

\[ y = c_1 e^{kx} + c_2 e^{-kx} \quad (4) \]

Notice that if we choose

\[ c_1 = c_2 = \frac{1}{2} \]

and

\[ c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2} \]

we get particular solutions

\[ y = \frac{1}{2} (e^{kx} + e^{-kx}) = \cosh (kx) \]
\[ y = \frac{1}{2} (e^{kx} - e^{-kx}) = \sinh (kx) \]
Since $\cosh(kx)$ and $\sinh(kx)$ are linear independent on any interval of the $x$-axis, an alternative form for the general solution of Eq. (2) is

\[ y = C_1 \cosh(kx) + C_2 \sinh(kx) \]

1.2 NONTRIVIAL SOLUTION OF A BOUNDARY-VALUE PROBLEM (BVP)

Given the BVP

\[ y'' + \lambda y = 0 \]

Boundary Values

\[ \begin{cases} y(0) = 0 \\ y(L) = 0 \end{cases} \]

We shall consider three cases:

1. $\lambda = 0$
2. $\lambda < 0$
3. $\lambda > 0$
CASE 1

\[ x = 0 \]

\[ y'' + xy' = 0 \]

\[ y'' = 0 \]

\[ y' = c_1 \]

\[ \Rightarrow y = c_1 x + c_2 \]

Apply the BV

\[ y(0) = c_1 0 + c_2 = 0 \Rightarrow c_2 = 0 \]

\[ y(L) = c_1 L = 0 \Rightarrow c_1 = 0 \]

Hence for \( x = 0 \) the only solution of the BVP is the trivial solution \( y = 0 \)
CASE 2 \( x < 0 \)

It is convenient to write

\[ x = -\alpha^2 \quad \alpha > 0 \]

Given this notation the roots of the auxiliary equation

\[ m^2 - \alpha^2 = 0 \quad \Rightarrow \quad m_1 = +\alpha \]
\[ m_2 = -\alpha \]

Since the interval on which we are working on is finite, we choose to write the general solution of

\[ y'' - \alpha^2 y = 0 \]

as

\[ y = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x) \]
FIRST BV \[ y(0) = c_1 \cosh(0) + c_2 \sinh(0) = 0 \]
\[ = c_1 \cdot 1 + c_2 \cdot 0 = 0 \]
\[ \Rightarrow c_1 = 0 \]
\[ \Rightarrow y = c_2 \sinh(\alpha x) \]

SECOND BV \[ y(L) = c_2 \sinh(\alpha L) = 0 \]
- for any \( \alpha \neq 0 \) \( \sinh(\alpha L) \neq 0 \)
- We are forced to choose \( c_2 = 0 \)
- the only solution of the BVP is the trivial solution \( y = 0 \)
CASE 3  \( \lambda > 0 \)

- we may again write

\[
J = +x^2 \quad \alpha > 0
\]

- The auxiliary equation has to complex roots

\[
m^2 + x^2 = 0 \quad \Rightarrow \quad m_1 = ix, \quad m_2 = -ix
\]

- The general solution is

\[
y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)
\]

**FIRST BV**  \( y(0) = c_1 \cos(0) + c_2 \sin(0) \)

\[
c_1 \cdot 1 + c_2 \cdot 0 = 0
\]

\[\Rightarrow c_1 = 0\]

\[\Rightarrow y = c_2 \sin(\alpha x)\]

**SECOND BV**  \( y(L) = c_2 \sin(\alpha L) = 0 \)

- It is satisfied by choosing \( c_2 = 0 \) but that means that \( y = 0 \)
- If we require \( c_2 \neq 0 \), then \( \sin \alpha L = 0 \) is satisfied whenever \( \alpha L \) is an integer multiple of \( \pi \).

\[
x = \frac{n \pi}{2}
\]

\[
\lambda = \omega_n = \left( \frac{n \pi}{L} \right)^2 \quad n = 1, 2, 3, \ldots
\]

- Therefore for any real non-zero \( c_2 \)

\[
y = c_2 \sin \left[ \frac{n \pi}{L} x \right] \quad n = 1, 2, 3, \ldots
\]

is a solution of the problem for each \( n \).

- Because the differential eq. is homogeneous, any constant multiple of a solution is also a solution, so we may simply take \( c_2 = 1 \).
For each number in the sequence

\[ J_1 = \frac{\pi^2}{L^2} \quad J_2 = \frac{4\pi^2}{L^2} \quad J_3 = \frac{9\pi^2}{L^2} \quad \ldots \]

\[ n = 1 \quad n = 2 \quad n = 3 \]

the corresponding function in the sequence

\[ y_1 = \sin \left( \frac{n \pi}{L} x \right) \quad y_2 = \sin \left( \frac{2n \pi}{L} x \right) \quad y_3 = \sin \left( \frac{3n \pi}{L} x \right) \quad \ldots \]

is a nontrivial solution of the problem

\[ y'' + \lambda_n y = 0 \]
\[ y(0) = 0 \]
\[ y(L) = 0 \]

\[ \lambda_n \rightarrow \text{EIGENVALUES} \]

\[ y_n = \sin \left( \frac{n \pi}{L} x \right) \rightarrow \text{EIGENFUNCTIONS} \]
2. BUCKLING OF A THIN VERTICAL COLUMN (BVP1)

In the 18th century Leonhard Euler was one of the first mathematicians to study an eigenvalue problem in analyzing how a thin elastic column buckles under a compressive axial force.

Consider a long, slender vertical column of a uniform cross section and length \( L \).

Let \( y(x) \) denote the deflection of the column when a constant vertical compressive force or load, \( P \), is applied to its top.
By comparing bending moments at any point along the column, we obtain

$$ \pm I \frac{d^2y}{dx^2} = -Py $$

or

$$ \pm I \frac{d^2y}{dx^2} + Py = 0 $$

where $E$ is the Young's modulus of elasticity and $I$ is the moment of inertia of the cross section about a vertical line through its centroid.

The BV conditions are

$$ \left\{ \begin{array}{l}
y(x=0) = 0 \\
y(x=L) = 0
\end{array} \right. $$
PROBLEM No 1

(a) Find the first three axial buckling loads $P_1, P_2, P_3$ that will cause the beam to buckle.

(b) Define the corresponding three deflections of the beam $y_1(t), y_2(t), y_3(t)$ corresponding to the three loads (also known as the buckling modes).

(c) Sketch schematically $y_1(t), y_2(t), y_3(t)$.

(d) If you had to put supports for the beam to prevent it from buckling, where would you put these supports for each one of the cases?

(e) How the deflection of the column will be affected if you put supports along the column when $y(x) = 0$ for $P_2, P_3$. 

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3. BUCKLING OF THIN VERTICAL COLUMN (BVP 2)

The critical loads of thin columns depend on the boundary (end) conditions of the column. Suppose that the thin vertical homogeneous column is embedded at its base \((x=0)\) and free at the top \((x=L)\). Note the difference in the frame assignment compared to the previous case.

\[\text{A constant axial load } P \text{ is applied to its free end. This load either causes a small deflection } \delta \text{ as shown in the figure or does not cause such a deflection at all}\]
In either case the differential eq. for the deflection $y(x)$ is defined by

$$EI \frac{d^2 y}{dx^2} + py = p\gamma$$
PROBLEM No. 2

Given buckling of thin vertical column described in section 3. (p 13-14) and its diff eq.

\[ EI \frac{d^4 y}{dx^4} + py = p \delta \]

\[ y(x) = 0 \]

\[ y'(0) = 0 \]

(a) What is the predicted deflection when \( \delta = 0 \)

(b) When \( \delta \neq 0 \) showed that the equal load for this column is one-fourth (1/4) the Euler load for the hinged column in section 2 (problem 1)

\[ P_n = \left( \frac{1}{4} \right) \left[ \frac{n^2 \pi^2 EI}{L^2} \right] \]
PROBLEM No. 3

In general, the diff. eq. governing the deflection of a column is given by

\[
\frac{d^2}{dx^2} \left( EI \frac{d^2 y}{dx^2} \right) + p \frac{d^2 y}{dx^2} = 0
\]

\[
EI y'''' + py'' = 0
\]

Assume that the column is uniform \((EI\) is constant) given the boundary values of \(y\):

\[
\begin{cases} 
y(0) = 0 \\
y''(0) = 0 \\
y(L) = 0 \\
y''(L) = 0
\end{cases}
\]

(a) Show that the solution of the fourth order diff. eq. subject to these boundary conditions is equivalent to the section 1.2 case 3 p. 7-9