

Class Notes 17:

**System of First Order  
Linear Differential Equations -  
Nonhomogenous System(2/2)**

**MAE 82 – Engineering Mathematics**

# Nonhomogeneous Linear Systems

- Linear system  $X' = AX + F(t)$

- General Solution  $X = X_c + X_p$

Complementary solution or  
General solution of the  
associated homogeneous linear  
system  $X'=AX$

Particular solution of the  
nonhomogeneous linear system  
 $X'=AX+F$

- Particular Solution – Methods
  - Undetermined Coefficients (Quick)
  - Variation of Parameters (More powerful)

# Undetermined Coefficients

- Assumptions

$$X' = AX + F(t)$$

Constants

- Constants
- Polynomials
- Exponential functions
- sin/cos
- finite sum/products of these functions

# Undetermined Coefficients – Example (Constants)

- Example ( $X_p$  - constants)

$$X' = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} X + \begin{Bmatrix} -8 \\ 3 \end{Bmatrix} \quad \text{on } (-\infty, \infty)$$

- Solve the associated homogeneous system

$$X' = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} X$$

$$\det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & 2 \\ -1 & 1 - \lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = i$$

$$\lambda_2 = \bar{\lambda}_1 = -i$$

$$\begin{cases} (-1 - i)K_1 + 2K_2 = 0 \\ -1 \cdot K_1 + (1 - i)K_2 = 0 \end{cases} \quad (*)$$

# Undetermined Coefficients – Example (Constants)

from (\*)  $K_1 = (1-i)K_2 \Rightarrow K_2 = 1, K_1 = 1-i$

$$K_1 = \begin{Bmatrix} 1-i \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} i \quad K_2 = \begin{Bmatrix} 1+i \\ 1 \end{Bmatrix}$$

$$B_1 = \operatorname{Re}(K_1) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad B_2 = \operatorname{Im}(K_1) = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

$$X_c = c_1 \left[ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos t - \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \sin t \right] e^{it} + c_2 \left[ \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \cos t + \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin t \right] e^{it}$$

$$X_c = c_1 \begin{Bmatrix} \cos t + \sin t \\ \cos t \end{Bmatrix} + c_2 \begin{Bmatrix} -\cos t + \sin t \\ \sin t \end{Bmatrix}$$

- Since  $F(t)$  is a constant vector, we assume a constant particular solution

$$X_p = \begin{Bmatrix} a \\ b \end{Bmatrix} \quad X'_p = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

# Undetermined Coefficients – Example (Constants)

Plug into

$$\underbrace{\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}_{X'_p} = \underbrace{\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}}_A \underbrace{\begin{Bmatrix} a \\ b \end{Bmatrix}}_{X_p} + \underbrace{\begin{Bmatrix} -8 \\ 3 \end{Bmatrix}}_F$$

$$\begin{cases} 0 = -a + 2b - 8 \\ 0 = -a + b + 3 \end{cases} \Rightarrow a = 14; b = 11$$

$$X_p = \begin{Bmatrix} 14 \\ 11 \end{Bmatrix}$$

$$X = c_1 \begin{Bmatrix} \cos t + \sin t \\ \cos t \end{Bmatrix} + c_2 \begin{Bmatrix} -\cos t + \sin t \\ \sin t \end{Bmatrix} + \begin{Bmatrix} 14 \\ 11 \end{Bmatrix}$$

# Undetermined Coefficients – Example (First Order Polynomial)

- Example ( $X_p$  - polynomial)

$$X' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} X + \begin{Bmatrix} 6t \\ -10t + 4 \end{Bmatrix} \quad \text{on } (-\infty, \infty)$$

- The associated homogeneous system

$$X' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} X \quad \lambda_1 = 2; \lambda_2 = 7 \quad K_1 = \begin{Bmatrix} 1 \\ -4 \end{Bmatrix}; K_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$X_c = c_1 \begin{Bmatrix} 1 \\ -4 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{7t}$$

since  $F(t) = \begin{Bmatrix} 6 \\ -10 \end{Bmatrix} t + \begin{Bmatrix} 0 \\ 4 \end{Bmatrix}$  the particular solution of the same form is

$$X_p = \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}$$

# Undetermined Coefficients – Example (First Order Polynomial)

$$\underbrace{\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix}}_{X'_p} = \underbrace{\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}}_A \underbrace{\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix}t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}}_{X_p} + \underbrace{\begin{Bmatrix} 6 \\ -10 \end{Bmatrix}t + \begin{Bmatrix} 0 \\ 4 \end{Bmatrix}}_F$$

$$0 = \overbrace{(6a_2 + b_2 + 6)}{=0}t + \overbrace{6a_1 + b_1 - a_2}{=0}$$

$$0 = \underbrace{(4a_2 + b_2 - 10)}{=0}t + \underbrace{4a_1 + 3b_1 - b_2 + 4}{=0}$$

$$\begin{cases} 6a_2 + b_2 + 6 = 0 \\ 4a_2 + b_2 - 10 = 0 \\ 6a_1 + b_1 - a_2 = 0 \\ 4a_1 + 3b_1 - b_2 + 4 = 0 \end{cases} \rightarrow \begin{cases} a_1 = -4/7 \\ b_1 = 10/7 \\ a_2 = -2 \\ b_2 = 6 \end{cases}$$

$$X_p = \begin{Bmatrix} -2 \\ 6 \end{Bmatrix}t + \begin{Bmatrix} -4/7 \\ 10/7 \end{Bmatrix}$$

$$X = X_c + X_p = c_1 \begin{Bmatrix} 1 \\ -4 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{7t} + \begin{Bmatrix} -2 \\ 6 \end{Bmatrix}t + \begin{Bmatrix} -4/7 \\ 10/7 \end{Bmatrix}$$



# Undetermined Coefficients – Example (Exponential Function)

- Example ( $X_p$  – exponential functions)

$$\frac{dy}{dt} = 5x + 3y - 2e^{-t} + 1$$

$$\frac{dy}{dt} = -x + y + e^{-t} - 5t + 7$$

$$X' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} X + \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} e^{-t} + \begin{Bmatrix} 0 \\ -5 \end{Bmatrix} t + \begin{Bmatrix} 1 \\ 7 \end{Bmatrix}$$

Assumption for a particular solution would be

$$X_p = \begin{Bmatrix} a_3 \\ b_3 \end{Bmatrix} e^{-t} + \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}$$

# Variation of Parameters

- Fundamental Matrix

If

$X_1, X_2, \dots, X_n$  is a fundamental set of solution of the homogeneous system  $X' = AX$  on the interval  $I$

Then

Its general solution on the interval is

$$X = c_1 \begin{Bmatrix} X_{11} \\ X_{21} \\ \vdots \\ X_{n1} \end{Bmatrix} + c_2 \begin{Bmatrix} X_{12} \\ X_{22} \\ \vdots \\ X_{n2} \end{Bmatrix} + \cdots + c_n \begin{Bmatrix} X_{1n} \\ X_{2n} \\ \vdots \\ X_{nn} \end{Bmatrix} = \begin{bmatrix} c_1 X_{11} + c_2 X_{12} + \cdots + c_n X_{1n} \\ c_1 X_{21} + c_2 X_{22} + \cdots + c_n X_{2n} \\ \vdots \\ c_1 X_{n1} + c_2 X_{n2} + \cdots + c_n X_{nn} \end{bmatrix}$$

The general solution can be written as a product

$$X = \Phi C$$

# Variations of Parameters – Fundamental Matrix

$C$  –  $n \times 1$  column vector of arbitrary constants  $c_1, c_2, \dots, c_n$

$\Phi$  –  $n \times n$  matrix, whose columns consist of the entries of the solution vectors of the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$\Phi$  – is called a fundamental matrix

$$\Phi = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}$$

# Variations of Parameters – Fundamental Matrix

## Properties of the fundamental matrix

1. A fundamental matrix  $\Phi(t)$  is nonsingular.
  2. If  $\Phi(t)$  is a fundamental matrix of the system  $\mathbf{x}' = \mathbf{A}\mathbf{x}$  then,  $\Phi'(t) = A\Phi(t)$
- 

## Variation of parameters

Is it possible to replace the vector of constants  $C$  in

$$X'(t) = \Phi(t)C$$

by a vector of functions.

## Variations of Parameters

$$U(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{Bmatrix} \quad \text{so, } x_p = \Phi(t)U(t)$$

is a particular solution of the nonhomogeneous system.

$$X'(t) = AX + F(t)$$

by the product rule the derivative of  $X_p = \Phi(t)U(t)$

$$X'_p = \Phi(t)U'(t) + \Phi'(t)U(t)$$

[Note that the order is important  $\Phi(t)U'(t)$  or  $\Phi'(t)U(t)$

## Variations of Parameters

$$\left. \begin{array}{l} \text{substitute } X_p = \Phi(t)U(t) \\ X'_p = \Phi(t)U'(t) + \Phi'(t)U(t) \end{array} \right\} \rightarrow X' = AX + F$$

$$\underbrace{\Phi(t)U'(t) + \Phi'(t)U(t)}_{X'_p} = A \underbrace{\Phi(t)U(t)}_{X_p} = F(t)$$

use the property that  $x' = Ax \rightarrow \Phi'(t) = A\Phi$

$$\underbrace{\Phi(t)U'(t) + \Phi'(t)U(t)}_{X'_p} = A \underbrace{\Phi(t)U(t)}_{X_p} = F(t)$$

$$\Phi(t)U'(t) + \cancel{A\Phi(t)U(t)} = A\cancel{\Phi(t)U(t)} + F(t)$$

$$\Phi^{-1} \cdot (\Phi(t)U'(t) = F(t))$$

$$U'(t) = \Phi^{-1}F(t)$$

# Variations of Parameters

$$U(t) = \int \Phi^{-1}(t)F(t)dt$$

Since

$$X_p = \Phi(t)U(t)$$

$$X_p = \Phi(t) \underbrace{\int \Phi^{-1}(t)F(t)dt}_{\text{Integrate each entry}}$$

$$x = x_c + x_p = \Phi(t)C + \Phi(t) \int \Phi^{-1}(t)F(t)dt$$

## Example - Variations of Parameters

$$x' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} x + \begin{Bmatrix} 3t \\ e^{-t} \end{Bmatrix} \text{ on } (-\infty, \infty)$$

The associated homogeneous system

$$x' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} x$$

$$\det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & 1 \\ 2 & -4 - \lambda \end{vmatrix} = (\lambda + 2)(\lambda + 5) = 0$$

$$k_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad k_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} \quad \lambda_1 = -2, \lambda_2 = -5$$

$$x_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-2t} = \begin{Bmatrix} e^{-2t} \\ e^{-2t} \end{Bmatrix} \quad x_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} e^{-5t} = \begin{Bmatrix} e^{-5t} \\ -2e^{-5t} \end{Bmatrix}$$



## Example - Variations of Parameters

$$\Phi(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \quad \Phi^{-1}(t) = \begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}$$

$$\begin{aligned} x_p &= \Phi(t) \int \Phi^{-1}(t) F(t) dt = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \int \begin{bmatrix} 2/3e^{2t} & 1/3e^{2t} \\ 1/3e^{5t} & -1/3e^{5t} \end{bmatrix} \begin{Bmatrix} 3t \\ e^{-t} \end{Bmatrix} dt \\ &= \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \int \begin{bmatrix} 2te^{2t} + 1/3e^t \\ te^{5t} - 1/3e^{4t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{bmatrix} te^{2t} - 1/2e^{2t} + 1/3e^t \\ 1/5te^{5t} - 1/25e^{5t} - 1/12e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} 6/5 \cdot t - 27/50 + 1/4e^{-t} \\ 3/5 \cdot t - 21/50 + 1/2e^{-t} \end{bmatrix} \end{aligned}$$

## Example - Variations of Parameters

$$\begin{aligned}x &= \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} + \begin{bmatrix} 6/5 \cdot t - 27/50 + 1/4 e^{-t} \\ 3/5 \cdot t - 21/50 + 1/2 e^{-t} \end{bmatrix} \\ &= c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-2t} + c_2 \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} e^{-5t} + \begin{Bmatrix} 6/5 \\ 3/5 \end{Bmatrix} t - \begin{Bmatrix} 27/50 \\ 21/50 \end{Bmatrix} + \begin{Bmatrix} 1/4 \\ 1/2 \end{Bmatrix} e^{-t}\end{aligned}$$

## Initial Value Problem

$$X = \Phi(t)C + \Phi(t) \int_{t_0}^t \Phi^{-1}(s)F(s)ds \quad (*)$$

$$I.C. \quad X(t_0) = X_0$$

Because the limits of integration are chosen so that the particular solution vanishes  $t = t_0$ , substituting  $t = t_0$  into (\*) yields

$$X = \Phi(t_0)C$$

for which we can get

$$C = \Phi^{-1}(t_0)X_0$$

$$X = \Phi(t) \overbrace{\left[ \Phi^{-1}(t_0)X_0 \right]}^C + \Phi(t) \int_{t_0}^t \Phi^{-1}(s)F(s)ds$$