HW 7

Part 1.

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Pr. 6

$$F(s) = \frac{8s^2 - 4s + 12}{s(s^2+4)} = 3\frac{1}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t$$

Pr. 15

$$\mathcal{L}\{y'' + w^2 y = \cos(2t)\} \quad \left(y(0) = 1 \quad y'(0) = 0\right)$$

$$\Rightarrow s^2 Y(s) - s \times 1 - 0 + w^2 Y(s) = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2 + w^2) Y(s) = \frac{s}{s^2+4} + s+1 \quad \Rightarrow Y(s) = \frac{s}{(s^2+w^2)(s^2+4)} + \frac{s}{s^2+w^2}$$

$$\frac{s}{(s^2+w^2)(s^2+4)} = \frac{1}{4-w^2}\left[\frac{s}{s^2+w^2} - \frac{s}{s^2+4}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+w^2}\right] = \cos wt \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$$

$$\Rightarrow y(t) = \frac{1}{4-w^2}\left[\cos(wt) - \cos(2t)\right] + \cos(wt)$$

Pr. 18 $f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < \infty \end{cases} \Rightarrow s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \mathcal{L}[f(t)]$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt = \int_0^1 t e^{-st} dt + \int_1^\infty e^{-st} dt = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}$$

P. 263 $\quad f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$

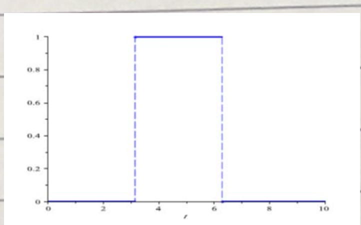
Pr. 22

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

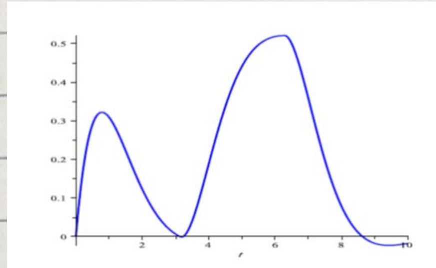
P. 268

Pr. 2

• (a)



(c)



(b) $\mathcal{L}[y'' + 2y' + 2y] = s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s)$

$$\mathcal{L}[u_\pi(t) - u_{2\pi}(t)] = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2 + 2s + 2)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + 1}\right] = e^{-t}\sin t$$

let $G(s) = \dfrac{1}{s(s^2 + 2s + 2)} = \dfrac{1}{2}\dfrac{1}{s} - \dfrac{1}{2}\dfrac{(s+1) + 1}{(s+1)^2 + 1}$

$$\Rightarrow \mathcal{L}^{-1}[G(s)] = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$$

$$\mathcal{L}^{-1}[e^{-cs}G(s)] = u_c(t)\, g(t-c)$$

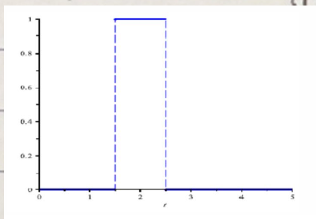
$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = e^{-t}\sin t + \frac{1}{2}[u_\pi(t) - u_{2\pi}(t)] + \frac{1}{2}e^{-(t-\pi)}[\cos t + \sin t]u_\pi(t)$$
$$+ \frac{1}{2}e^{-(t-2\pi)}[\cos t + \sin t]u_{2\pi}(t)$$

(d) the solution starts out as free oscillation with increasing amplitude. When the forcing term absent, the solution decays rapidly.

P.269 $\quad u'' + \frac{1}{4}u' + u = k\,g(t) \qquad u(0)=0\ ,\ u'(0)=0$

Pr 11 $\qquad g(t) = u_{\frac{3}{2}}(t) - u_{\frac{5}{2}}(t) \qquad (k>0)$

(a)



(b) $\mathcal{L}[u''+\frac{1}{4}u'+u] = s^2 U(s) + \frac{1}{4}s\,U(s) + U(s)$

$$\mathcal{L}[g(t)] = \frac{e^{-\frac{3}{2}s}}{s} - \frac{e^{-\frac{5}{2}s}}{s}$$

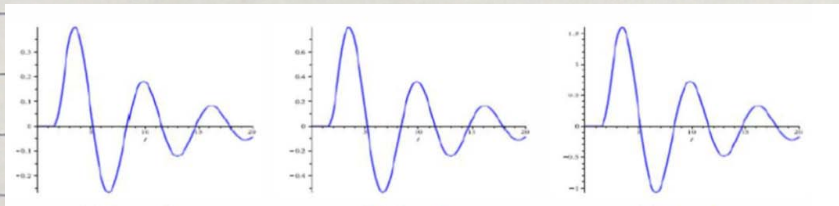
$$\Rightarrow U(s) = k\,\frac{e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}}{(s^2+\frac{1}{4}s+1)s} = k\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right)\left[\frac{1}{s} - \frac{(s+\frac{1}{8})+\frac{1}{8}}{(s+\frac{1}{8})^2+\frac{63}{64}}\right]$$

$$\text{let } h(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{(s+\frac{1}{8})}{(s+\frac{1}{8})^2+\frac{63}{64}} - \frac{\frac{\sqrt{63}}{8}}{(s+\frac{1}{8})^2+\frac{63}{64}} \times \frac{1}{\sqrt{63}}\right]$$

$$= 1 - e^{-\frac{t}{8}}\cos\left(\frac{3\sqrt{7}}{8}t\right) - e^{-\frac{t}{8}} \times \frac{1}{3\sqrt{7}} \times \sin\left(\frac{3\sqrt{7}}{8}t\right)$$

$$u(t) = k\,u_{\frac{3}{2}}(t)\,h(t-\tfrac{3}{2}) - k\,u_{\frac{5}{2}}(t)\,h(t-\tfrac{5}{2})$$

(c)

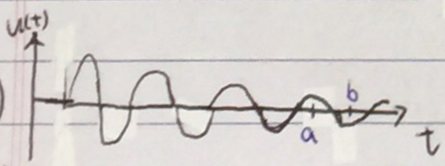


(a) $k = 1/2$ (b) $k = 1$ (c) $k = 2$

(d) from (c), we find the maximum of $u(t)$ occur in first peak ($t \to 3.46748$ by matlab)

Solve $u(t\to 3.46748, k) = 2 \overset{\text{matlab}}{\Rightarrow} k \doteq 2.50241$, $(k \approx 2.51)$

(e)



K=2

$a \approx 22.4661 \Rightarrow u(a) \approx 0.1487$

$b \approx 25.6325 \Rightarrow u(b) \approx -0.1001$

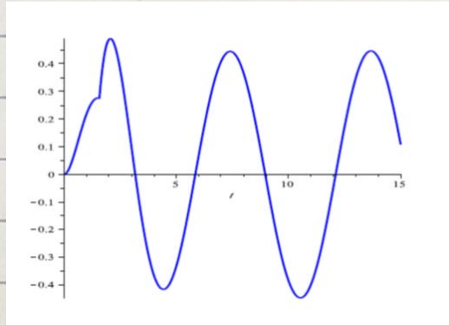
Solve $u(t, k{=}2) = -0.1$ when $t > 25.63$

$\Rightarrow \boxed{t \approx 25.6773}$

P. 273 (a)
Pr 7

$$y'' + 2y' + 2y = \cos t + \delta(t - \tfrac{\pi}{2}) \qquad y(0) = 0, \; y'(0) = 0$$

- $$\mathcal{L}[y'' + 2y' + 2y] = s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + 2s Y(s) - \cancel{2 f(0)} + 2Y(s)$$



$$\mathcal{L}[\cos t + \delta(t - \tfrac{\pi}{2})] = \frac{s}{s^2+1} + e^{-\frac{\pi}{2}s}$$

$$\Rightarrow Y(s) = \frac{s}{(s^2+2s+2)(s^2+1)} + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}$$

$$= \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{(s+1)+3}{(s+1)^2+1}\right] + \frac{e^{-\frac{\pi}{2}s}}{s^2+2s+2}$$

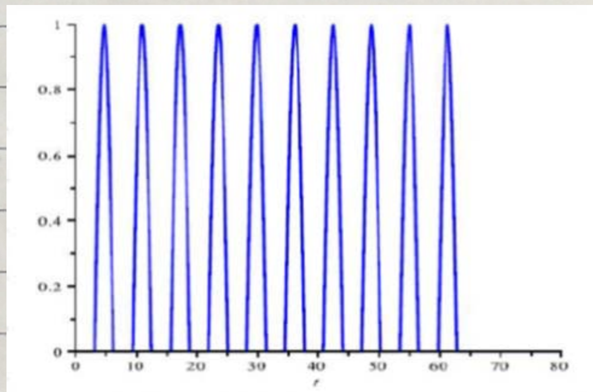
$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}[\cos t + 3\sin t] - e^{-(t-\frac{\pi}{2})}\cos t \, U_{\frac{\pi}{2}}(t)$$

P. 274
Pr 13.

$$y'' + y = \sum_{k=1}^{20} \delta(t - k\pi) \qquad y(0) = 0 \quad y'(0) = 0$$

- (b) $$\mathcal{L}[y'' + y] = s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + Y(s)$$

$$\mathcal{L}\left[\sum_{k=1}^{20} \delta(t - k\pi)\right] = \sum_{k=1}^{20} e^{-k\pi s}$$



$$Y(s) = \frac{1}{s^2+1} \sum_{k=1}^{20} e^{-k\pi s}$$

$$y(t) = \sum_{k=1}^{20} \sin(t - k\pi) U_{k\pi}(t)$$

(c) After the sequence of impulses ends, the oscillation return to its equilibrium

P. 279

Pr. 6 $f(t) = g * h \Rightarrow g(t) = \sin t$; $h(t) = \cos t$

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] \times \mathcal{L}[h(t)] = \frac{s}{(s^2+1)^2}$$

Part 2

P. 255

Pr. 19.

$$f(t)=\begin{cases} t & 0\le t<1 \\ 2-t & 1\le t<2 \\ 0 & 2\le t<\infty \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 t e^{-st}\,dt + \int_1^2 (2-t)\, e^{-st}\,dt =$$

IBP:

$$= \left[-\frac{e^{-st}}{s} t \Big|_0^1 - \int_0^1 -\frac{e^{-st}}{s}\,dt \right] + \left[\frac{(2-t)e^{-st}}{-s} \Big|_1^2 - \int_1^2 \frac{(-1)}{(-s)} e^{-st}\,dt \right]$$

$$= -\frac{e^{-s}}{s} + \frac{-e^{-st}}{s^2}\Big|_0^1 + \frac{e^{-s}}{s} + \frac{e^{-st}}{s^2}\Big|_1^2$$

$$= \frac{-e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} = \frac{1}{s^2}\left(1-2e^{-s}+e^{-2s}\right)$$

$$\mathcal{L}[y''+y] = s^2Y(s) - sy(0) - y'(0) + Y(s)$$

$$\Rightarrow Y(s) = \frac{1-2e^{-s}+e^{-2s}}{(s^2+1)(s^2)}$$

P. 268

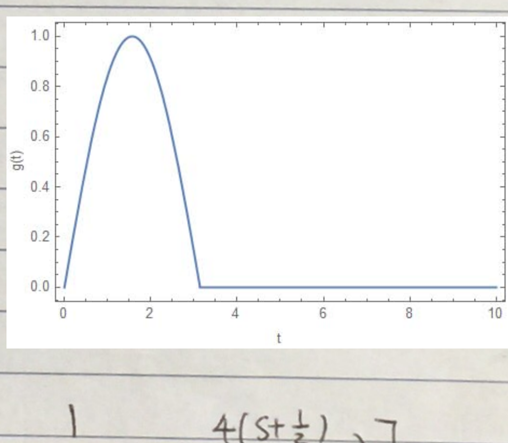
Pr. 6.

(b)

$$\mathcal{L}[y''+y'+\tfrac{5}{4}y] = s^2Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0} + sY(s) - \cancel{y(0)}^{\,0} + \frac{5}{4}Y(s)$$

$$g(t) = \left(u_0(t) - u_\pi(t)\right)\sin t$$

(a)



$$\mathcal{L}[g(t)] = \frac{1}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1}$$

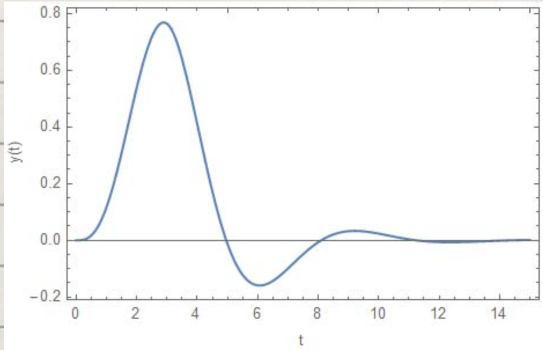
$$Y(s) = \frac{1-e^{-\pi s}}{(s^2+s+\frac{5}{4})(s^2+1)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+s+\frac{5}{4})(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{4}{17}\times\left(\frac{1}{s^2+1} - \frac{4s}{s^2+1} + \frac{1}{(s+\frac{1}{2})^2+1} + \frac{4(s+\frac{1}{2})}{(s+\frac{1}{2})^2+1}\right)\right]$$

$$= \frac{4}{17}\left(-4\cos t + \sin t + 4e^{-t/2}\cos t + e^{-t/2}\sin t\right) \equiv h(t)$$

$\Rightarrow y = h(t) + u_\pi(t)\, h(t-\pi)$

(c)

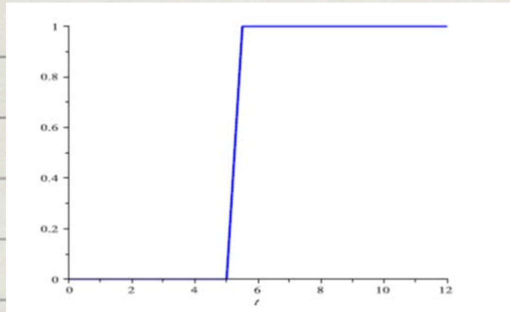


(d)

The solution oscillates for a while and then decays to zero after forcing term ceasing.

P. 269 (a)
Pr 12

$$f(t) = \begin{cases} 0 & 0 \le t < 5 \\ \frac{1}{k}(t-5) & 5 \le t < 5+k \\ 1 & t \ge 5+k \end{cases}$$



When $k=5 \Rightarrow f(t) = g(t)$

b) $\mathcal{L}[y'' + 4y] = s^2 Y(s) - s y(0) - y'(0) + 4 Y(s)$

$$\mathcal{L}[f(t)] = \frac{e^{-5s}}{ks^2} - \frac{e^{-(5+k)s}}{ks^2}$$

$$\Rightarrow Y(s) = \frac{e^{-5s}}{ks^2(s^2+4)} - \frac{e^{-(5+k)s}}{ks^2(s^2+4)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)\right] = \frac{t}{4} - \frac{1}{8}\sin 2t \equiv h(t)$$

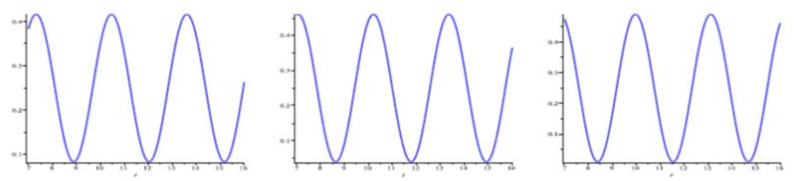
$$\Rightarrow y(t) = \frac{1}{k}\left[h(t-5)\,u_5(t) - h(t-5-k)\,u_{5+k}(t)\right]$$

• (c) for $t > 5+k$

$$y(t) = \frac{1}{4} - \frac{1}{8k}\sin(2t-10) + \frac{1}{8k}\sin(2t-10-2k) = \frac{1}{4} - \frac{\sin k}{4k}\cos(2t-10-k)$$

→ the solution oscillates about $y_m = \frac{1}{4}$

with an amplitude of $A = \frac{|\sin(k)|}{4k}$



(a) $k = 3/2$ (b) $k = 1$ (c) $k = 1/2$

P.279

Pr. 9 $F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{1}{(s+1)^2} \times \frac{1}{s^2+2^2}$

$$\Rightarrow f(t) = \int_0^t (t-\tau)e^{-(t-\tau)} \cdot \frac{1}{2}\sin 2\tau \, d\tau \left(= \frac{1}{2}\int_0^t \sin(2t-2\tau)\,\tau e^{-\tau}\, d\tau \right)$$

$$\frac{1}{2}\int_0^t (t-\tau)e^{-(t-\tau)}\sin 2\tau \, d\tau$$

$$= \frac{1}{2}\,\mathrm{Im}\left[\int_0^t (t-\tau)e^{-(t-\tau)} e^{2\tau i}\, d\tau\right]$$

$$= \frac{1}{2}\,\mathrm{Im}\left[t\int_0^t e^{-t}\cdot e^{\tau+2\tau i}\, d\tau - \int_0^t \tau e^{-t}\cdot e^{\tau+2\tau i}\, d\tau\right]$$

↓ integration by part

$$= \frac{1}{2}\,\mathrm{Im}\left[t\times\frac{e^{-t}\times e^{(1+2i)\tau}}{1+2i}\Big|_0^t - e^{-t}\left(\frac{\tau e^{(1+2i)\tau}}{1+2i}\Big|_0^t - \int_0^t \frac{e^{(1+2i)\tau}}{1+2i}\, d\tau\right)\right]$$

$$= \frac{1}{2}\,\mathrm{Im}\left[te^{-t}\times\frac{e^{(1+2i)t}-1}{1+2i}\times\left(\frac{1-2i}{1-2i}\right) - e^{-t}\frac{te^{(1+2i)t}}{1+2i}\cdot\frac{1-2i}{1-2i} + e^{-t}\cdot\frac{e^{(1+2i)\tau}}{(1+2i)(1+2i)}\Big|_0^t\times\frac{-3-4i}{-3-4i}\right]$$

cancel!

$$= \frac{1}{2}\,\mathrm{Im}\left[te^{-t}\times\frac{-1+2i}{5} + e^{-t}\times\frac{e^{(1+2i)t}-1}{25}\times(-3-4i)\right]$$

$$= \frac{e^{-t}}{50}\cdot\mathrm{Im}\left[-5t + 10ti + (e^{t}\cos 2t - 1 + ie^{t}\sin 2t)(-3-4i)\right]$$

$$= \frac{e^{-t}}{50}\times\mathrm{Im}\left[-5t+10ti + (e^{t}\cos 2t - 1)(-3) + 4e^{t}\sin 2t - i\,4(e^{t}\cos 2t - 1) - i\,3e^{t}\sin 2t\right]$$

$$= \frac{e^{-t}}{50}\times\left(4 + 10t - 4e^{t}\cos 2t - 3e^{t}\sin 2t\right)$$