Class Notes 17:

System of First Order Linear Differential Equations - Nonhomogenous System(2/2)

MAE 82 – Engineering Mathematics

Nonhomogeneous Linear Systems

- Linear system $X' = AX + F(t)$

- General Solution $X = X_c + X_p$

Complementary solution or General solution of the associated homogeneous linear system X'=AX

Particular solution of the nonhomogeneous linear system X'=AX+F

- Particular Solution – Methods
  - Undetermined Coefficients (Quick)
  - Variation of Parameters (More powerful)

Undetermined Coefficients

- Assumptions

$$X' = AX + F(t)$$

Constants

- Constants
- Polynomials
- Exponential functions
- sin/cos
- finite sum/products of these functions

Undetermined Coefficients – Example (Constants)

- Example (X_p - constants)

$$X' = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} X + \begin{Bmatrix} -8 \\ 3 \end{Bmatrix} \quad \text{on } (-\infty, \infty)$$

- Solve the associated homogeneous system

$$X' = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} X$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = i$$

$$\lambda_2 = \overline{\lambda_1} = -i$$

$$\begin{cases} (-1-i)K_1 + 2K_2 = 0 \\ -1 \cdot K_1 + (1-i)K_2 = 0 \end{cases} \quad (*)$$

Undetermined Coefficients – Example (Constants)

from (*) $K_1 = (1-i)K_2 \Rightarrow K_2 = 1, K_1 = 1-i$

$$K_1 = \begin{Bmatrix} 1-i \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} i \qquad K_2 = \begin{Bmatrix} 1+i \\ 1 \end{Bmatrix}$$

$$B_1 = \mathrm{Re}(K_1) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad B_2 = \mathrm{Im}(K_1) = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

$$X_c = c_1 \left[\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos t - \begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \sin t \right] \cancel{e^{0t}} + c_2 \left[\begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \cos t + \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin t \right] \cancel{e^{0t}}$$

$$X_c = c_1 \begin{Bmatrix} \cos t + \sin t \\ \cos t \end{Bmatrix} + c_2 \begin{Bmatrix} -\cos t + \sin t \\ \sin t \end{Bmatrix}$$

- Since F(t) is a constant vector, we assume a constant particular solution

$$X_p = \begin{Bmatrix} a \\ b \end{Bmatrix} \qquad X_p' = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Undetermined Coefficients – Example (Constants)

Plug into

$$\underbrace{\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}_{X_p'} = \underbrace{\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} a \\ b \end{Bmatrix}}_{X_p} + \underbrace{\begin{Bmatrix} -8 \\ 3 \end{Bmatrix}}_{F}$$

$$\begin{cases} 0 = -a + 2b - 8 \\ 0 = -a + b + 3 \end{cases} \Rightarrow a = 14; b = 11$$

$$X_p = \begin{Bmatrix} 14 \\ 11 \end{Bmatrix}$$

$$X = c_1 \begin{Bmatrix} \cos t + \sin t \\ \cos t \end{Bmatrix} + c_2 \begin{Bmatrix} -\cos t + \sin t \\ \sin t \end{Bmatrix} + \begin{Bmatrix} 14 \\ 11 \end{Bmatrix}$$

Undetermined Coefficients – Example (First Order Polynomial)

- Example (X_p - polynomial)

$$X' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} X + \begin{Bmatrix} 6t \\ -10t+4 \end{Bmatrix} \quad \text{on } (-\infty, \infty)$$

- The associated homogeneous system

$$X' = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} X \qquad \lambda_1 = 2; \lambda_2 = 7 \qquad K_1 = \begin{Bmatrix} 1 \\ -4 \end{Bmatrix}; K_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$X_c = c_1 \begin{Bmatrix} 1 \\ -4 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{7t}$$

since $F(t) = \begin{Bmatrix} 6 \\ -10 \end{Bmatrix} t + \begin{Bmatrix} 0 \\ 4 \end{Bmatrix}$ the particular solution of the same form is

$$X_p = \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}$$

Undetermined Coefficients – Example (First Order Polynomial)

$$\underbrace{\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix}}_{X_p'} = \underbrace{\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}}_{A} \underbrace{\left[\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} \right]}_{X_p} + \underbrace{\begin{Bmatrix} 6 \\ -10 \end{Bmatrix} t + \begin{Bmatrix} 0 \\ 4 \end{Bmatrix}}_{F}$$

$$0 = \overbrace{(6a_2 + b_2 + 6)}^{=0} t + \overbrace{6a_1 + b_1 - a_2}^{=0}$$

$$0 = \underbrace{(4a_2 + b_2 - 10)}_{=0} t + \underbrace{4a_1 + 3b_1 - b_2 + 4}_{=0}$$

$$\begin{cases} 6a_2 + b_2 + 6 = 0 \\ 4a_2 + b_2 - 10 \\ 6a_1 + b_1 - a_2 = 0 \\ 4a_1 + 3b_1 - b_2 + 4 = 0 \end{cases} \rightarrow \begin{cases} a_1 = -4/7 \\ b_1 = 10/7 \\ a_2 = -2 \\ b_2 = 6 \end{cases}$$

$$X_p = \begin{Bmatrix} -2 \\ 6 \end{Bmatrix} t + \begin{Bmatrix} -4/7 \\ 10/7 \end{Bmatrix}$$

$$X = X_c + X_p = c_1 \begin{Bmatrix} 1 \\ -4 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{7t} + \begin{Bmatrix} -2 \\ 6 \end{Bmatrix} t + \begin{Bmatrix} -4/7 \\ 10/7 \end{Bmatrix}$$

Undetermined Coefficients – Example (Exponential Function)

- Example (X_p – exponential functions)

$$\frac{dy}{dt} = 5x + 3y - 2e^{-t} + 1$$

$$\frac{dy}{dt} = -x + y + e^{-t} - 5t + 7$$

$$X' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} X + \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} e^{-t} + \begin{Bmatrix} 0 \\ -5 \end{Bmatrix} t + \begin{Bmatrix} 1 \\ 7 \end{Bmatrix}$$

Assumption for a particular solution would be

$$X_p = \begin{Bmatrix} a_3 \\ b_3 \end{Bmatrix} e^{-t} + \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} t + \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}$$

Variation of Parameters

- Fundamental Matrix

  If

  X1, X2,… Xn is a fundamental set of solution of the homogeneous system X'=AX on the interval I

  Then

  Its general solution on the interval is

$$X = c_1\begin{pmatrix} X_{11} \\ X_{21} \\ \vdots \\ X_{n1} \end{pmatrix} + c_2\begin{pmatrix} X_{12} \\ X_{22} \\ \vdots \\ X_{n2} \end{pmatrix} + \cdots + c_n\begin{pmatrix} X_{1n} \\ X_{2n} \\ \vdots \\ X_{nn} \end{pmatrix} = \begin{bmatrix} c_1X_{11} + c_2X_{12} + \cdots + c_nX_{1n} \\ c_1X_{21} + c_2X_{22} + \cdots + c_nX_{2n} \\ \vdots \\ c_1X_{n1} + c_2X_{n2} + \cdots + c_nX_{nn} \end{bmatrix}$$

  The general solution can be written as a product

$$X = \Phi C$$

Variations of Parameters – Fundamental Matrix

$C -$ n x 1 column vector of arbitrary constants $c_1, c_2, \cdots, c_n$

$\Phi -$ n x n matrix, whose columns consist of the entries of the solution vectors of the system **x**' = **Ax**

$\Phi -$ is called a fundamental matrix

$$\Phi = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}$$

Variations of Parameters – Fundamental Matrix

Properties of the fundamental matrix

1. A fundamental matrix $\Phi(t)$ is nonsingular.
2. If $\Phi(t)$ is a fundamental matrix of the system **x' = Ax**

   then, $\Phi'(t) = A\Phi(t)$

Variation of parameters

Is it possible to replace the vector of constants *C* in

$$X'(t) = \Phi(t)C$$

by a vector of functions.

Variations of Parameters

$$U(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{Bmatrix} \quad so, \quad x_p = \Phi(t)U(t)$$

is a particular solution of the nonhomogeneous system.

$$X'(t) = AX + F(t)$$

by the product rule the derivative of $X_p = \Phi(t)U(t)$

$$X_p' = \Phi(t)U'(t) + \Phi'(t)U(t)$$

[Note that the order is important $\Phi(t)U'(t)$ or $\Phi'(t)U(t)$

Variations of Parameters

substitute $\left.\begin{aligned} X_p &= \Phi(t)U(t) \\ X_p' &= \Phi(t)U'(t) + \Phi'(t)U(t) \end{aligned}\right\} \to X' = AX + F$

$$\underbrace{\Phi(t)U'(t) + \Phi'(t)U(t)}_{X_p'} = A\underbrace{\Phi(t)U(t)}_{X_p} + F(t)$$

use the property that $x' = Ax \to \Phi'(t) = A\Phi$

$$\underbrace{\Phi(t)U'(t) + \Phi'(t)U(t)}_{X_p'} = A\underbrace{\Phi(t)U(t)}_{X_p} + F(t)$$

$$\Phi(t)U'(t) + \cancel{A\Phi(t)U}(t) = \cancel{A\Phi(t)U}(t) + F(t)$$

$$\Phi^{-1} \cdot (\Phi(t)U'(t) = F(t))$$

$$U'(t) = \Phi^{-1}F(t)$$

Variations of Parameters

$$U(t) = \int \Phi^{-1}(t) F(t) dt$$

Since

$$X_p = \Phi(t) U(t)$$

$$X_p = \Phi(t) \underbrace{\int \Phi^{-1}(t) F(t) dt}_{\textit{Integrate each entry}}$$

$$x = x_c + x_p = \Phi(t) C + \Phi(t) \int \Phi^{-1}(t) F(t) dt$$

Example - Variations of Parameters

$$x' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} x + \begin{Bmatrix} 3t \\ e^{-t} \end{Bmatrix} \; on \; (-\infty, \infty)$$

The associated homogeneous system

$$x' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} x$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda + 2)(\lambda + 5) = 0$$

$$\lambda_1 = -2, \; \lambda_2 = -5$$

$$k_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad k_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

$$x_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-2t} = \begin{Bmatrix} e^{-2t} \\ e^{-2t} \end{Bmatrix} \quad x_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} e^{-5t} = \begin{Bmatrix} e^{-5t} \\ -2e^{-5t} \end{Bmatrix}$$

Example - Variations of Parameters

$$\Phi(t)=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \qquad \Phi^{-1}(t)=\begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}$$

$$x_p=\Phi(t)\int \Phi^{-1}(t)F(t)dt=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\int\begin{bmatrix} 2/3e^{2t} & 1/3e^{2t} \\ 1/3e^{5t} & -1/3e^{5t} \end{bmatrix}\begin{Bmatrix} 3t \\ e^{-t} \end{Bmatrix}dt$$

$$=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\int\begin{bmatrix} 2te^{2t}+1/3e^{t} \\ te^{5t}-1/3e^{4t} \end{bmatrix}dt$$

$$=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\begin{bmatrix} te^{2t}-1/2e^{2t}+1/3e^{t} \\ 1/5te^{5t}-1/25e^{5t}-1/12e^{4t} \end{bmatrix}$$

$$=\begin{bmatrix} 6/5\cdot t-27/50+1/4e^{-t} \\ 3/5\cdot t-21/50+1/2e^{-t} \end{bmatrix}$$

Example - Variations of Parameters

$$x = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} + \begin{bmatrix} 6/5 \cdot t - 27/50 + 1/4e^{-t} \\ 3/5 \cdot t - 21/50 + 1/2e^{-t} \end{bmatrix}$$

$$= c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-2t} + c_2 \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} e^{-5t} + \begin{Bmatrix} 6/5 \\ 3/5 \end{Bmatrix} t - \begin{Bmatrix} 27/50 \\ 21/50 \end{Bmatrix} + \begin{Bmatrix} 1/4 \\ 1/2 \end{Bmatrix} e^{-t}$$

Initial Value Problem

$$X = \Phi(t)C + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)ds \quad (*)$$

$$I.C. \;\; X(t_0) = X_0$$

Because the limits of integration are chosen so that the particular solution vanishes $t = t_0$, substituting $t = t_0$ into (*) yields

$$X = \Phi(t_0)C$$

for which we can get

$$C = \Phi^{-1}(t_0)X_0$$

$$X = \Phi(t)\overbrace{\left[\Phi^{-1}(t_0)X_0\right]}^{C} + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)F(s)ds$$