

Project No.1

Linear & Nonlinear Differential Equations

1. Introduction

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing.

From the first scientific investigations of the pendulum around 1602 by Galileo Galilei, the regular motion of pendulums was used for timekeeping, and was the world's most accurate timekeeping technology until the 1930s. The pendulum clock invented by Christian Huygens in 1658 became the world's standard timekeeper, used in homes and offices for 270 years, and achieved accuracy of about one second per year before it was superseded as a time standard by the quartz clock in the 1930s. Pendulums are also used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length. The word "pendulum" is new Latin, from the Latin *pendulus*, meaning 'hanging'.

The Foucault's pendulum (English: /fu:'kɔʊ/ foo-KOH; French pronunciation: [fu'ko]) is a simple device named after French physicist Léon Foucault and conceived as an experiment to demonstrate the Earth's rotation. The pendulum was introduced in 1851 and was the first experiment to give simple, direct evidence of the earth's rotation.



Figure 1 : Foucault pendulum

2. Questions

- a. Using the parameters defined in the following figure draw a free body diagram of the mass (bob)

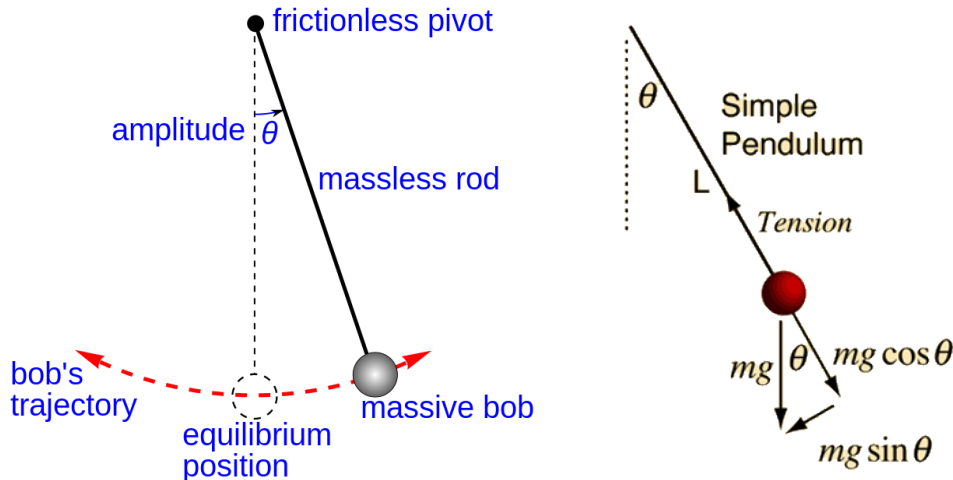


Figure : The Pendulum and its parameters

- b. Derive the differential equations of the of the pendulum using newton's second law (Eq.1) and show that the differential equation takes the form of Eq. 2

$$\sum M = I\alpha = I \frac{d^2\theta}{dt^2} \quad (1)$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin\theta = 0 \quad (2)$$

- c. Assume that you rotate the bob from it equilibrium position to some initial angle (θ_0) and let go without giving it any initial angular velocity. Define the two initial conditions ($\theta(t=0)$ and $d\theta/dt(t=0)$). Note that since the differential equation is of the second order it has two initial conditions
- d. Is the differential equation (Eq. 2) linear or nonlinear and why?
- e. Eq. 2 may be linearized and it takes the form of Eq. 3 by using the Taylor series (Eq. 4). Explain how it is done

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0 \quad (3)$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \quad (4)$$

- f. The solution of the linearized differential equation (Eq.3) is given by Eq. 5. Show that the solution satisfies the differential equation (Eq.3) given the initial conditions that you found in section c

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}} t\right) \quad (5)$$

- g. The small angles approximation is based on Eq. 6. The relative error expressed in % (δ_θ) is defined by Eq.7. Calculate and plot the relative error expressed in % (δ_θ) for the interval $0 \leq \theta \leq \pi/2$ with increments of 1° (Note that $1^\circ = \pi/180$). Based on this calculation what is the largest angle for which the relative error is less than 15%

$$\sin \theta \approx \theta \quad (6)$$

$$\delta_\theta = 100\% \cdot \left| \frac{\sin \theta - \theta}{\sin \theta} \right| \quad (7)$$

- h. Develop two Simulink models for solving the nonlinear differential equation (Eq. 2) and the linear differential equation (Eq. 3) using the initial conditions $\theta(t=0) = \theta_0$ and $d\theta/dt(t=0) = 0$. Note that these initial conditions are used as inputs to the two integrals blocks in simulink. For simulation purposes you may use Foucault's pendulum original values including:

$$m = 25 \text{ kg} \quad l = 33 \text{ m} \quad g = 9.8 \text{ m/s}^2.$$

- i. Run the two simulations for various initial conditions of $\theta(t=0) = \theta_0$. Each simulation would start by setting the initial angular velocity to be zero $d\theta/dt(t=0) = 0$ and a selected one of the following initial angular displacement $\theta_0 = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90$ Deg. Note that you will run 20 simulations running each model in 10 different configuration. In each simulation measure the time period expressed in sec. which is the time it will take the pendulum to complete one cycle. You may export the data to Matlab and check the exact time for the angular displacement waveform to reach it peaks.
- j. For each initial angular displacement (θ_0) calculate the relative error expressed in % (δ_T) of the period exhibit by the pendulum and defined by the linear differential equation (T_L) and the nonlinear differential equation (T_{NL})

$$\delta_T = 100\% \cdot \left| \frac{T_L - T_{NL}}{T_{NL}} \right| \quad (7)$$

As a reference the analytical expression of the time period of the linear differential equation is defined by

$$T_L = 2\pi \sqrt{\frac{l}{g}} \quad (7)$$

Plot relative error expressed in % (δ_T) as a function of the initial angular displacement θ_0

- k. Base on your calculations in section j, what is the largest angular initial displacement (θ_0) for which the relative error (δ_T) is less than 1%.
- l. Explain how Foucault used the pendulum to prove that earth is rotated around an axis.
- m. Visit the Griffith Observatory in LA and examine the Foucault's pendulum (not for credit but just as a scientific adventure) – Take a selfy with the pendulum.

3. References

- Pendulum
<https://en.wikipedia.org/wiki/Pendulum>
[https://en.wikipedia.org/wiki/Pendulum_\(mathematics\)](https://en.wikipedia.org/wiki/Pendulum_(mathematics))
- Foucault pendulum
https://en.wikipedia.org/wiki/Foucault_pendulum
- Small Angle Approximation
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- Approximation error
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