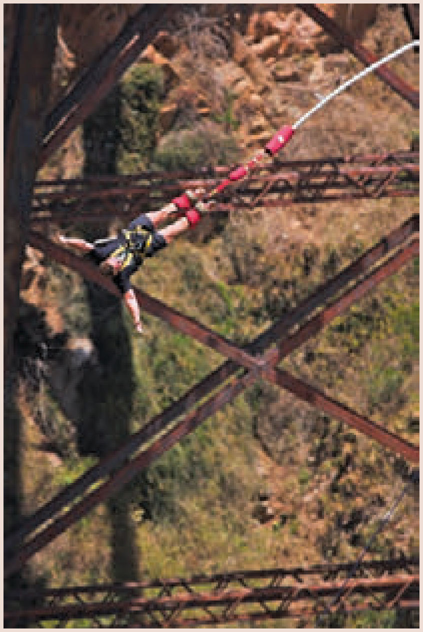
Bungee Jumping

by Kevin Cooper



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Bungee jumping from a bridge

Suppose that you have no sense. Suppose that you are standing on a bridge above the Malad River canyon. Suppose that you plan to jump off that bridge. You have no suicide wish. Instead, you plan to attach a bungee cord to your feet, to dive gracefully into the void, and to be pulled back gently by the cord before you hit the river that is 174 feet below. You have brought several different cords with which to affix your feet, including several standard bungee cords, a climbing rope, and a steel cable. You need to choose the stiffness and length of the cord so as to avoid the unpleasantness associated with an unexpected water landing. You are undaunted by this task, because you know math!

Each of the cords you have brought will be tied off so as to be 100 feet long when hanging from the bridge. Call the position at the bottom of the cord 0, and measure the position of your feet below that "natural length" as $x(t)$, where x increases as you go down and is a function of time t. See Figure 1. Then, at the time you jump, $x(0) = -100$, while if your six-foot frame hits the water head first, at that time $x(t) = 174 - 100 - 6 = 68$. Notice that distance increases as you fall, and so your velocity is positive as you fall and negative when you bounce back up. Note also that you plan to dive so your head will be six feet below the end of the chord when it stops you.

You know that the acceleration due to gravity is a constant, called g, so that the force pulling downwards on your body is mg. You know that when you leap from the bridge, air resistance will increase proportionally to your speed, providing a force in the opposite direction to your motion of about βv, where β is a constant and v is your velocity. Finally, you know that Hooke's law describing the action of springs says that the bungee cord will eventually exert a force on you proportional to its distance past its natural length. Thus, you know that the force of the cord pulling you back from destruction may be expressed as

$$b(x) = \begin{cases} 0 & x \leq 0 \\ -kx & x > 0 \end{cases}$$

The number k is called the *spring constant*, and it is where the stiffness of the cord you use influences the equation. For example, if you used the steel cable, then k would be very large, giving a tremendous stopping force very suddenly as you passed the natural length of the cable. This could lead to discomfort, injury, or even a Darwin award. You want to choose the cord with a k value large enough to stop you above or just touching the water, but not too suddenly. Consequently, you are interested in finding the distance you fall below the natural length of the cord as a function of the spring constant. To do that, you must solve the differential equation that we have derived in words above: The force mx'' on your body is given by

$$mx'' = mg + b(x) - \beta x'.$$

Here mg is your weight, 160 lb., and x' is the rate of change of your position below the equilibrium with respect to time; i.e., your velocity. The constant β for air resistance depends on a number of things, including whether you wear your skin-tight pink spandex or your skater shorts and XXL T-shirt, but you know that the value today is about 1.0.

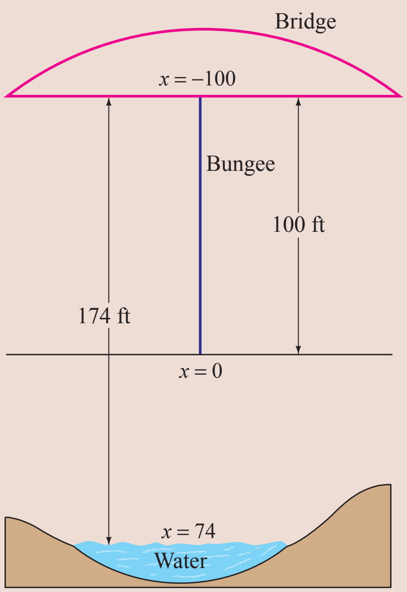


FIGURE 1 The bungee setup

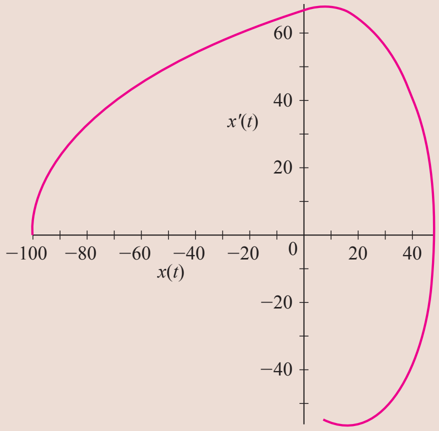


FIGURE 2 An example plot of $x(t)$ against $x'(t)$ for a bungee jump

This is a nonlinear differential equation, but inside it are two linear differential equations, struggling to get out. We will work with such equations more extensively in later chapters, but we already know how to solve such equations from our past experience. When $x < 0$, the equation is $mx'' = mg - \beta x'$, while after you pass the natural length of the cord it is $mx'' = mg - kx - \beta x'$. We will solve these separately, and then piece the solutions together when $x(t) = 0$.

In Problem 1 you find an expression for your position t seconds after you step off the bridge, before the bungee cord starts to pull you back. Notice that it does not depend on the value for k, because the bungee cord is just falling with you when you are above $x(t) = 0$. When you pass the natural length of the bungee cord, it does start to pull back, so the differential equation changes. Let t_1 denote the first time for which $x(t_1) = 0$, and let v_1 denote your speed at that time. We can thus describe the motion for $x(t) > 0$ using the problem $x'' = g - kx - \beta x'$, $x(t_1) = 0$, $x'(t_1) = v_1$. An illustration of a solution to this problem in phase space can be seen in Figure 2.

This will yield an expression for your position as the cord is pulling on you. All we have to do is to find out the time t_2 when you stop going down. When you stop going down, your velocity is zero, i.e., $x'(t_2) = 0$.

As you can see, knowing a little bit of math is a dangerous thing. We remind you that the assumption that the drag due to air resistance is linear applies only for low speeds. By the time you swoop past the natural length of the cord, that approximation is only wishful thinking, so your actual mileage may vary. Moreover, springs behave nonlinearly in large oscillations, so Hooke's law is only an approximation. Do not trust your life to an approximation made by a man who has been dead for 200 years. Leave bungee jumping to the professionals.

Related Problems

1. Solve the equation $mx'' + \beta x' = mg$ for $x(t)$, given that you step off the bridge—no jumping, no diving! Stepping off means $x(0) = -100$, $x'(0) = 0$. You may use $mg = 160$, $\beta = 1$, and $g = 32$.

2. Use the solution from Problem 1 to compute the length of time t_1 that you freefall (the time it takes to go the natural length of the cord: 100 feet).

3. Compute the derivative of the solution you found in Problem 1 and evaluate it at the time you found in Problem 2. Call the result v_1. You have found your downward speed when you pass the point where the cord starts to pull.

4. Solve the initial-value problem

$$mx'' + \beta x' + kx = mg, \quad x(t_1) = 0, \quad x'(t_1) = v_1.$$

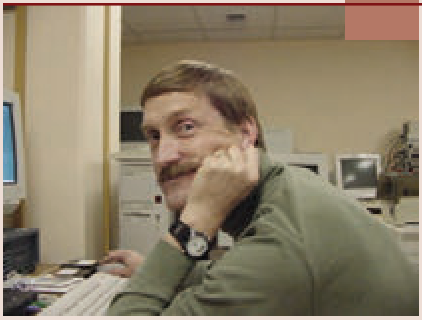
   For now, you may use the value $k = 14$, but eventually you will need to replace that with the actual values for the cords you brought. The solution $x(t)$ represents the position of your feet below the natural length of the cord after it starts to pull back.

5. Compute the derivative of the expression you found in Problem 4 and solve for the value of t where it is zero. This time is t_2. Be careful that the time you compute is greater than t_1—there are several times when your motion stops at the top and bottom of your bounces! After you find t_2, substitute it back into the solution you found in Problem 4 to find your lowest position

6. You have brought a soft bungee cord with $k = 8.5$, a stiffer cord with $k = 10.7$, and a climbing rope for which $k = 16.4$. Which, if any, of these may you use safely under the conditions given?

7. You have a bungee cord for which you have not determined the spring constant. To do so, you suspend a weight of 10 lb. from the end of the 100-foot cord, causing the cord to stretch 1.2 feet. What is the k value for this cord? You may neglect the mass of the cord itself.

ABOUT THE AUTHOR



Courtesy of Kevin Cooper

Kevin Cooper, PhD, Colorado State University, is the Computing Coordinator for Mathematics at Washington State University, Pullman, Washington. His main interest is numerical analysis, and he has written papers and one textbook in that area. Dr. Cooper also devotes considerable time to creating mathematical software components, such as *DynaSys*, a program to analyze dynamical systems numerically.