PROJECT

COLUMN BUCKLING

HIGH ORDER DIF EQ.

PROJECT - COLUMN BUCKLING

# 1 MATH BACKGROUND

## 1.1 TWO EQUATION WORTH KNOWING

$$\left\{\begin{array}{l} y'' + k^2 y = 0 \quad (1) \\ y'' - k^2 y = 0 \quad (2) \end{array}\right] \quad k \text{ is real}$$

The two differentia equations (1), (2) where k is real, are important in applied mathematics

Eq 1

The auxiliary equation

$$m^2 + k^2 = 0$$

$$m^2 = -k^2$$

roots $\quad m = \pm k\sqrt{-1} = \begin{cases} +ki \\ -ki \end{cases}$

The general solution of the DE is

$$y = c_1 \cos(kx) + c_2 \sin(kx) \quad (3)$$

Eq (2)

The auxiliary equation

$$m^2 - k^2 = 0$$

$$m^2 = k^2$$

$$m = \pm k \begin{cases} +k \\ -k \end{cases}$$

The general solution of the DE is

$$y = c_1 e^{kx} + c_2 e^{-kx} \qquad (4)$$

Notice that if we choose

$$c_1 = c_2 = \frac{1}{2}$$

and

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2}$$

we get particular solutions

$$y = \frac{1}{2}\left(e^{kx} + e^{-kx}\right) = \cosh(kx)$$

$$y = \frac{1}{2}\left(e^{kx} - e^{-kx}\right) = \sinh(kx)$$

since $\cosh(kx)$ and $\sinh(kx)$ are linear independent on any interval of the x-axis, an alternative form for the general solution of eq. (2) is

$$y = c_1 \cosh(kx) + c_2 \sinh(kx)$$

## 1.2 NONTRIVIAL SOLUTION OF A BOUNDARY-VAVUE PROBLEM (BVP)

Given the BVP

$$y'' + \lambda y = 0$$

$$\text{Boundary Values} \begin{cases} y(0) = 0 \\ y(L) = 0 \end{cases}$$

We shall consider three cases:

1. $\lambda = 0$
2. $\lambda < 0$
3. $\lambda > 0$

CASE 1 $\quad \lambda = 0$

$$y'' + \cancel{\lambda y} = 0$$

$$y'' = 0$$

$$y' = c_1$$

$$\rightarrow \; y = c_1 x + c_2$$

Apply the BV

$$y(0) = c_1 0 + c_2 = 0 \quad \Rightarrow c_2 = 0$$
$$y(L) = c_1 L \qquad = 0 \quad \Rightarrow c_1 = 0$$

Hence for $\lambda = 0$ the only solution of the BVP is the trival solution

$$y = 0$$

## CASE 2 $\lambda < 0$

It is convenient to write

$$\lambda = -\alpha^2 \qquad \alpha > 0$$

Given this notation the roots of the auxiliary equation

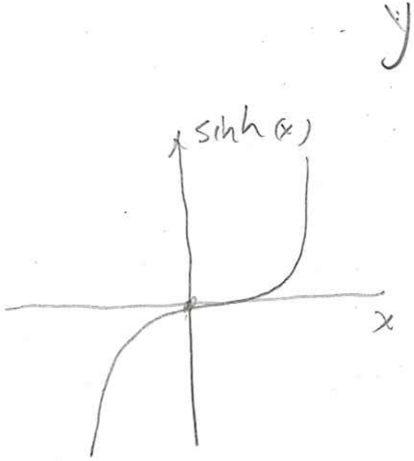
$$m^2 - \alpha^2 = 0 \quad m_1 = +\alpha, \quad m_2 = -\alpha$$

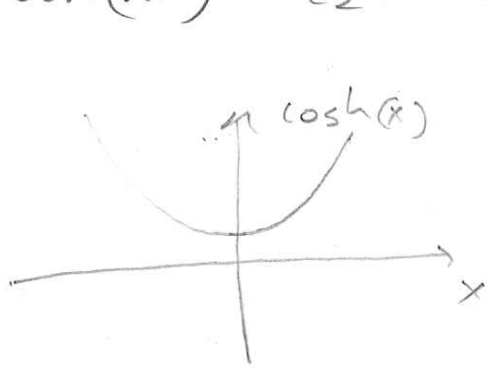
Since the interval on which we are working on is finite, we choose to write the general solution of

$$y'' - \alpha^2 y = 0$$

as

$$y = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$



FIRST BV $y(0) = c_1 \cosh(0) + c_2 \sinh(0) = 0$

$$= c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$\Rightarrow c_1 = 0$$

$$\Rightarrow y = c_2 \sinh(\alpha x)$$

SECOND BV $y(L) = c_2 \sinh(\alpha L) = 0$

- for any $\alpha \neq 0$ $\sinh \alpha L \neq 0$
- We are force to choose $c_2 = 0$
- the only solution of the BVP is the the trivial solution

$$y = 0$$

## CASE 3 $\quad \lambda > 0$

- We may again write

$$\lambda = +\alpha^2 \qquad \alpha > 0$$

- The axiliary equation has to complex roots

$$m^2 + \alpha^2 = 0 \quad \begin{cases} m_1 = i\alpha \\ m_2 = -i\alpha \end{cases}$$

- The general solution is

$$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

FIRST BV $\quad y(0) = c_1 \cos(0) + c_2 \sin(0)$

$$c_1 \cdot 1 + c_2 \cdot 0 \quad = 0$$

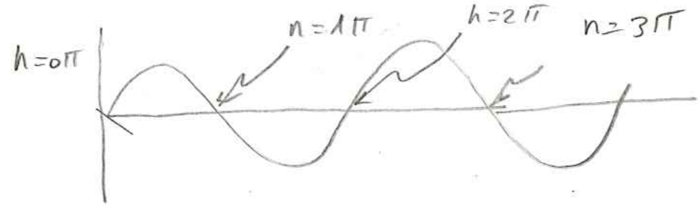
$$\Rightarrow c_1 = 0$$

$$\Rightarrow y = c_2 \sin(\alpha x)$$

SECOND BV $\quad y(L) = c_2 \sin(\alpha L) = 0$

- It is satisfied by choosing $c_2 = 0$ but that means that $y = 0$

- If we require $c_2 \neq 0$, then $\sin \alpha L = 0$ is satisfied whenever $\alpha L$ is an integer multiple of $\pi$




$$\alpha L = n\pi$$

$$\alpha = \frac{n\pi}{L}$$

$$\lambda = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots$$

- Therefore for any real non zero $c_2$

$$y = c_2 \sin\left[\frac{n\pi}{L} x\right] \qquad n = 1, 2, 3, \ldots$$

is a solution of the problem for each $n$

- Because the differential eq. is homogeneous, any constant multiple of a solution is also a solution, so we may simply take $c_2 = 1$

- For each number in the sequence

$$\lambda_1 = \frac{\pi^2}{L^2} \;;\quad \lambda_2 = \frac{4\pi^2}{L^2} \;;\quad \lambda_3 = \frac{9\pi^2}{L^2}, \; \ldots$$

$$\underset{n=1}{\uparrow} \qquad\qquad \underset{n=2}{\uparrow} \qquad\qquad \underset{n=3}{\uparrow}$$

the corresponding function in the sequance

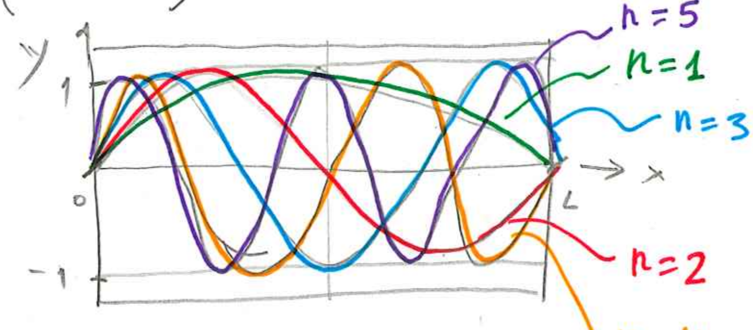
$$y_1 = \sin\frac{\pi}{L}x \;;\quad y_2 = \sin\frac{2\pi}{L}x \;;\quad y_3 = \sin\frac{3\pi}{L}x, \; \ldots$$

is a nontrivial solution of the problem

$$y'' + \lambda_n y = 0$$
$$y(0) = 0$$
$$y(L) = L$$

$\lambda_n \rightarrow$ EIGENVALUES

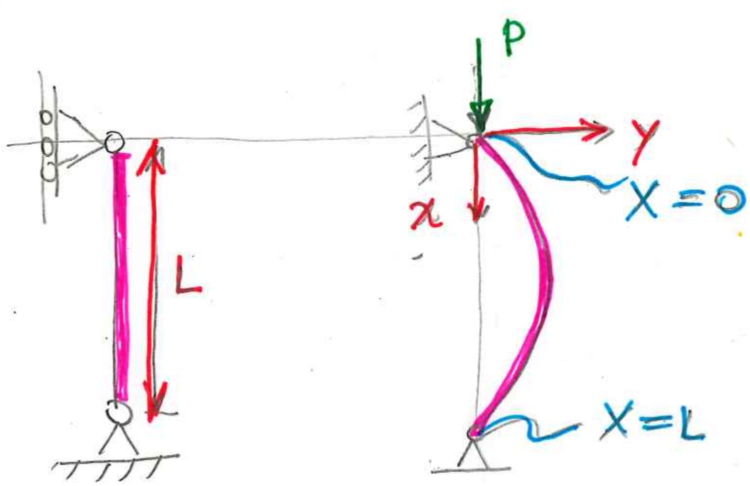
$y_n = \sin\left(\frac{n\pi}{L}x\right) \rightarrow$ EIGENFUNCTIONS



## 2. BUCKLING OF A THIN VERTICAL COLUMN (BVP1)

In the $18^{th}$ century Leonhard Euler was one of the first mathematicians to study an eigenvalue problem in analyzing how a thin elastic column buckles under a compressive axial force.

Consider a long, slender vertical column of a uniform cross section and Length $L$




Let $y(x)$ denote the diffection of the column when a constant vertical compressive fore or load, P is applied to its top

By compering bending moments at any point along the column, we obtain

$$EI \frac{d^2y}{dx^2} = -Py$$

or

$$EI \frac{d^2y}{dx^2} + Py = 0$$

where $E$ is the Young's modulus of elasticity and $I$ is the moment of inertia of the cross section about a vertical line throug it centroid.

The BV conditions are

$$\begin{cases} y(x=0) = 0 \\ y(x=L) = 0 \end{cases}$$

# PROBLEM No 1

(a) Find the first three axial buckling loads $P_1, P_2, P_3$ that will couse the beam to buckle

(b) Define the coresponding three difflections of the beam $y_1(t)$, $y_2(t)$, $y_3(t)$ coresponding to the three loads (also known as the buckling modes)

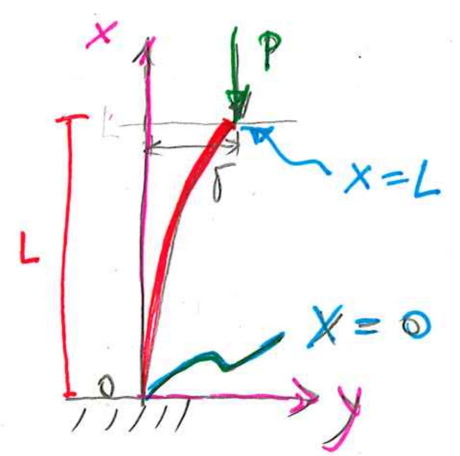
(c) sketch schamatically $y_1(t), y_2(t), y_3(t)$

(d) If you had to put supports for the beam to prevent it from bucling, where would you put these subports for each one of the cases

(e) How the difflection of the column will be affected if you put supports along the column where $y(x)=0$ for $P_2$, $P_3$

## 3. BUCKLING OF THIN VERTICAL COLUMN (BVP 2)

The critical loads of thin columns depend on the boundery (end) conditions of the column. Suppose that the thin vertical homogeneous column is embedded at is its base $(x=0)$ and free at the top $(x=L)$. Note the differance in the frame assignment compare to the previous case



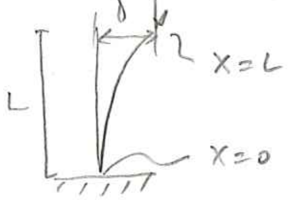

A constant axial load P is applied to its free end. This load either cause a small deflection $\delta$ as shown in the figure or does not couse such a deflection at all

In either case the differential eq. for the diffection $y(x)$ is defined by

$$EI\frac{d^2y}{dx^2} + Py = P\delta$$

# PROBLEM No. 2

Given buckling of thin vertical column described in section 3. (p 13 – 14) and it diff eq.




$$EI \frac{d^2y}{dx^2} + Py = P\delta \qquad \begin{cases} y(x) = 0 \\ y'(0) = 0 \end{cases}$$

(a) What is the predicted deflection when $\delta = 0$

(b) When $\delta \neq 0$ showed that the equler load for this column is one-fourth $(1/4)$ the Euler load for the hinged colume in section 2 (problem 1)

$$P_n = \frac{1}{4}\left[\frac{n^2 \pi^2 EI}{L^2}\right]$$

PROBLEM No. 3

In general, the diff. eq. governing the deflection of a column is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} = 0$$

$$EI\,y'''' + P\,y'' = 0$$

Assume that the column is uniform (EI is constant)

Given the boundary values of

$$\begin{cases} y(0) = 0 \\ y''(0) = 0 \\ y(L) = 0 \\ y'(L) = 0 \end{cases}$$

(a) Show that the solution of the fourth order diff eq. subject to these boundary conditions is equivalent to the section 1.2 case 3 p. 7-9