Class Notes 16:

System of First Order Linear Differential Equations - Homogenous System(1/2)

MAE 82 – Engineering Mathematics

System of First Order Linear Differential Equations – Motivation

- Example 1 - 2 DOF Mass Spring Damper
- Example 2 – Lotka-Voltera Eq. (Preys / Predators)

Lotka-Voltera Eq. (Preys / Predators)

LOTKA - VOLTERRA EQUATION

The rate of change of the Rabit pop (Preys)

- The Rabit reproduce proportional to the current Rabit population (Exponentitional Growth)
- The rate at which the Rabit and the fox meet (Loss of Rabit pop)

$$\frac{dR}{dt} = \alpha R - \beta RF$$

R - Rabit

F - Fox

$$\frac{dF}{dt} = \Gamma RF - \gamma F$$

- Loss of Fox due to natural death or emigration
- The Fox reproduce proportional to the Rabit/Fox meeting

The rate of change of the Fox pop (Predators)

Assumptions

1. The prey population finds ample food at all times
2. The food supply of the predator population depends entirely on the size of the prey population
3. The rate of change of population is proportional to its size
4. During the process, the environment does not change in favor of one species and genetic adaptation is inconsequential
5. Predators have limitless appetite

(1) Prey find food – All the time

(2) Predator food – function of – size of the prey population

(3) $\frac{dP}{dt} \propto kP$

(4) Fix environment (No genetic adaptation)

(5) Predators → limited appetite

HISTORY

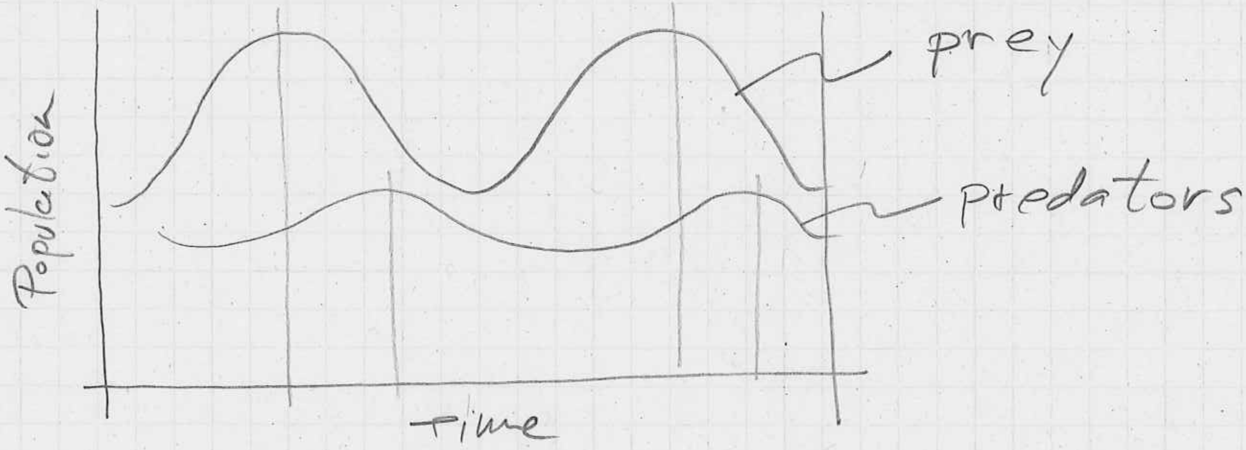
- Alfred J. Lotka
  - 1910 Autocatalytic chemical reaction
  - 1925 Predator-Prey interaction

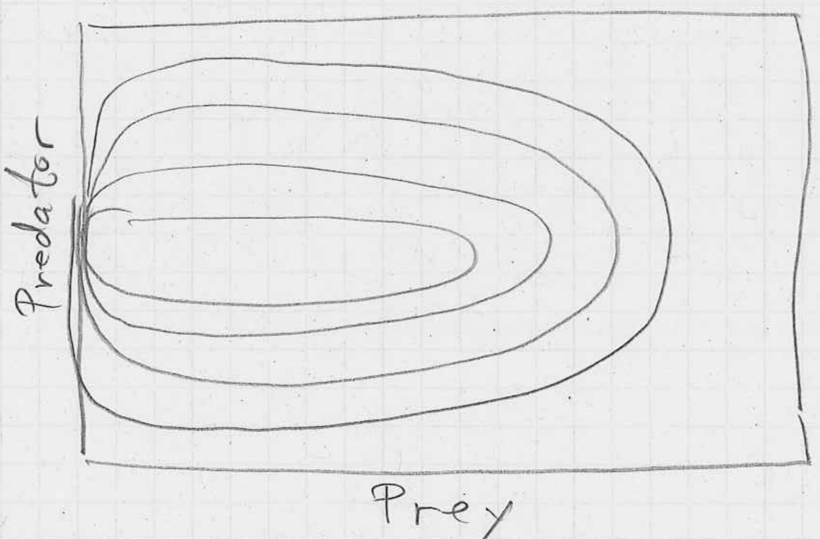
- Vito Volterra
  - 1926 Fish catches (WW I)

Richard Goodwin
1969/67 Economic Theory

SOLUTIONS TO THE EQUATIONS

- Periodic solution ($\alpha, \beta, \gamma, \delta > 0$)





Population Equilibrium

$$\frac{dR}{dt} = 0 \qquad \frac{dF}{dt} = 0$$

$$\begin{cases} x(\alpha - \beta y) = 0 \\ -y(\gamma - \delta x) = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad\qquad x = \frac{\gamma}{\delta}, \quad y = \frac{\alpha}{\beta}$$

Predator - Prey Model - Analytic Solution

Exponential Growth

y negative interaction with x ⇒ y → predator

$$\dot{x} = \frac{dx}{dt} = \boxed{a x} \boxed{- b x y}$$

$$\dot{y} = \frac{dy}{dt} = \boxed{-c x} \boxed{+ d x y}$$

x postive interaction with y ⇒ x → prey

Exponential Decay

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-cy + dxy}{ax - bxy} = \frac{y(-c + dx)}{x(a - by)}$$

Sparation of variables

$$\frac{a - by}{y} dy = \frac{-c + dx}{x} dx$$

Integrate both sides of the eq.

$$a \int \frac{1}{y} dy - b \int 1 \, dy = -c \int \frac{1}{x} dx + d \int 1 \, dx$$

$$a \ln|y| - by = -c \ln|x| + dx + \bar{C}$$

$$\underbrace{\ln|y^a| + \ln|x^c|} - by - dx = \bar{c}$$

$$\ln|x^c y^a| - by - dx = \bar{c}$$

$$a \ln|y| - by + c \ln|x| - dx = c$$

$x \to R$

$y \to F$

$a \to \alpha$

$b \to \beta$

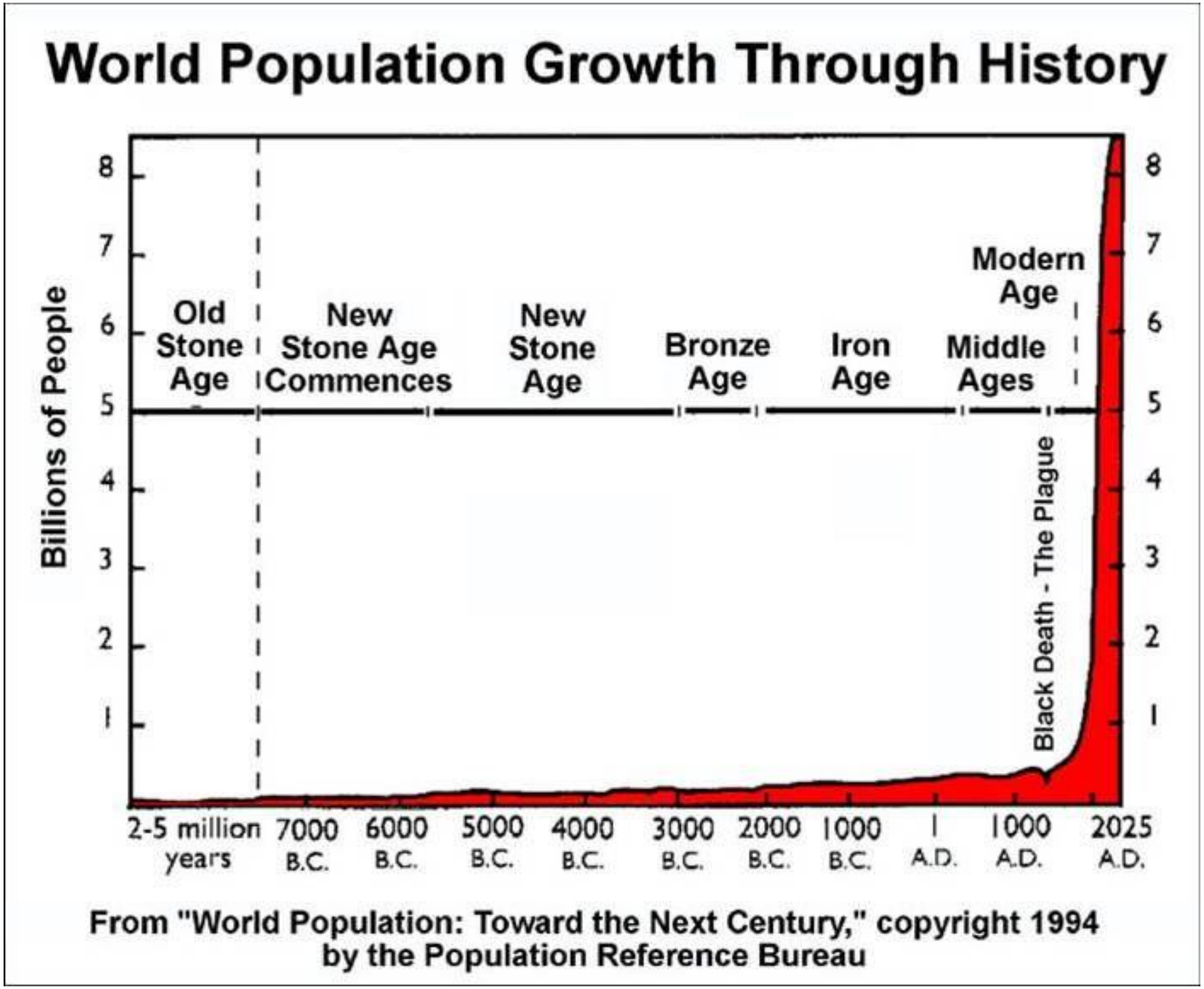
$c \to \delta$

$d \to \gamma$

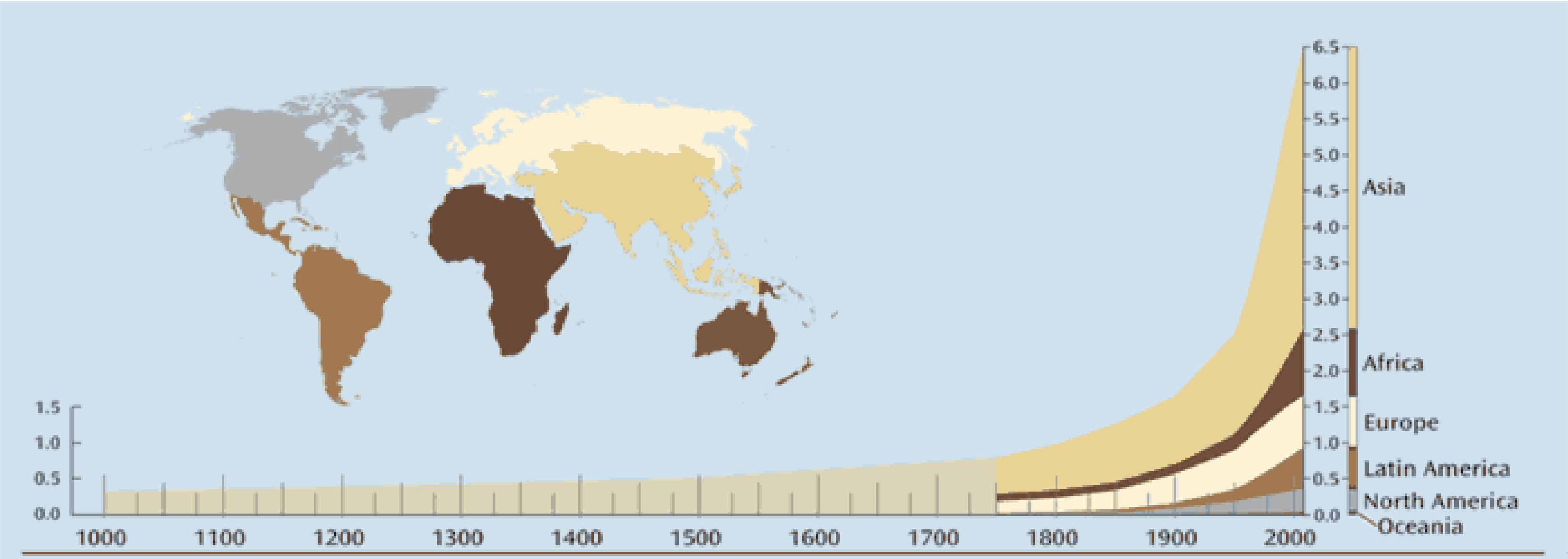
$$\ln |R^{\delta} F^{\alpha}| - \beta F - \gamma R = \bar{C}$$

$$\alpha \ln|F| - \beta F + \delta \ln|R| - \gamma R = \bar{C}$$

Lotka-Voltera Eq. (Preys / Predators)



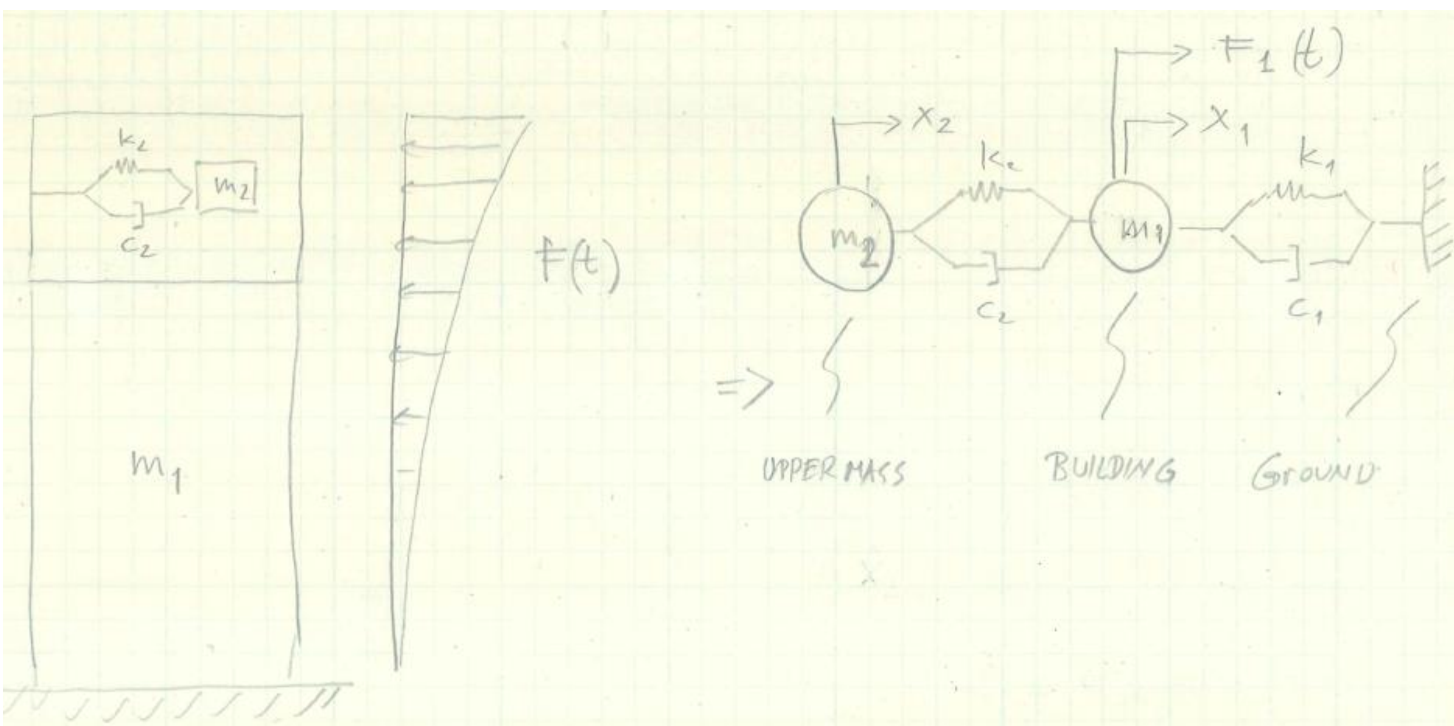
Lotka-Voltera Eq. (Preys / Predators)



Sources: 1 - The World at Six Billion; Population Division of the Department of Economic and Social Affairs of the United Nations Secretariat, World Population Prospects: The 2004 Revision and World Urbanization Prospects: The 2003 Revision, <http://esa.un.org/unpp> 2 - United Nations, 1973. "The Determinants and Consequences of Population Trends, Vol.1" (United Nations, New York). United Nations, (forthcoming). "World Population Prospects: The 1998 Revision" (United Nations, New York). <http://www.geohive.com/global/>

System of Linear Differential Equation

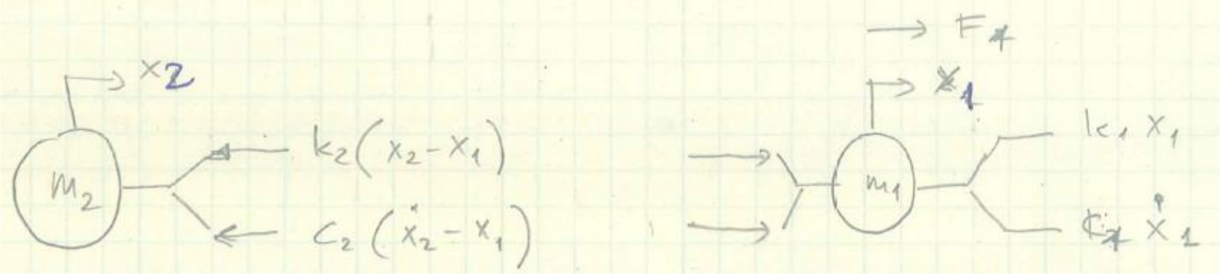
Differential Equation → Laplace Transform → Algebraic Equation



System of Linear Differential Equation

- Assumption $\begin{cases} c_1 = c_2 = 0 \\ x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0 \end{cases}$ No dampingIC

$x_2 > x_1$



$$+ \rightarrow \sum F = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2$$

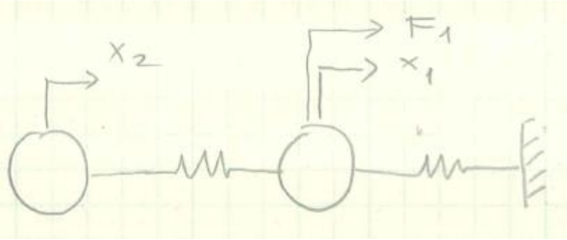
$$+ \rightarrow \sum F = k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - c_1\dot{x}_1 + F_1 = m_1\ddot{x}_1$$

set $c_1 = c_2 = 0$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t)$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

System of Linear Differential Equation



$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + F_1 \\ m_2 x_2'' = + \qquad k_2 x_1 - k_2 x_2 \end{cases}$$

$$\begin{cases} x_1'' = -\dfrac{(k_1 + k_2)}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 + \dfrac{F_1}{m_1} \\ x_2'' = + \qquad \dfrac{k_2}{m_2} x_1 - \dfrac{k_2}{m_2} x_2 \end{cases}$$

Four first order linear Diff. Eq.

$$\begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -\dfrac{(k_1 + k_2)}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 + \dfrac{F_1}{m_1} \\ x_4' = + \qquad \dfrac{k_2}{m_2} x_1 - \dfrac{k_2}{m_2} x_2 \end{cases}$$

System of Linear Differential Equation

$$x' = P(t)x + g(t)$$

$$\underbrace{\begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{Bmatrix}}_{\underset{x'}{\uparrow}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & 0 \end{bmatrix}}_{P(t)} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}}_{x} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ F_1 / m_1 \\ 0 \end{Bmatrix}}_{g(t)}$$

$$\begin{cases} \text{Homogeneous} & g(t) = 0 \\ \text{Nonhomogeneous} & g(t) \neq 0 \end{cases}$$

System of Linear Differential Equation

Laplace

$$m_1 s^2 X_1(s) + (k_1 + k_2) X_1(s) - k_2 X_2(s) = F_1(s)$$
$$m_2 s^2 X_2(s) - k_2 X_1(s) + k_2 X_2(s) = 0$$

$$\begin{bmatrix} k_1 + k_2 + m_1 s^2 & -k_2 \\ -k_2 & k_2 + m_2 s^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(s) \\ 0 \end{Bmatrix}$$

$$X_1(s) = \frac{F_1(m_2 s^2 + k_2)}{(m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$$
$$X_2(s) = \frac{+k_2 F_1}{(m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$$

System of Linear Differential Equation

Check

$$\left[1-\left(\frac{w}{w_a}\right)^2\right]X_{ST}=\left[1-w^2\frac{m_2}{k_2}\right]\frac{F_1}{k_1}=\frac{F_1}{k_1}-w^2\frac{m_2F_1}{k_1k_2}$$

$$=\left(\frac{F_1}{k_1}-w^2\frac{m_2F_1}{k_1k_2}\right)k_1k_2$$

$$=F_1k_2-w^2m_2F_1$$

$$\left.\begin{array}{l} X_{ST}=\dfrac{F_1}{k_1}\left(k_1k_2\right)=F_1k_2 \\ (k_1k_2)=(w_n^2m_1)(w_a^2m_2) \end{array}\right] \text{NUM}$$

$$\left.\left\{\left[1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2\right]\left[1-\left(\frac{w}{w_n}\right)^2\right]-\mu\left(\frac{w_a}{w_n}\right)^2\right\}k_1k_2\right] \text{DEN}$$

System of Linear Differential Equation

$$s = jw, \quad s^2 = (jw)^2 = -w^2$$

$$|X_1| = \frac{(k_2 - w^2 m_2)F_1}{(k_1 + k_2 - w^2 m_1)(k_2 - w^2 m_2) - k_2^2}$$

$$|X_2| = \frac{k_2 F_1}{(k_1 + k_2 - w^2 m_1)(k_2 - w^2 m_2) - k_2^2}$$

$$* \frac{\overbrace{k_1 k_2}^{extract\ from\ Num}}{\underbrace{k_1 k_2}_{divide\ by}}$$

$$w_n = \sqrt{\frac{k_1}{m_1}}, \quad w_a = \sqrt{\frac{k_2}{m_2}}, \quad X_{st} = \frac{F_1}{k_1}, \quad \mu = \frac{m_2}{m_1}$$

$$|X_1| = \frac{\left[1 - \left(\frac{w}{w_a}\right)^2\right] X_{ST}}{\left[1 + \mu\left(\frac{w_a}{w_n}\right)^2 - \left(\frac{w}{w_n}\right)^2\right]\left[1 - \left(\frac{w}{w_a}\right)^2\right] - \mu\left(\frac{w_a}{w_n}\right)^2}$$

$$|X_2| = \frac{X_{ST}}{\left[1 + \mu\left(\frac{w_a}{w_n}\right)^2 - \left(\frac{w}{w_n}\right)^2\right]\left[1 - \left(\frac{w}{w_a}\right)^2\right] - \mu\left(\frac{w_a}{w_n}\right)^2}$$

System of Linear Differential Equation

$$\mu\left(\frac{w_a}{w_n}\right)^2 k_1 k_2 = \frac{\cancel{m_2}}{\cancel{m_1}}\,\frac{k_2}{\cancel{m_2}}\,\frac{\cancel{m_1}}{\cancel{k_1}}\,\cancel{k_1}k_2 = k_2^2$$

$$\left[1-\left(\frac{w}{w_a}\right)^2\right]k_2 = k_2 - \left(\frac{w^2}{\cancel{k_2}}m_2\right)\cancel{k_2} = k_2 w^2 m_2$$

$$\left[1+\mu\left(\frac{w_a}{w_n}\right)^2 - \frac{w}{w_n}\right]k_1 = \left[k_1 + \frac{\cancel{m_2}}{\cancel{m_1}}\,\frac{k_2}{\cancel{m_2}}\,\frac{\cancel{m_1}}{\cancel{k_1}}\,\cancel{k_1} - w^2\,\frac{m_1}{\cancel{k_1}}\,\cancel{k_1}\right]$$

$$= k_1 + k_2 - w^2 m_1$$

Den

System of Linear Differential Equation

From Eq for $\left|X_1\right| = \dfrac{\left[1-\left(\dfrac{w}{w_a}\right)^2\right]X_s}{\Delta}$

when $w = w_a \rightarrow x_1 = 0$

The amplitude of the main mass reduces to zero

Hence the absorber can indeed perform the task for which it is designed, namely to eliminate the vibration of the main mass, provided the natural frequency of the absorber is the same as the frequency of the external exitation.

From Eq for x_2 $w = w_a$

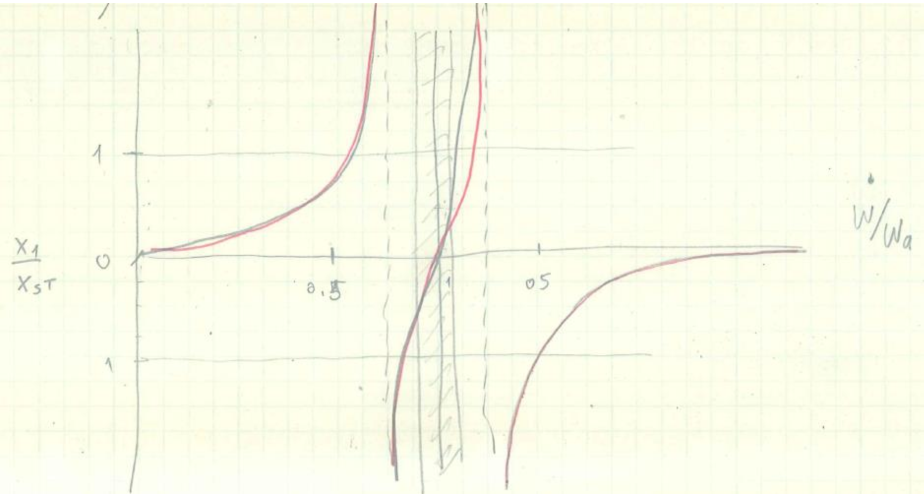
$$\left|X_2\right| = \frac{X_{ST}}{\left[1+\mu\left(\dfrac{w_a}{w_n}\right)^2 - \left(\dfrac{w_a}{w_n}\right)\right]\underbrace{\left[1-\left(\dfrac{w_a}{w_a}\right)^2\right]}_{=0} - \mu\left(\dfrac{w_a}{w_n}\right)^2} = -\frac{X_{ST}}{\mu}\left(\frac{w_n}{w_a}\right)^2$$

System of Linear Differential Equation

$$\left|X_2\right| = -\underbrace{\frac{F_1}{k_1}}_{X_{ST}}\underbrace{\frac{m_1}{m_2}}_{\frac{1}{\mu}}\underbrace{\frac{k_1}{m_1}}_{w_n^2}\underbrace{\frac{m_2}{k_2}}_{\frac{1}{w_a^2}} = -\frac{F_1}{k_2}$$

$$F_1 = \left|X_2\right| k_2$$ max force generated by the absorber

$$For \;\; \mu = 0.2\left(=\frac{m_2}{m_1}\right)$$



System of Linear Differential Equation

$$\frac{X_1}{X_{ST}} = \frac{\left[1-\left(\frac{w}{w_a}\right)^2\right]}{\underbrace{\left[1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2\right]\left[1-\left(\frac{w}{w_a}\right)^2\right]-\mu\left(\frac{w_a}{w_n}\right)^2}_{=\Delta}}$$

Resonance $\Delta = 0$

$$1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2-\left(\frac{w}{w_a}\right)^2-\mu\left(\frac{w_a}{w_n}\right)^2\left(\frac{w}{w_a}\right)^2+\left(\frac{w}{w_n}\right)^2\left(\frac{w}{w_a}\right)^2-\mu\left(\frac{w_a}{w_n}\right)^2=0$$

$$1-\frac{w^2}{w_n^2}-\frac{w^2}{w_a^2}-\mu\frac{w_a^2}{w_n^2}\frac{w^2}{w_a^2}+\frac{w^4}{w_n^2 w_a^2}=0$$

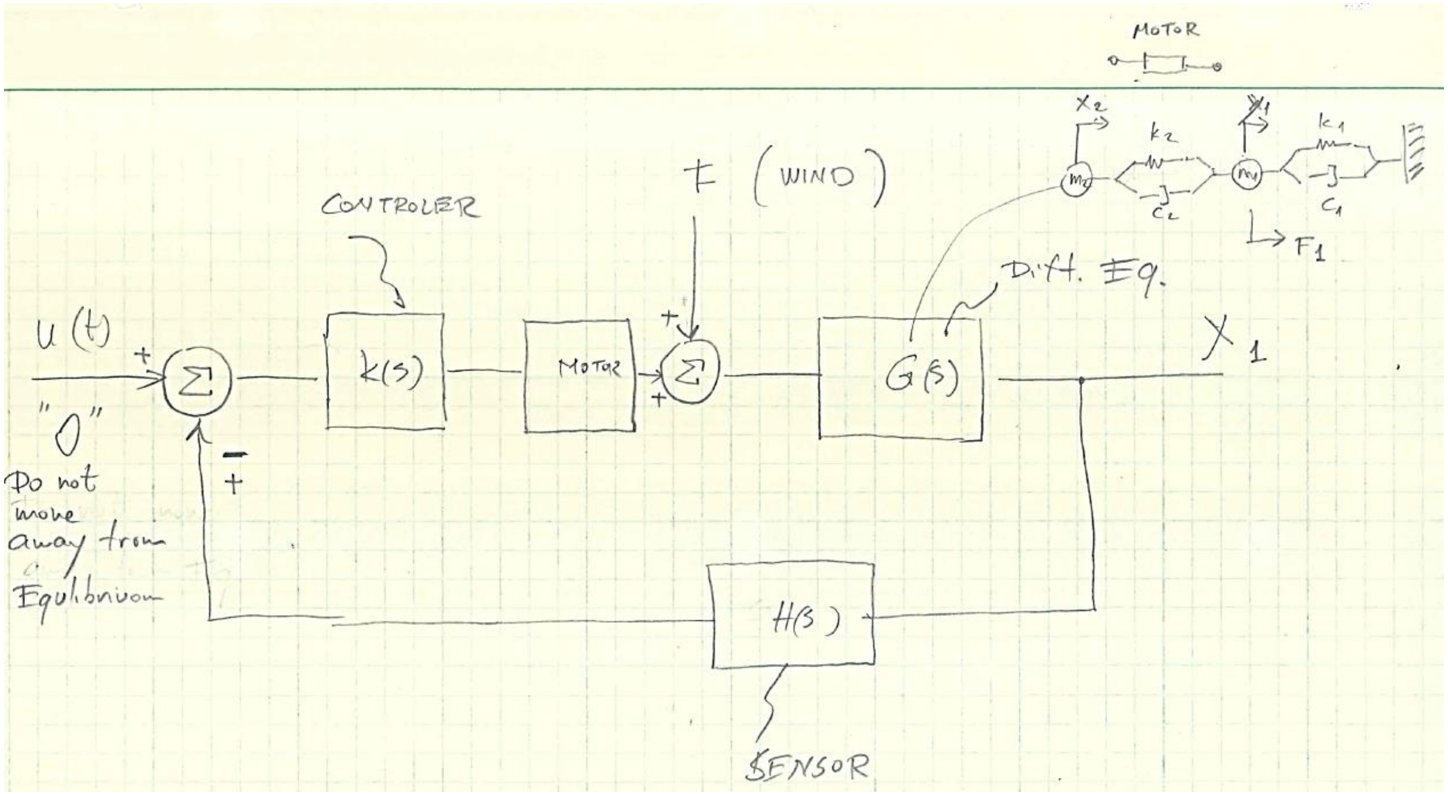
$$\frac{w_n^2 w_a^2 - w^2 w_a^2 - w^2 w_n^2 - \mu w^2 w_a^2 + w^4}{w_n^2 w_a^2}=0$$

$$w^4 + w^2\left(\underbrace{-w_a^2 - w_n^2 - \mu w_a^2}_{b}\right)+\underbrace{w_n^2 w_a^2}_{c}=0$$

$$w=\sqrt{\frac{-b\pm\sqrt{b^2-4c}}{2}}$$

$$=\sqrt{\frac{w_a^2+w_n^2-\mu w_a^2\pm\sqrt{\left(w_a(1+\mu)-w_n\right)^2-4w_n^4 w_a^2}}{2}}$$

System of Linear Differential Equation



$$\frac{Y(s)}{U(s)} = \frac{K(s)G(s)}{1 + K(s)G(s)H(s)}$$

Preliminary Theory – Linear System

- A system of simultaneous first order ordinary differential equations has the general form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

  where each x_k is a function of t. If each F_k is a linear function of x_1, x_2, …, x_n, then the system of equations is said to be **linear**, otherwise it is **nonlinear**.
- Systems of higher order differential equations can similarly be defined.

Preliminary Theory – Linear System

- Linear System (Explicit)

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2(t)$$

$$\dots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n(t)$$

Preliminary Theory – Linear System

- Linear System – Matrix Form

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ \dots \\ f_n \end{Bmatrix}$$

$x' = Ax + F$ (Nonhomogeneos)

$x' = Ax$ (Homogeneos)

Preliminary Theory – Linear System – Solution Vector

- A solution vector on an interval (I) of a linear system – Matrix Form

$$x = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{Bmatrix}$$

whose entries are differentiable functions satisfying the system

$$x' = Ax + F$$

on the interval

Preliminary Theory – Linear System – Solution Vector

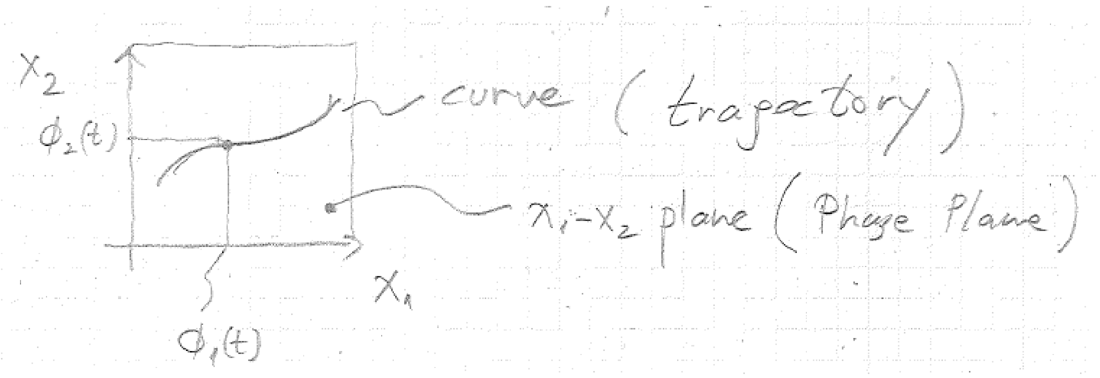
- Solution vector of

$$x' = Ax + F \longrightarrow \begin{cases} x_1 = \phi_1(t) \\ x_2 = \phi_2(t) \\ \quad\vdots \\ x_n = \phi_n(t) \end{cases}$$

A set of parametric equation of a space curve.

Special Case n=2

$$\begin{cases} x_1 = \phi_1(t) \\ x_2 = \phi_2(t) \end{cases}$$



Verification of the Solution

$$x' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} x$$

$$x_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-2t} = \begin{Bmatrix} e^{-2t} \\ -e^{-2t} \end{Bmatrix} \qquad x_2 = \begin{Bmatrix} 3 \\ 5 \end{Bmatrix} e^{6t} = \begin{Bmatrix} 3e^{6t} \\ 5e^{6t} \end{Bmatrix}$$

Verify the solution on the interval $(-\infty, \infty)$

$$x_1' = \begin{Bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{Bmatrix} \qquad x_2' = \begin{Bmatrix} 18e^{6t} \\ 30e^{6t} \end{Bmatrix}$$

$$Ax_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{Bmatrix} e^{-2t} \\ 2e^{-2t} \end{Bmatrix} = \begin{Bmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{Bmatrix} = \begin{Bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{Bmatrix} = x_1'$$

$$Ax_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{Bmatrix} 3e^{6t} \\ 5e^{6t} \end{Bmatrix} = \begin{Bmatrix} 3e^{6t} + 15e^{6t} \\ 15e^{6t} + 15e^{6t} \end{Bmatrix} = \begin{Bmatrix} 18e^{6t} \\ 30e^{6t} \end{Bmatrix} = x_2'$$

Initial Value Problem

- Let to denote a point on the interval I and

$$x(t_0) = \begin{Bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{Bmatrix} \quad and \quad x_0 = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{Bmatrix}$$

where $\gamma_i,\ i = 1, 2, \cdots n$ are given. Then the problem

$$\begin{aligned} &\text{Solve} && : x' = A(t)x + F(t) \\ &\text{Subject to} && : x(t_0) = x_0 \end{aligned}$$

is the initial problem on the interval

Theorem 7.1.1 – Existence of a Solution

- Suppose $F_1, \ldots, F_n$ and $\partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n$, are continuous in the region R of $t\, x_1\, x_2 \ldots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, …, $\alpha_n < x_n < \beta_n$, and let the point

$$\left(t_0, x_1^0, x_2^0, \ldots, x_n^0\right)$$

be contained in R. Then in some interval $(t_0 - h, t_0 + h)$ there exists a unique solution

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

that satisfies the IVP.

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Theorem 7.1.2 – Uniqueness of the Solution

- Suppose p_{11}, p_{12},…, p_{nn}, g_1,…, g_n are continuous on an interval I: $\alpha < t < \beta$ with t_0 in I, and let

$$x_1^0, x_2^0, \ldots, x_n^0$$

prescribe the initial conditions. Then there exists a unique solution

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

that satisfies the IVP, and exists throughout I.

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

Theorem 7.4.1 (Homogeneous System) – Supper Position

- **If** the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$,
- **Then** the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

- Note: By repeatedly applying the result of this theorem, it can be seen that every finite linear combination

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_k\mathbf{x}^{(k)}(t)$$

of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k)}$ is itself a solution to $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Theorem 7.4.2 (Homogeneous System) – General Solution

- **If** $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ for each point in I: $\alpha < t < \beta$,
- **Then** each solution $\mathbf{x} = \phi(t)$ can be expressed uniquely in the form

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

- **If** solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent for each point in I: $\alpha < t < \beta$,
- **Then** they are **fundamental solutions on *I***, and the **general solution** is given by

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \ldots + c_n\mathbf{x}^{(n)}(t)$$

Example – Supper Position Principle

System

$$x' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} x$$

$$x_1 = \begin{Bmatrix} \cos t \\ -1/2\cos t + 1/2\sin t \\ -\cos t - \sin t \end{Bmatrix} \qquad x_2 = \begin{Bmatrix} 0 \\ e^t \\ 0 \end{Bmatrix}$$

By the superposition the linear combination is also a solution of the system

$$x = c_1 x_1 + c_2 x_2 = c_1 \begin{Bmatrix} \cos t \\ -1/2\cos t + 1/2\sin t \\ -\cos t - \sin t \end{Bmatrix} + c_2 \begin{Bmatrix} 0 \\ e^t \\ 0 \end{Bmatrix}$$

Linear Dependency

Let $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$,…, $\mathbf{x}^{(n)}$ be a set of solution vectors of the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval I.

$$c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0$$

For k=2

The two solution vectors $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are linearly dependent if one is a constant multiple of the other.

$$c\ x_1 = x_2$$

Linear Independency Solution Criterion (Wronskian)

- Theorem – criterion for linearly independent solutions

Let

$$x_1 = \begin{Bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{Bmatrix},\ x_2 = \begin{Bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{Bmatrix},\ \cdots, x_n = \begin{Bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{Bmatrix}$$

solution of the homogeneous system **x' = Ax** on the interval I

- Then the set of solution vectors is linearly independent on I if and only if the Wronskian

$$W(x_1, x_2, \cdots x_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0$$

Example - Linear Independency Solution & General Solution (Homogeneous System)

$$x' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} x$$

$$x_1 = \begin{Bmatrix} e^{-2t} \\ -e^{-2t} \end{Bmatrix} \qquad x_2 = \begin{Bmatrix} 3e^{6t} \\ 5e^{6t} \end{Bmatrix}$$

$$W(x_1, x_2) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{2t} & 5e^{6t} \end{vmatrix} = 8e^{4t} \neq 0$$

$\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ are linearly independent on the interval $(-\infty, \infty)$ since neither vector is a constant multiple of the other in addition $W(x_1, x_2) \neq 0$

Example - General Solution

$$x' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} x$$

$$x_1 = \begin{Bmatrix} \cos t \\ -1/2\cos t + 1/2\sin t \\ -\cos t - \sin t \end{Bmatrix} \quad x_2 = \begin{Bmatrix} 0 \\ e^t \\ 0 \end{Bmatrix} \quad x_3 = \begin{Bmatrix} \sin t \\ -1/2\sin t - 1/2\cos t \\ -\sin t + \cos t \end{Bmatrix}$$

$$W(x_1, x_2, x_3) = \begin{vmatrix} \cos t & 0 & \sin t \\ -1/2\cos t + 1/2\sin t & e^t & -1/2\sin t - 1/2\cos t \\ -\cos t - \sin t & 0 & -\sin t + \cos t \end{vmatrix} = e^t \neq 0$$

for all real value of t, we conclude that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ form a fundamental set of solutions on $(-\infty, \infty)$. The general solution of the system on the interval is the linear combination.

Example - General Solution

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 + c_3 x_3 \\
&= c_1 \begin{Bmatrix} \cos t \\ -1/2\cos t + 1/2\sin t \\ -\cos t - \sin t \end{Bmatrix} + c_2 \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} e^t + c_3 \begin{Bmatrix} \sin t \\ -1/2\sin t - 1/2\cos t \\ -\sin t + \cos t \end{Bmatrix}
\end{aligned}
$$

General Solution – Nonhomogeneous System

The general solution of the homogeneous system **x'** = **Ax+F**

Complimentary Solution

$$\underline{x} = \overbrace{c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}^{x_c} + \underline{x_p}$$

General solution of the nonhomogeneous system **x'** = **Ax+F**

Particular solution of the nonhomogeneous system **x'** = **Ax+F**

Example – Nonhomogeneous System

$$x' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} x + \begin{Bmatrix} 12t - 11 \\ -3 \end{Bmatrix}$$

$$x = x_c + x_p = \underbrace{c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-2t} + c_2 \begin{Bmatrix} 3 \\ 5 \end{Bmatrix} e^{6t}}_{x_c} + \underbrace{\begin{Bmatrix} 3t - 4 \\ -5t + 6 \end{Bmatrix}}_{x_p}$$

Homogeneous Linear Systems

Homogeneous Linear Systems

- Diff. Eq. **x'** = **Ax**

  A = n x n matrix of constants

- General solution

$$x = \begin{Bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{Bmatrix} e^{\lambda t} = k e^{\lambda t}$$

$$x' = k\lambda e^{\lambda t}$$

$$x' = Ax \qquad \rightarrow \qquad k\lambda e^{\lambda t} = Ake^{\lambda t}$$

$$k\lambda = Ak$$

$$Ak - \lambda k = 0$$

since $k = IK$

Homogeneous Linear Systems

$$
\begin{aligned}
&(A-\lambda I)k = 0 \quad (*) \\
&(a_{11}-\lambda)k_1 + a_{12}k_2 + \cdots + a_{1n}k_n = 0 \\
&a_{21}k_1 + (a_{22}-\lambda)k_2 + \cdots + a_{2n}k_n = 0 \\
&\qquad\qquad\vdots \\
&a_{n1}k_1 + k_{n2}k_2 + \cdots + (a_{nn}-\lambda)k_n = 0
\end{aligned}
$$

- In order to find a nontrivial solution, we need to find a nontrivial vector k that satisfy (*). We must have

$$\det(A-\lambda I) = 0$$

- The polynomial equation in λ is called the characteristic eq. of the matrix **A**. It solutions are the eigenvalues of **A**.
- A solution k≠0 of (*) corresponding to an eigenvalue λ is called an eigenvector. The solution is then $x = ke^{\lambda t}$

Case 1: Distinct Real Eigenvalues

n x n matrix **A** → **District real eigenvalues** $\lambda_1, \lambda_2 \cdots, \lambda_n$

→ A set of linear independent eigenvectors $k_1, k_2 \cdots, k_n$ can always be found

→ The fundamental set of solutions

$$x_1 = k_1 e^{\lambda_1 t}, x_2 = k_2 e^{\lambda_2 t}, \cdots, x_n = k_n e^{\lambda_n t}$$

is a set of solutions of **x'** = **Ax** on the interval $(-\infty, \infty)$

Case 1: Distinct Real Eigenvalues

- Let $\lambda_1, \lambda_2 \cdots, \lambda_n$ be n distinct real eigenvalue of the coefficient matrix A of an homogeneous system **x**' = **Ax**

- Let $k_1, k_2 \cdots, k_n$ be the corresponding eigenvectors.

- The general solution of **x**' = **Ax** on the interval $(-\infty, \infty)$ is given by

$$x = c_1 k_1 e^{\lambda_1 t} + c_2 k_2 e^{\lambda_2 t} + \cdots + c_n k_n e^{\lambda_n t}$$

Example – Distinct Eigenvalues

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\ \dfrac{dy}{dt} = 2x + y \end{cases} \qquad X' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} X$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0 \qquad \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 4 \end{cases}$$

$$(A - \lambda I)k = 0$$

For $\lambda = -1$ $\begin{cases} 3k_1 + 3k_2 = 0 \\ 2k_1 + 2k_2 = 0 \end{cases}$

$k_1 = -k_2$ when $k_2 = -1$ the related eigenvector $K_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

Example – Distinct Eigenvalues

For $\lambda_2 = 4$ $\quad \begin{cases} -2k_1 + 3k_2 = 0 \\ 2k_1 - 3k_2 = 0 \end{cases}$

$k_1 = \frac{3}{2} k_2$ when $k_2 = 2$ the related eigenvector $K_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$

- Since the matrix of coefficients **A** is a 2x2 matrix and since we have found two linearly independent solutions

$$x_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-t} \qquad x_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} e^{4t}$$

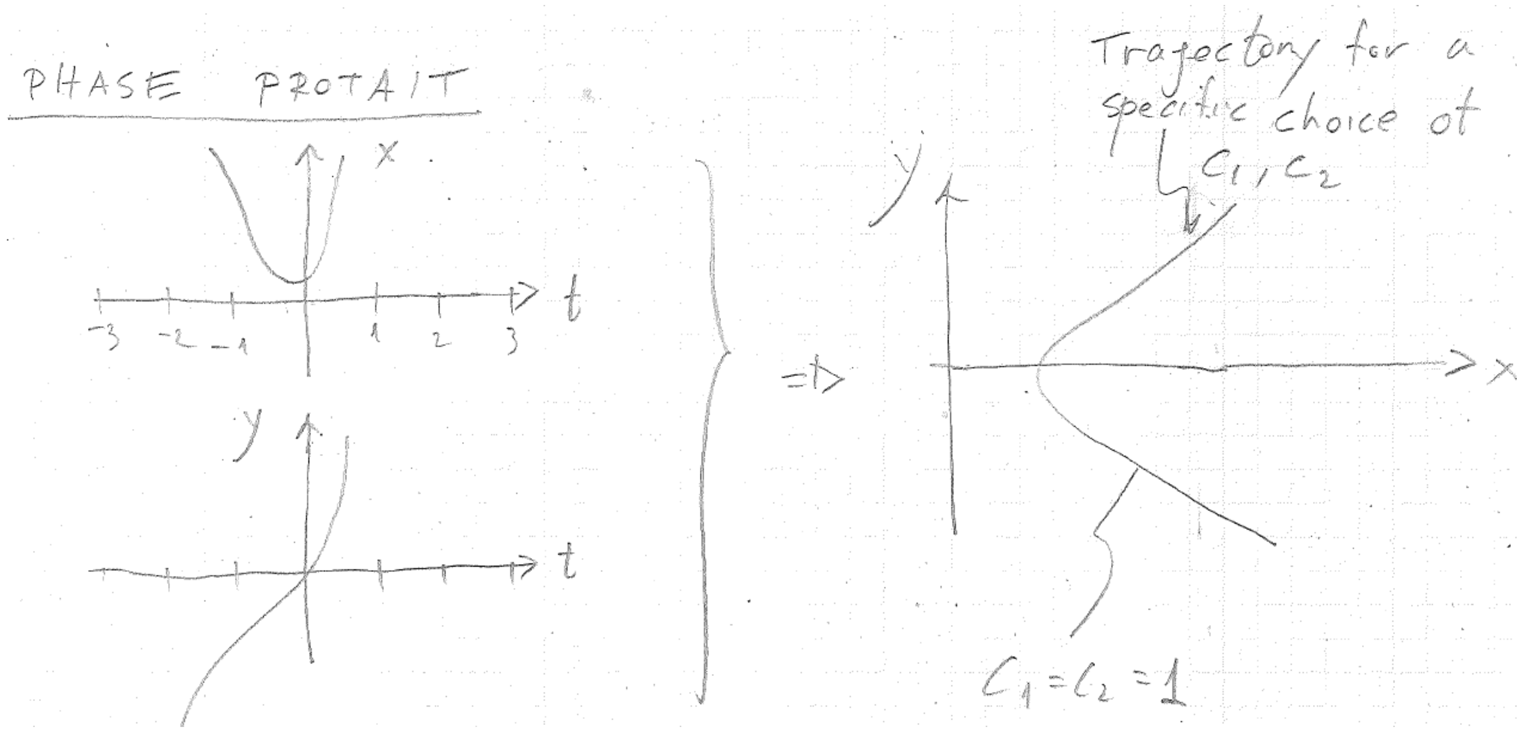
- The general solution of the system is

$$x = c_1 x_1 + c_2 x_2 = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} e^{4t}$$

Example – Distinct Eigenvalues

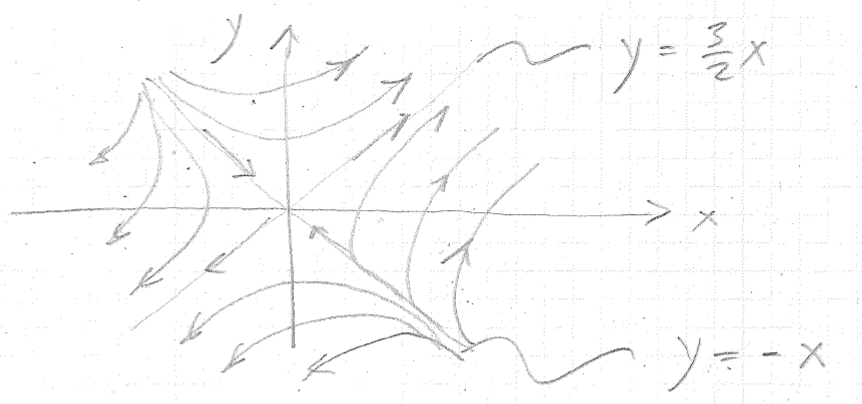
- The solution can be rewritten

$$x = c_1 e^{-t} + 3c_2 e^{4t}$$
$$y = -c_1 e^{-t} + 2c_2 e^{4t}$$



Case 1: Distinct Real Eigenvalues

$$X = \begin{Bmatrix} x \\ y \end{Bmatrix} = c_1 x_1 + c_2 x_2 = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} e^{4t}$$



- The origin is a constant solution $\begin{cases} x = 0 \\ y = 0 \end{cases}$ of every 2x2 homogeneous linear system $X' = AX$
- The arrowheads on each trajectory indicate the direction that a particle with coordinates x(t), y(t) on the trajectory at time t moves as time increases
- The eigenvector $K_2 = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$ lies along $y = \frac{3}{2}x$
- The eigenvector $K_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ lies along $y = -x$

Case 2: Repeated Eigen Value

- Suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value
- The first solution is defined similar to case 1 as

$$X_1 = Ke^{\lambda_1 t}$$

- The second solution can be found of the form

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$

$$K = \begin{Bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{Bmatrix} \quad \text{and} \quad P = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix}$$

Case 2: Repeated Eigen Value

- In general – if there is only one eigenvector corresponding to the eigenvalue λ_2 of multiplicity m then m linearly independent solutions of the form

$$
\begin{aligned}
X_1 &= K_{11}e^{\lambda_1 t} \\
X_2 &= K_{21}te^{\lambda_1 t} + K_{22}e^{\lambda_1 t} \\
X_3 &= K_{31}\frac{t^2}{2}e^{\lambda_1 t} + K_{32}te^{\lambda_1 t} + K_{33}e^{\lambda_1 t} \\
&\vdots \\
X_m &= K_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + K_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \cdots + K_{mm}e^{\lambda_1 t}
\end{aligned}
$$

Case 3: Complex Eigen Value

- <u>Second Solution</u>

  Suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value, a second solution can be found from

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$

$$K = \begin{Bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{Bmatrix} \qquad P = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{Bmatrix}$$

Substitute into the system $x' = Ax$

$$X_2' = Ke^{\lambda_1 t} + Kt\lambda_1 e^{\lambda_1 t} + P\lambda_1 e^{\lambda_1 t} = AKte^{\lambda_1 t} + APe^{\lambda_1 t}$$

$$\underbrace{(AK - \lambda_1 K)}_{=0} te^{\lambda_1 t} + \underbrace{(AP - P\lambda_1 - K)}_{=0} e^{\lambda_1 t} = 0$$

Case 2: Repeated Eigen Value

$$\begin{cases} (A-\lambda_1 I)K = 0 & (*) \\ (A-\lambda_1 I)P = K & (**) \end{cases}$$

- Equation (*) states that K must be an eigenvector at A associated with λ_1
- By solving (*) we find one solution $X_1 = Ke^{\lambda_1 t}$
- To find the second solution X_2 we need only solve the additional system (**) for the vector P

Case 2: Repeated Eigen Value

- Example

$$X' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} X$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (3-\lambda)(-9-\lambda) + 36 = -27 - 3\lambda + 9\lambda + \lambda^2 + 36$$

$$= \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$$

$\lambda_1 = -3, \lambda_2 = -3$ Root of multiplicity two

$$\begin{cases} 0k_1 - 18k_2 = 0 \\ 2k_1 - 6k_2 = 0 \end{cases} \Rightarrow k_1 = 3k_2$$

select $k_2 = 1 \Rightarrow k_1 = 3$

$$K_1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} \Rightarrow X_1 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} e^{-3t}$$

Case 2: Repeated Eigen Value

- Assume $P = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$

  Solve $(A + 3I)P = K$

$$\begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 2 & -6 \end{bmatrix}$$

$$(\mathrm{R}_1 = 3\mathrm{R}_2) \qquad \begin{cases} 6p_1 - 18p_2 = 3 \\ 2p_1 - 6p_2 = 1 \end{cases} \Rightarrow p_1 = \frac{1 + 6p_2}{2}$$

$$p_2 = 0 \rightarrow p_1 = 1/2$$

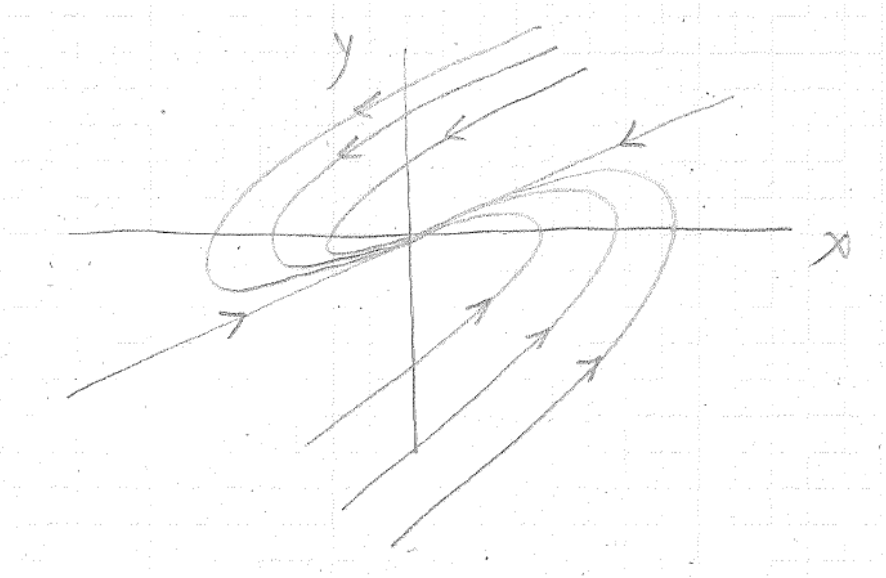
$$X_2 = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} te^{-3t} + \begin{Bmatrix} 1/2 \\ 0 \end{Bmatrix} e^{-3t}$$

- The general solution

$$X = c_1 X_1 + c_2 X_2 = c_1 \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} e^{-3t} + c_2 \left[\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} te^{-3t} + \begin{Bmatrix} 1/2 \\ 0 \end{Bmatrix} e^{-3t} \right]$$

Case 2: Repeated Eigen Value

$$x = c_1 3e^{-3t} + c_2\left[3te^{-3t} + \frac{1}{2}e^{-3t}\right]$$
$$y = c_1 3e^{-3t} + c_2\left[te^{-3t}\right]$$



For various values of c_1, c_2

Case 2: Repeated Eigen Value

- Eigenvalue of Multiplicity Three

  When the coefficient matrix A has only eigenvector associated with an eigenvalue λ_1 of multiplicity three

$$x_3 = K\frac{t^2}{2}e^{\lambda_1 t} + Pte^{\lambda_1 t} + Qe^{\lambda_1 t}$$

  substituting into the system $x' = Ax$, the vector K, P, Q

$$\begin{cases} (A - \lambda_1 I)K = 0 \\ (A - \lambda_1 I)P = K \\ (A - \lambda_1 I)Q = P \end{cases}$$

Case 3: Complex Eigen Value

- Theorem – Solutions corresponding to a complex eigenvalue

  Let A be the coefficient matrix having real entries of the homogeneous system

  $$x' = Ax$$

  Let K_1 be an eigenvector corresponding to the complex eigenvalue $\lambda_1=\alpha+i\beta$ and α, β are real

  Then $x_1 = K_1 e^{\lambda_1 t}, x_2 = \overline{K}_1 e^{\overline{\lambda}_1 t}$

Case 3: Complex Eigen Value

- Example

$$\begin{cases} \dfrac{dx}{dt} = 6x - y \\ \dfrac{dy}{dt} = 5x - 4y \end{cases}$$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0$$

$$\lambda_1 = 5 + 2i$$
$$\lambda_2 = 5 - 2i$$

- For λ_1

$$\begin{cases} 6 - (5+2i)k_1 - k_2 = 0 \\ 5k_1 + 4 - (5+2i)k_2 = 0 \end{cases}$$

$$\begin{cases} (1-2i)k_1 - k_2 = 0 \\ 5k_1 - (1+2i)k_2 = 0 \end{cases} \qquad (*)$$

Case 3: Complex Eigen Value

- From (*) $k_2 = (1-2i)k_1$
- For $k_1 = 1 \rightarrow k_2 = 1-2i$

$$K_1 = \begin{Bmatrix} 1 \\ 1-2i \end{Bmatrix} \rightarrow X_1 = \begin{Bmatrix} 1 \\ 1-2i \end{Bmatrix} e^{(5+2i)t}$$

- Note: when the characteristic equation has real coefficients, complex eigenvalues always appear in conjugate pair
- In like manner for $\lambda_2 = 5-2i$

$$K_2 = \begin{Bmatrix} 1 \\ 1+2i \end{Bmatrix} \rightarrow X_2 = \begin{Bmatrix} 1 \\ 1+2i \end{Bmatrix} e^{(5-2i)t}$$

$$X = c_1 \begin{Bmatrix} 1 \\ 1-2i \end{Bmatrix} e^{(5+2i)t} + c_2 \begin{Bmatrix} 1 \\ 1+2i \end{Bmatrix} e^{(5-2i)t}$$

Note: K_2 corresponding to λ_2 are the conjugates of K_1 corresponding to λ_1

$$\lambda_2 = \overline{\lambda_1} \qquad K_2 = \overline{K_1}$$

Case 3: Complex Eigen Value

- The two solution can be generalized for $\lambda_1 = \alpha + i\beta$ using Euler's Equation

$$K_1 e^{\lambda_1 t} = K_1 e^{\alpha t + i\beta t} = K_1 e^{\alpha t} e^{i\beta t} = K_1 e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)$$

$$\overline{K}_1 e^{\overline{\lambda}_1 t} = \overline{K}_1 e^{\alpha t - i\beta t} = \overline{K}_1 e^{\alpha t} e^{-i\beta t} = \overline{K}_1 e^{\alpha t}\left(\cos\beta t - i\sin\beta t\right)$$

- By super position principle the following vectors are also solutions

$$X_1 = \frac{1}{2}\left(K_1 e^{\lambda_1 t} + \overline{K}_1 e^{\lambda_1 t}\right) = \underbrace{\frac{1}{2}\left(K_1 + \overline{K}_1\right)}_{B_1} e^{\alpha t}\cos\beta t - \frac{i}{2}\left(-K_1 + \overline{K}_1\right) e^{\alpha t}\sin\beta t$$

$$X_2 = \frac{i}{2}\left(-K_1 e^{\lambda_1 t} + \overline{K}_1 e^{\lambda_1 t}\right) = \underbrace{\frac{i}{2}\left(-K_1 + \overline{K}_1\right)}_{B_2} e^{\alpha t}\cos\beta t + \frac{1}{2}\left(K_1 + \overline{K}_1\right) e^{\alpha t}\sin\beta t$$

Note

$$\left.\begin{aligned} &\frac{1}{2}\left(Z + \overline{Z}\right) = \frac{1}{2}(2 + 2i + 2 - 2i) = 2 \\ &\frac{1}{2}i\left(-Z + \overline{Z}\right) = \frac{1}{2}i(-2 - 2i + 2 - 2i) = \frac{-4i^2}{2} = 2 \end{aligned}\right\} \text{-real numbers}$$

Case 3: Complex Eigen Value

- Define $$B_1 = \frac{1}{2}\left(K_1 + \overline{K_1}\right) \qquad B_2 = \frac{i}{2}\left(-K_1 + \overline{K_1}\right)$$

- Theorem – real solutions corresponding to a complex eigenvalue

$$X_1 = \left[B_1 \cos \beta t - B_2 \sin \beta t\right]e^{\alpha t}$$
$$X_2 = \left[B_2 \cos \beta t + B_1 \sin \beta t\right]e^{\alpha t}$$

$$B_1 = \mathrm{Re}\left(K_1\right) \qquad B_2 = \mathrm{Im}\left(K_1\right)$$

Case 3: Complex Eigen Value

- Back to the example

$$K_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$B_1 = \mathrm{Re}(K_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$B_2 = \mathrm{Im}(K_1) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$X_1 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin 2t\right]e^{5t}$$

$$X_2 = \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t\right]e^{5t}$$

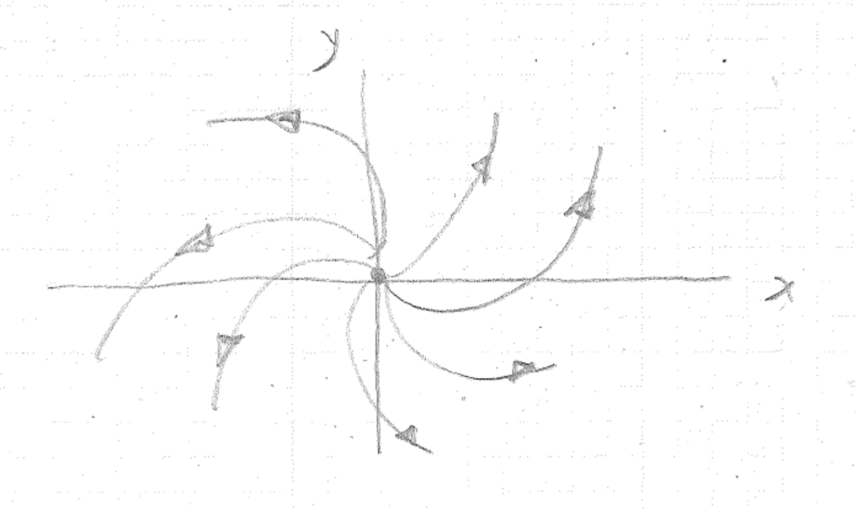
Case 3: Complex Eigen Value

$$\underset{\begin{pmatrix} x \\ y \end{pmatrix}}{\underset{\downarrow}{X}} = \tilde{c}_1 X_1 + \tilde{c}_2 X_2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tilde{c}_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t \right] e^{5t} + \tilde{c}_2 \left[\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t \right] e^{5t}$$

$$x = \tilde{c}_1 e^{5t} \cos 2t + \tilde{c}_2 e^{5t} \sin 2t$$

$$y = \left(\tilde{c}_1 - 2c_1\right) e^{5t} \cos 2t + \left(2c_1 + c_2\right) e^{5t} \sin 2t$$



Homogeneous Linear Systems
Summary

Homogenous Linear System – Summary
Case 1: Distinct Real Eigenvalues

- Homogeneous Linear System $x' = Ax$

$$\det(A - \lambda I) = 0 \to \lambda_1, \lambda_2, \ldots, \lambda_n$$

solve $(A - \lambda I)K = 0$ for each λ_i, i=1,…,n

solution $X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \cdots + c_n K_n e^{\lambda_n t}$

Homogenous Linear System – Summary
Case 2: Repeated Eigen Value

- Homogeneous Linear System $x' = Ax$

$$\det(A-\lambda I)=0 \rightarrow \underbrace{\lambda_1,\lambda_1,\lambda_1}_{\text{m repeated eigenvalues}},\lambda_2,\lambda_3,\ldots$$

$$\text{for three } \lambda_1,\lambda_1,\lambda_1 \begin{cases} \text{for two } \lambda_1,\lambda_1 \begin{cases} \det(A-\lambda_1 I)K = 0 \\ \det(A-\lambda_1 I)P = K \end{cases} \\ \det(A-\lambda_1 I)Q = P \end{cases}$$

$$X = c_1\{K\}e^{\lambda_1 t} + c_2\left[\{K\}te^{\lambda_1 t} + \{P\}e^{\lambda_1 t}\right] + c_3\left[\{K\}\frac{t^2}{2}e^{\lambda_1 t} + \{P\}e^{\lambda_1 t} + \{Q\}e^{\lambda_1 t}\right]$$

Homogenous Linear System – Summary
Case 3: Complex Eigen Value

- Homogeneous Linear System $x' = Ax$

$$\det(A-\lambda I)=0 \rightarrow \begin{cases} \lambda_1 = \alpha + \beta i \\ \lambda_1 = \alpha - \beta i \end{cases}$$

$$(A-\lambda I)K = 0 \rightarrow \begin{cases} K_1 = \begin{Bmatrix} \vdots \end{Bmatrix} \\ K_2 = \begin{Bmatrix} \vdots \end{Bmatrix} \end{cases}$$

- Complex $X = c_1 K_1 e^{\lambda_1 t} + c_2 \overline{K}_1 e^{\overline{\lambda}_1 t}$

- Non-complex $\begin{cases} x_1 = \left[B_1 \cos\beta t - B_2 \sin\beta t\right]e^{\alpha t} \\ x_1 = \left[B_2 \cos\beta t + B_1 \sin\beta t\right]e^{\alpha t} \end{cases}$ $\begin{cases} B_1 = \mathrm{Re}(K_1) \\ B_2 = \mathrm{Im}(K_1) \end{cases}$