Class Notes 18:

Numerical Methods (1/2)

82 – Engineering Mathematics

Numerical Methods

Differential Equation

Analytical Techniques:
- Integrating Factor & Exact
  (1st order)
- $y = y_h + y_p$
  (2nd or higher order, linear, constant coefficients)
- Power Series
  (2nd order, linear)
- Laplace Transform
  (Linear, constant coefficient, I.C.)

Numerical Techniques:
1)Fix (single) step
  - Euler
  - Runge-Kutta
2)Multi-step
  - Adams-Bashforth

Idea: approximate the value of y at a specified time t_n

Numerical Methods: the Euler Method

WHAT is Euler Method

↓

WHY Euler method can work

↓

HOW to apply Euler method

↓

What is the PROBLEM with Euler method

↓

How to IMPROVE Euler method

The Euler Method (Tangent Line Method) - WHAT

- First order initial value problem

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0$$

- Euler's Formula: $$y_{n+1} = y_n + hf_n$$

- Repeatedly evaluating Euler's formula using the result of each step to execute the next step
- Obtain a sequence of values y_0, y_1, y_2, … y_n that approximate the value of the solution at points t_0, t_1, t_2, … t_n

- Euler's formula can be derived in three ways

The Euler Method (Tangent Line Method) - WHY

- Approach 1: Tangent Line

  Assume an unique solution of the form $y = \phi(t)$

  Write the differential equation at the point $t=t_n$

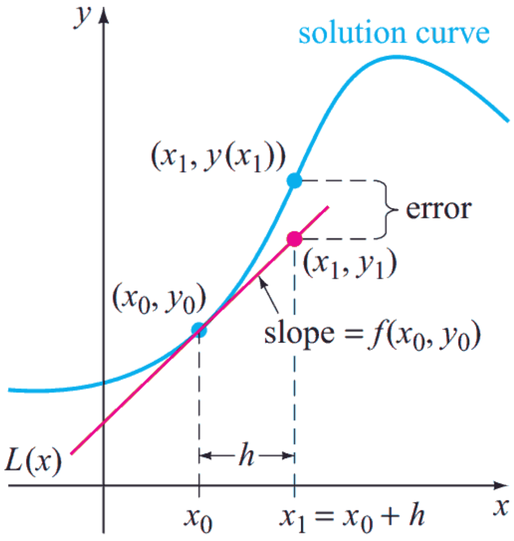
$$\frac{d\phi}{dt}(t_n) = f(t_n, \phi_n)$$

- Approximate the derivation by the corresponding forward difference quotient

$$\frac{\phi(t_{n+1}) - \phi(t_n)}{t_{n+1} - t_n} \cong f[t_n, \phi(t_n)]$$

The Euler Method (Tangent Line Method) - WHY

$$\frac{\phi(t_{n+1})-\phi(t_n)}{t_{n+1}-t_n} \cong f\left[t_n,\phi(t_n)\right]$$



Replace $\phi(t_{n+1}) \rightarrow y_{n+1}$

$\phi(t_n) \rightarrow y_n$

Solve for y_{n+1}

h → step size

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

$$\boxed{y_{n+1} = y_n + hf_n}$$

The Euler Method (Tangent Line Method) - WHY

- Approach 2: Integration

  Let $y=\phi(t)$ be the solution of the initial value problem

$$\int_{t_n}^{t_{n+1}} \phi' dt = \int_{t_n}^{t_{n+1}} f\left[t_n, \phi(t_n)\right] dt$$

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f\left[t_n, \phi(t_n)\right] dt$$

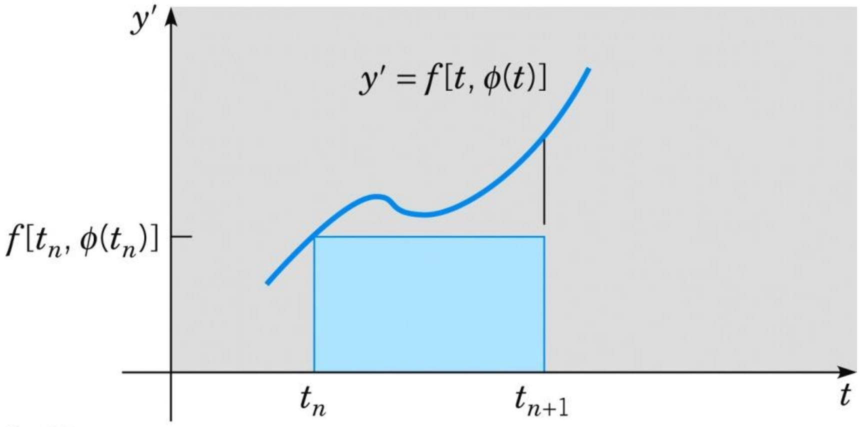


Figure 8.1.1

The Euler Method (Tangent Line Method) - WHY

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f[t_n, \phi(t_n)]dt$$

$$\cong \phi(t_n) + f[t_n, \phi(t_n)](t_{n+1} - t_n)$$

$$= \phi(t_n) + hf[t_n, \phi(t_n)]$$

shaded rectangle

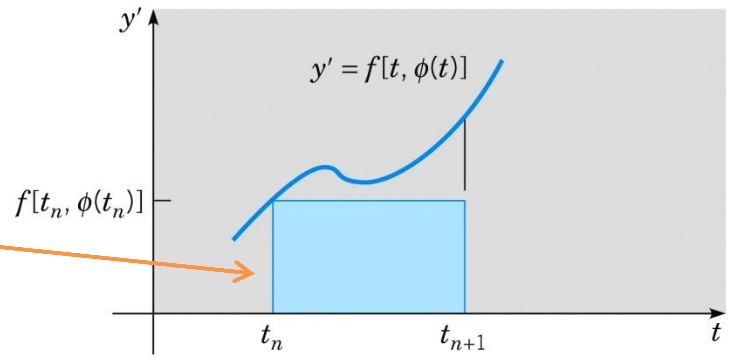


Figure 8.1.1

Replace $\phi(t_{n+1}) \to y_{n+1}$

$\phi(t_n) \to y_n$

$$y_{n+1} = y_n + hf_n$$

The Euler Method (Tangent Line Method) - WHY

- Approach 3: Taylor Series

The solution $y = \phi(t)$ has a Taylor series about t_n

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \phi''(t_n)\frac{h^2}{2!} + \cdots$$

or $$\phi(t_{n+1}) = \phi(t_n) + f\left[t_n, \phi(t_n)\right]h + \phi''(t_n)\frac{h^2}{2!} + \cdots$$

replace $\begin{cases} \phi(t_{n+1}) \to y_{n+1} \\ \phi(t_n) \to y_n \end{cases}$ and take the first two terms (linearization)

$$\boxed{y_{n+1} = y_n + hf_n}$$

The Euler Method (Tangent Line Method) - HOW

- How to apply Euler method

```
step 1          define f(t,y)
step 2          input initial values t0 and y0
step 3          input step size h and the number of steps n
step 4          output t0 and y0
step 5          for j from 1 to n do
                      k1 = f(t, y)      \
                      y = y + h*k1       }  n times
                      t = t + h         /
                output t and y
                end
```

The Euler Method (Tangent Line Method) - HOW

- Example $$y' = 1 - t + 4y$$ $$y(0) = 1$$

Exact solution $$y = \phi(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}$$

TABLE 8.1.1 A Comparison of Results for the Numerical Approximation of the Solution of $y' = 1 - t + 4y$, $y(0) = 1$ Using the Euler Method for Different Step Sizes h

t	$h = 0.05$	$h = 0.025$	$h = 0.01$	$h = 0.001$	Exact
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5475000	1.5761188	1.5952901	1.6076289	1.6090418
0.2	2.3249000	2.4080117	2.4644587	2.5011159	2.5053299
0.3	3.4333560	3.6143837	3.7390345	3.8207130	3.8301388
0.4	5.0185326	5.3690304	5.6137120	5.7754845	5.7942260
0.5	7.2901870	7.9264062	8.3766865	8.6770692	8.7120041
1.0	45.588400	53.807866	60.037126	64.382558	64.897803
1.5	282.07187	361.75945	426.40818	473.55979	479.25919
2.0	1745.6662	2432.7878	3029.3279	3484.1608	3540.2001

Table 8.1.1

The Euler Method (Tangent Line Method) - PROBLEM

- Error in numerical approximations

  Convergence

  Numerical approximation → y_1 ; Analytical solution → $\phi(t_1)$

  Step size → h

$$h \to 0 \quad \Rightarrow \quad \begin{cases} y_1 \to \phi(t_1) \\ y_2 \to \phi(t_2) \\ \vdots \\ y_n \to \phi(t_n) \end{cases}$$

- How small a step size is needed in order to guarantee a given level of accuracy?

- Small step size - { slow down calculations; may cause loss of accuracy }

The Euler Method (Tangent Line Method) - PROBLEM

- Fundamental sources of errors

  1) Formula/Algorithm - approximation

  e.g. Euler → straight line approximations

  2) The input data (except for the first step) are only approximations to the actual values of the solution at the specific points

  3) Finite precision of the computer

The Euler Method (Tangent Line Method) - PROBLEM

- Error type 1) and 2)

  Assume that the computer can execute all computations exactly. It can retain infinitely many digits (if necessary) at each step

$$E_n = \phi(t_n) - y_n$$

- E_n: Global Truncation Error (source of error: 1 and 2)
- $\phi(t_n)$: Solution
- y_n: Numerical Approximation

- Error type 3)

$$R_n = y_n - Y_n$$

- R_n: Round off error
- y_n: Numerical Approximation
- Y_n: Actual computed solution With round off error

The Euler Method (Tangent Line Method) - PROBLEM

- The absolute value of the total error in computing $\phi(t_n)$

$$|\phi(t_n) - Y_n| = |\phi(t_n) - y_n + y_n - Y_n|$$

based on $|a+b| \le |a| + |b|$

$$|\phi(t_n) - Y_n| \le |\phi(t_n) - y_n| + |y_n - Y_n|$$
$$\le |E_n| + |R_n|$$

Global Truncation Error ($|E_n|$)

Round off error ($|R_n|$)

The Euler Method (Tangent Line Method) - PROBLEM

- Example – Round off Error

  Computer with four digits

$$\frac{1}{3} \to 0.3333 \qquad \frac{1}{9} \to 0.1111$$

Compute $\dfrac{x^2 - \frac{1}{9}}{x - \frac{1}{3}}$ for x=0.3334

$$\frac{(0.3334)^2 - 0.1111}{0.3334 - 0.3333} = \frac{0.1112 - 0.1111}{0.3334 - 0.3333} = 1$$

$$\frac{x^2 - \frac{1}{9}}{x - \frac{1}{3}} = \frac{\cancel{\left(x - \frac{1}{3}\right)}\left(x + \frac{1}{3}\right)}{\cancel{\left(x - \frac{1}{3}\right)}} = x + \frac{1}{3} \approx 0.3334 + 0.3333 = 0.6667$$

The Euler Method (Tangent Line Method) - PROBLEM

- How to reduce round-off error

  1) Minimize the number of calculations

  2) Use double-precision arithmetic

The Euler Method (Tangent Line Method) - PROBLEM

- Global versus Local Truncation Error

$$E_n \rightarrow e_n$$

Global Truncation Error

Local Truncation Error
(In each step)

Assume that the solution is

$$y = \phi(t)$$

to the initial value problem

$$\phi'(t) = f\left[t, \phi(t)\right]$$

The Euler Method (Tangent Line Method) - PROBLEM

- Expand $\phi(t)$ about t_n using polynomial with a remainder

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\bar{t}_n)h^2$$

where $\bar{t}_n$ is some point $t_n < \bar{t}_n < t_n + h$

$$\phi(t_n + h) = \phi(t_{n+1})$$

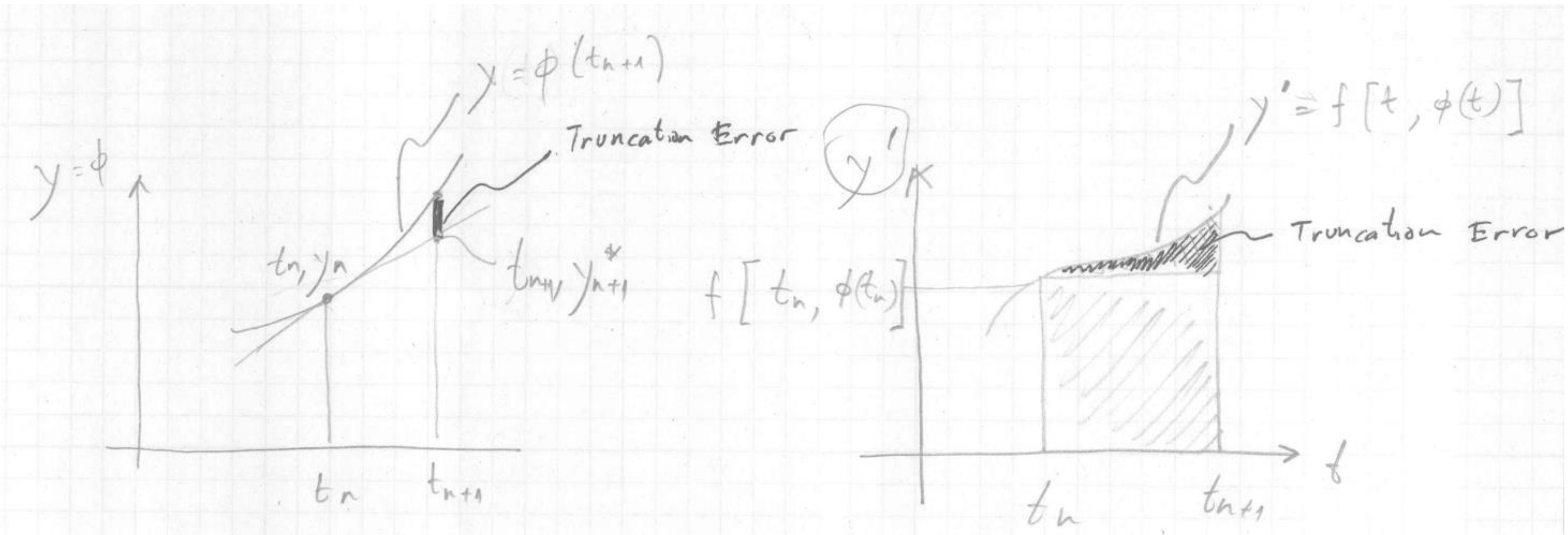
$$\phi(t_{n+1}) = \phi(t_n) + hf\left[t_n, \phi(t_n)\right] + \frac{1}{2}\phi''(\bar{t}_n)h^2$$

using the Euler formula to calculate an approximation to $\phi(t_{n+1})$

$$y_{n+1}^* = \phi(t_n) + hf\left[t_n, \phi(t_n)\right]$$

the difference between $\phi(t_{n+1})$ and y_{n+1}^* is the local truncation error

The Euler Method (Tangent Line Method) - PROBLEM



$$e_{n+1} = \phi(t_{n+1}) - y^*_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$$

$$e_{n+1} \propto h^2$$

$$e_{n+1} \le M\,\frac{h^2}{2}$$

- M is the maximum of $|\phi''(t_n)|$ on the interval [a b]

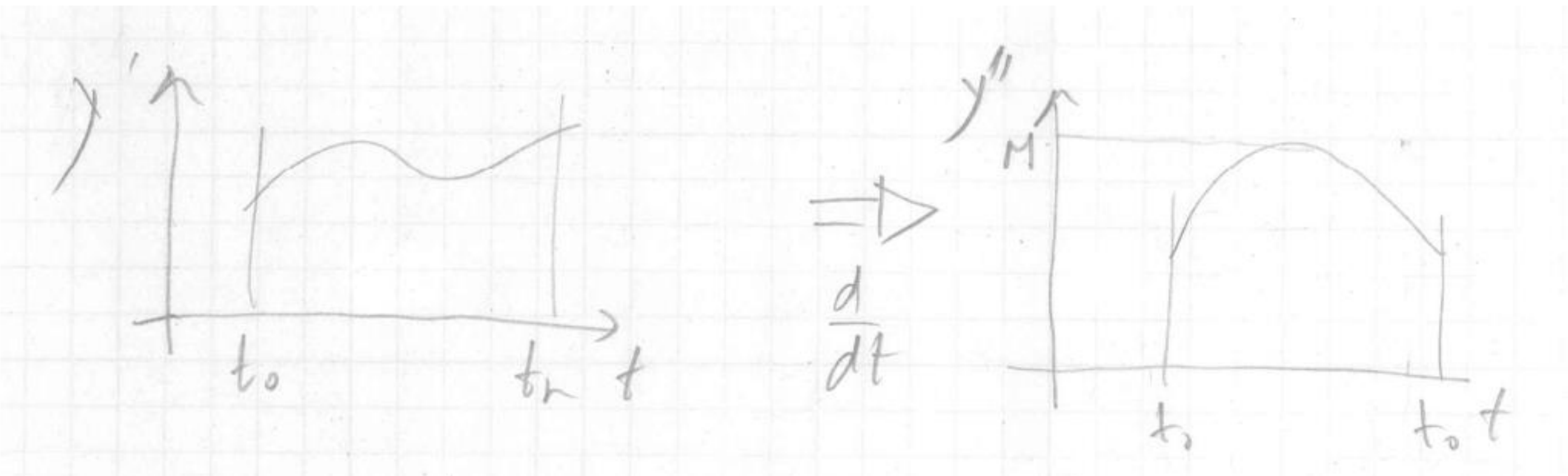
The Euler Method (Tangent Line Method) - PROBLEM

- One use of the equation is to choose a step size that will result in a local truncation error no greater than some given tolerance level

  For example if the local truncation error must be no greater than ε

$$M\frac{h^2}{2} \le \varepsilon \to h \le \sqrt{2\varepsilon/M}$$

  Difficulty: estimate $|\phi''(t_n)|$ or M



The Euler Method (Tangent Line Method) - PROBLEM

$$e_{n+1} \propto h^2$$
$$\bar{e}_{n+1} \propto \left(\frac{h}{2}\right)^2 = \frac{h^2}{4} = \frac{e_n}{4}$$

- Reducing the interval by ½ reduces the error by ¼

- The global truncation error

$$|E_n| \le Kh$$ K: some constant

- Euler method is called a first order method because its global truncation error is proportional to the first power of the step size h

The Euler Method (Tangent Line Method) - IMPROVE

- Improved Euler formula (Heun formula)

- Idea: replace integrand by the average of the two endpoints

$$\phi(t_{n+1}) = \phi(t_n) + \int_{t_n}^{t_{n+1}} f[t_n, \phi(t_n)]dt$$
$$\cong \phi(t_n) + f[t_n, \phi(t_n)](t_{n+1} - t_n)$$
$$= \phi(t_n) + hf[t_n, \phi(t_n)]$$

⇓

$$y_{n+1} = y_n + hf_n$$

Euler formula

$$\phi(t_{n+1}) \cong \phi(t_n) + \frac{f[t_n, \phi(t_n)] + f[t_{n+1}, \phi(t_{n+1})]}{2}$$
$$= \phi(t_n) + \frac{f[t_n, \phi(t_n)] + f[t_n + h, \phi(t_n + h)]}{2}(t_{n+1} - t_n)$$
$$= \phi(t_n) + \frac{f[t_n, \phi(t_n)] + f[t_n + h, \phi(t_n + h)]}{2}h$$

⇓

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f[t_n + h, y_n + hf(t_n, y_n)]}{2}h_n$$

Heun formula

The Euler Method (Tangent Line Method) - IMPROVE

- Improved Euler formula (Heun formula)

Euler formula

$$y_{n+1} = y_n + hf_n$$

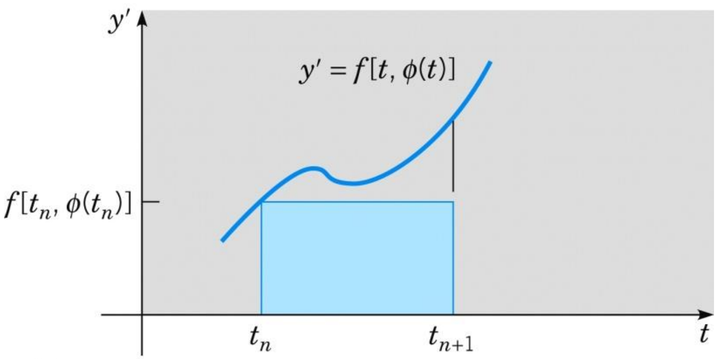


Figure 8.1.1

Local Truncation Error: $e_{n+1} \propto h^2$

Global truncation error: First order method

Heun formula

$$y_{n+1} = y_n + \frac{f\left(t_n, y_n\right) + f\left[t_n + h, y_n + hf\left(t_n, y_n\right)\right]}{2} h_n$$

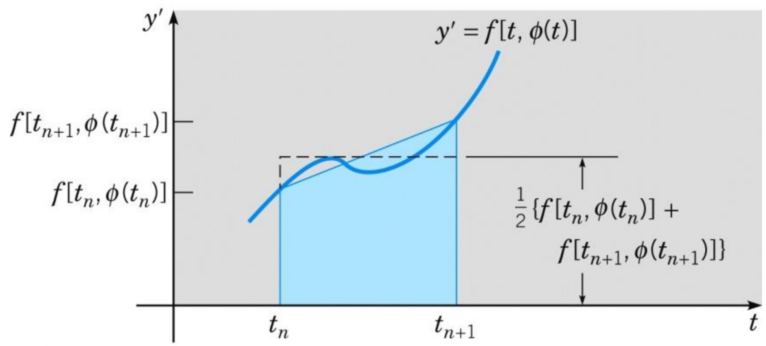


Figure 8.2.1

Local Truncation Error: $e_{n+1} \propto h^3$

Global Truncation Error: Second order method

Cost: more computational work

The Euler Method (Tangent Line Method) - IMPROVE

- If $f(t, y)$ depends only on t and not on y, solving the differential equation $y' = f(t, y)$ reduces to integrating $f(t)$ in this case

$$y_{n+1} - y_n = \frac{h}{2}\left[f(t_n) + f(t_n + h)\right]$$

- Trapezoid rule of numerical integration

The Euler Method (Tangent Line Method) - IMPROVE

- How to apply improved Euler method

step 1	define f(t,y)
step 2	input initial values t0 and y0
step 3	input step size h and the number of steps n
step 4	output t0 and y0
step 5	for j from 1 to n do

```
for j from 1 to n do
        k1 = f(t, y)
        k2 = f(t + h, y + h*k1)
        y = y (h/2)*(k1 + k2)
        t = t + h
output t and y
end
```

The Euler Method (Tangent Line Method) - IMPROVE

- The improved Euler method is an example of a predictor-corrector method

$$y^*_{n+1} = \phi(t_n) + hf\left[t_n, \phi(t_n)\right] \qquad \text{(Predictor)}$$

predicts the value of yn+1

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f\left[t_{n+1}, y^*_{n+1}\right]}{2} h_n \qquad \text{(Corrector)}$$

corrects the estimate