Systems of First Order Linear Equations

Review of Matrices

Systems of First Order Linear Equations

- A system of simultaneous first order ordinary differential equations has the general form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

  where each x_k is a function of t. If each F_k is a linear function of x_1, x_2, …, x_n, then the system of equations is said to be **linear**, otherwise it is **nonlinear**.
- Systems of higher order differential equations can similarly be defined.

*N*th Order ODEs and Linear 1st Order Systems

- The method illustrated in the previous example can be used to transform an arbitrary *n*th order equation

  $$y^{(n)} = F\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

  into a system of *n* first order equations, first by defining

  $$x_1 = y,\ x_2 = y',\ x_3 = y'',\ \ldots, x_n = y^{(n-1)}$$

  Then

  $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Solutions of First Order Systems

- A system of simultaneous first order ordinary differential equations has the general form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n). \end{aligned}$$

  It has a **solution** on *I*: $\alpha < t < \beta$ if there exists n functions

$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

  that are differentiable on *I* and satisfy the system of equations at all points t in *I*.

- Initial conditions may also be prescribed to give an IVP:

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0$$

Theorem 7.1.1

- Suppose $F_1,\ldots, F_n$ and $\partial F_1/\partial x_1,\ldots, \partial F_1/\partial x_n,\ldots, \partial F_n/\partial x_1,\ldots, \partial F_n/\partial x_n$, are continuous in the region R of $t\, x_1\, x_2\ldots x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, …, $\alpha_n < x_n < \beta_n$, and let the point
$$\left(t_0, x_1^0, x_2^0, \ldots, x_n^0\right)$$
be contained in R. Then in some interval $(t_0 - h, t_0 + h)$ there exists a unique solution
$$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$
that satisfies the IVP.
$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots x_n) \\ x_2' &= F_2(t, x_1, x_2, \ldots x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots x_n) \end{aligned}$$

Linear Systems

- If each F_k is a linear function of $x_1, x_2, \ldots, x_n$, then the system of equations has the general form

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

- If each of the $g_k(t)$ is zero on I, then the system is **homogeneous**, otherwise it is **nonhomogeneous**.

Theorem 7.1.2

- Suppose $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an interval I: $\alpha < t < \beta$ with t_0 in I, and let

  $$x_1^0, x_2^0, \ldots, x_n^0$$

  prescribe the initial conditions. Then there exists a unique solution

  $$x_1 = \phi_1(t),\ x_2 = \phi_2(t), \ldots, x_n = \phi_n(t)$$

  that satisfies the IVP, and exists throughout I.

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

Review of Matrices

Matrix

$$
\underset{\sim}{A} = \{ a_{\underset{\text{row}}{i}\underset{\text{col}}{j}} \} =
\begin{array}{rc}
 & \begin{array}{cccccc} \phantom{a_{11}} & \phantom{a_{12}} & & \downarrow \text{j-th col}^{\underline{m}} & & \phantom{a_{1m}} \end{array} \\
\begin{array}{r} \\ \\ \\ \text{i-th row} \rightarrow \\ \\ \\ \end{array} &
\left(\begin{array}{cccccc}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\
\vdots & \vdots & & \vdots & & \vdots \\
\cdots & \cdots & \cdots & \boxed{a_{ij}} & & \vdots \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n1} & a_{n2} & & a_{nj} & & a_{nm}
\end{array} \right)
\end{array}
= (\underset{\substack{\uparrow \\ \text{\# Rows}}}{n} \times \underset{\substack{\uparrow \\ \text{\# col}^{\underline{m}}}}{m})
$$

Vector

$$
\underset{\sim}{a} = \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \text{single col}^{\underline{m}} \text{ matrix}
$$

Review of Matrices

- For theoretical and computational reasons, we review results of matrix theory in this section and the next.
- A **matrix A** is an *m* x *n* rectangular array of elements, arranged in *m* rows and *n* columns, denoted

$$\mathbf{A} = \left(a_{ij}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- Some examples of 2 x 2 matrices are given below:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3-2i \\ 4+5i & 6-7i \end{pmatrix}$$

Transpose

- The **transpose** of $\mathbf{A} = (a_{ij})$ is $\mathbf{A}^T = (a_{ji})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{B} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2\times 3} \Rightarrow \mathbf{B}^T = \underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}_{3\times 2}$$

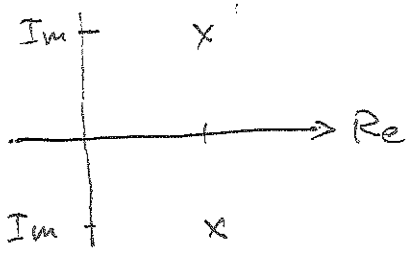
Conjugate

- The **conjugate** of $\mathbf{A} = (a_{ij})$ is $\overline{\mathbf{A}} = (\overline{a_{ij}})$.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \overline{\mathbf{A}} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{m1} & \overline{a}_{m2} & \cdots & \overline{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A} = \begin{pmatrix} \textcircled{1} & \boxed{2+3i} \\ \boxed{3-4i} & \textcircled{4} \end{pmatrix} \Rightarrow \overline{\mathbf{A}} = \begin{pmatrix} \textcircled{1} & \boxed{2-3i} \\ \boxed{3+4i} & \textcircled{4} \end{pmatrix}$$



Adjoint

conjugate Transpose

- The **adjoint** of **A** is $\overline{\mathbf{A}}^T$, and is denoted by $\mathbf{A}^*$

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}^*=\begin{pmatrix} \overline{a}_{11} & \overline{a}_{21} & \cdots & \overline{a}_{m1} \\ \overline{a}_{12} & \overline{a}_{22} & \cdots & \overline{a}_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{1n} & \overline{a}_{2n} & \cdots & \overline{a}_{mn} \end{pmatrix}$$

- For example,

$$\mathbf{A}=\begin{pmatrix} 1 & 2+3i \\ 3-4i & 4 \end{pmatrix} \Rightarrow \mathbf{A}^*=\begin{pmatrix} 1 & 3+4i \\ 2-3i & 4 \end{pmatrix}$$

Square Matrices

- A **square matrix A** has the same number of rows and columns. That is, **A** is *n* x *n*. In this case, **A** is said to have order *n*.

$$\mathbf{A} = \left. \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}^{n} \right\} n$$

- For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Vectors

- A **column vector x** is an n x 1 matrix. For example,

$$\mathbf{x} = \left.\overbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}^{1}\right\}3 \qquad\qquad 3\times 1$$

- A **row vector x** is a 1 x n matrix. For example,

$$\mathbf{y} = {}^{1}\left\{\overbrace{\begin{pmatrix}1 & 2 & 3\end{pmatrix}}^{3}\right. \qquad\qquad 1\times 3$$

- Note here that $\mathbf{y} = \mathbf{x}^T$, and that in general, if **x** is a column vector **x**, then $\mathbf{x}^T$ is a row vector.

The Zero Matrix

- The **zero matrix** is defined to be $\mathbf{0} = (0)$, whose dimensions depend on the context. For example,

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \ldots$$

Matrix Equality

- Two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are **equal** if $a_{ij} = b_{ij}$ for all i and j. For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{A} = \mathbf{B}$$

Matrix – Scalar Multiplication

- The product of a matrix **A** = (a_{ij}) and a constant k is defined to be k**A** = (ka_{ij}). For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow -5\mathbf{A} = \begin{pmatrix} -5 & -10 & -15 \\ -20 & -25 & -30 \end{pmatrix}$$

Matrix Addition and Subtraction

must have the same dimentions

- The **sum** of two $\overbrace{m \text{ x } n}$ matrices **A** = (a_{ij}) and **B** = (b_{ij}) is defined to be **A** + **B** = (a_{ij} + b_{ij}). For example,

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow \mathbf{A}+\mathbf{B}=\begin{pmatrix}6 & 8\\ 10 & 12\end{pmatrix}$$

must have the same dimention

- The **difference** of two $\overbrace{m \text{ x } n}$ matrices **A** = (a_{ij}) and **B** = (b_{ij}) is defined to be **A** - **B** = (a_{ij} - b_{ij}). For example,

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix},\ \mathbf{B}=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\ \Rightarrow \mathbf{A}-\mathbf{B}=\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}$$

Matrix Multiplication

- The **product** of an *m* x *n* matrix **A** = (a_{ij}) and an *n* x *r* matrix **B** = (b_{ij}) is defined to be the matrix **C** = (c_{ij}), where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Examples (note **AB** does not necessarily equal **BA**):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow \mathbf{AB} = \begin{pmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

$$\Rightarrow \mathbf{BA} = \begin{pmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \Rightarrow \mathbf{CD} = \begin{pmatrix} 3+2+0 & 0+4-3 \\ 12+5+0 & 0+10-6 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 17 & 4 \end{pmatrix}$$

Example 1: Matrix Multiplication

- To illustrate matrix multiplication and show that it is not commutative, consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

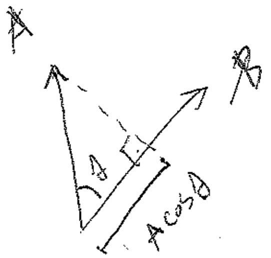
- From the definition of matrix multiplication we have:

$$\mathbf{AB} = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+1 \\ 2-2 & -2+1 & -1 \\ 4+1+2 & 2-1-1 & -2+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 2-2 & -4+2-1 & 2-1-1 \\ 1 & -2-2 & 1+1 \\ 2+2 & -4-2+1 & 2+1+1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix} \neq \mathbf{AB}$$

$A \cdot B = \|A\| \cdot \|B\| \cos\theta$

Vector Multiplication



- The **dot product** of two n x 1 vectors **x** & **y** is defined as

$$\mathbf{x}^T\mathbf{y} = \sum_{k=1}^{n} x_i y_j \longrightarrow \text{Scalar}$$

- The **inner product** of two n x 1 vectors **x** & **y** is defined as

$$(\mathbf{x},\mathbf{y}) = \mathbf{x}^T\overline{\mathbf{y}} = \sum_{k=1}^{n} x_i \overline{y}_j \longrightarrow \text{Scalar}$$

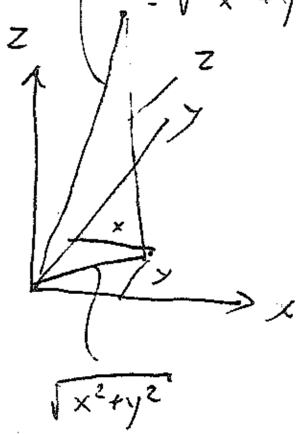
- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 \\ 2-3i \\ 5+5i \end{pmatrix} \Rightarrow \mathbf{x}^T\mathbf{y} = (1)(-1) + (2)(2-3i) + (3i)(5+5i) = -12+9i$$

$$\Rightarrow (\mathbf{x},\mathbf{y}) = \mathbf{x}^T\overline{\mathbf{y}} = (1)(-1) + (2)(2+3i) + (3i)(5-5i) = 18+21i$$

$$\|A\| = \sqrt{\left(\sqrt{x^2+y^2}\right)^2 + z^2}$$

$$= \sqrt{x^2+y^2+z^2}$$



Vector Length

- The **length** of an *n* x 1 vector **x** is defined as

$$\|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2} = \left[\sum_{k=1}^{n} \overset{\text{inner product}}{\boxed{x_k \bar{x}_k}}\right]^{1/2} = \left[\sum_{k=1}^{n} |x_k|^2\right]^{1/2}$$

- Note here that we have used the fact that if $x = a + bi$, then

$$x \cdot \bar{x} = (a+bi)(a-bi) = a^2 + b^2 = |x|^2$$

- Example:

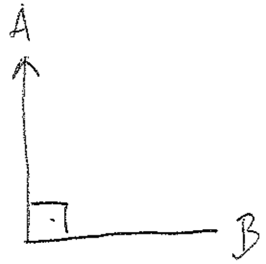
$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3+4i \end{pmatrix} \Rightarrow \|\mathbf{x}\| = (\mathbf{x},\mathbf{x})^{1/2} = \sqrt{(1)(1)+(2)(2)+(3+4i)(3-4i)}$$

$$= \sqrt{1+4+(9+16)} = \sqrt{30}$$

Orthogonality

- Two *n* x 1 vectors **x** & **y** are **orthogonal** if $\overbrace{(\mathbf{x},\mathbf{y})}^{\text{Inner product}} = 0$.
- Example:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} \Rightarrow (\mathbf{x},\mathbf{y}) = (1)(11) + (2)(-4) + (3)(-1) = 0$$



$A \perp B = \|A\| \|B\| \cos 90 = 0$

Identity Matrix

- The multiplicative **identity matrix I** is an $\boxed{n \text{ x } n}$ matrix given by

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

ORDER DOES NOT IMPORTANT

- For any square matrix **A**, it follows that $\underline{\mathbf{AI} = \mathbf{IA} = \mathbf{A}}$.
- The dimensions of **I** depend on the context. For example,

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{IB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Inverse Matrix

- A square matrix **A** is **nonsingular**, or **invertible**, if there exists a matrix **B** such that that **AB** = **BA** = **I**. Otherwise **A** is **singular**.
- The matrix **B**, if it exists, is unique and is denoted by $\mathbf{A}^{-1}$ and is called the **inverse** of **A**.
- It turns out that $\mathbf{A}^{-1}$ exists iff det**A** $\neq 0$, and $\mathbf{A}^{-1}$ can be found using **row reduction** (also called Gaussian elimination) on the augmented matrix (**A**|**I**), see example on next slide.
- The three elementary row operations:
  - Interchange two rows.
  - Multiply a row by a nonzero scalar.
  - Add a multiple of one row to another row.

Example 2: Finding the Inverse of a Matrix (1 of 2)

- Use row reduction to find the inverse of the matrix **A** below, if it exists.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

- Solution: If possible, use elementary row operations to reduce **(A|I)**,

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right),$$

such that the left side is the identity matrix, for then the right side will be $\mathbf{A}^{-1}$. (See next slide.)

Example 2: Finding the Inverse of a Matrix (2 of 2)

$r_2 - 3\cdot r_1$

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ \circled{0} & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

$\frac{1}{2} r_2$

$$\rightarrow \left(\begin{array}{cccccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & \circled{1} & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right) \xrightarrow{r_1 + r_2} \left(\begin{array}{cccccc} 1 & \circled{0} & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & \circled{0} & -5 & 4 & -2 & 1 \end{array}\right)$$

$-4r_2 + r_3$

$\left(-\frac{1}{5}\right) r_3$

$$\rightarrow \left(\begin{array}{cccccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccccc} 1 & 0 & \circled{0} & 7/10 & -1/10 & 3/10 \\ 0 & 1 & \circled{0} & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right)$$

$r_3 \cdot \left(-\frac{3}{2}\right)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$$

$r_3\left(-\frac{5}{2}\right)$

- Thus $\mathbf{A}^{-1} = \left(\begin{array}{ccc} 7/10 & -1/10 & 3/10 \\ 1/2 & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{array}\right)$

Matrix Functions

- The elements of a matrix can be functions of a real variable. In this case, we write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

- Such a matrix is continuous at a point, or on an interval (a, b), if each element is continuous there. Similarly with differentiation and integration:

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt} \right), \quad \int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt \right)$$

Example & Differentiation Rules

- Example:

$$\mathbf{A}(t) = \begin{pmatrix} 3t^2 & \sin t \\ \cos t & 4 \end{pmatrix} \Rightarrow \frac{d\mathbf{A}}{dt} = \begin{pmatrix} 6t & \cos t \\ -\sin t & 0 \end{pmatrix},$$

$$\Rightarrow \int_0^{\pi} \mathbf{A}(t)dt = \begin{pmatrix} \pi^3 & 0 \\ -1 & 4\pi \end{pmatrix}$$

- Many of the rules from calculus apply in this setting. For example:

$$\frac{d(\mathbf{CA})}{dt} = \mathbf{C}\frac{d\mathbf{A}}{dt}, \quad \text{where } \mathbf{C} \text{ is a constant matrix}$$

$$\frac{d(\mathbf{A}+\mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

$$\frac{d(\mathbf{AB})}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)\mathbf{B} + \mathbf{A}\left(\frac{d\mathbf{B}}{dt}\right)$$

Systems of Linear Equations, Linear Independence, Eigenvalues

- A system of n linear equations in n variables,

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n ,
\end{aligned}
$$

  can be expressed as a matrix equation **Ax** = **b**:

$$
\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}
$$

- If **b** = **0**, then system is **homogeneous**; otherwise it is **nonhomogeneous**.

$$A x = b$$

	NON SINGULAR CASE	SINGULAR CASE
SOLUTION $Ax = b$	UNIQUE	NON - UNIQUE (MORE THEN ONE SOLUTIONS)
	$Ax = b$ $\underbrace{A^{-1} A}_{I} \, x = A^{-1} b$ $x = A^{-1} b$	$Ax = b$ $X = X^{(0)} - \xi$ $X^{(0)}$ - particular solution of $Ax = b$ ξ - any solution of $Ax = 0$
A^{-1}	EXIST	DOESN'T EXIST
SOLUTION $Ax = 0$	TRIVIAL SOLUTION $X = 0$	INFINITELY MANY NON TRIVIAL SOLUTION $X = 0$
det A $\lvert A \rvert$	$\lvert A \rvert \neq 0$	$\lvert A \rvert = 0$
A (columns) or (Row)	Independent	dependent

Nonsingular Case

- If the coefficient matrix **A** is nonsingular, then it is invertible and we can solve **Ax** = **b** as follows:

$$\mathbf{Ax}=\mathbf{b} \;\Rightarrow \underbrace{\boxed{\mathbf{A}^{-1}}\mathbf{A}}_{I}\mathbf{x}=\boxed{\mathbf{A}^{-1}}\mathbf{b} \;\Rightarrow \mathbf{Ix}=\mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$$

- This solution is therefore unique. Also, if **b** = **0**, it follows that the unique solution to **Ax** = **0** is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$.

- Thus if **A** is nonsingular, then the only solution to **Ax** = **0** is the trivial solution **x** = **0**.

Example 1: Nonsingular Case (1 of 3)

- From a previous example, we know that the matrix **A** below is nonsingular with inverse as given.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix}$$

- Using the definition of matrix multiplication, it follows that the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 1: Nonsingular Case (2 of 3)

- Now let's solve the nonhomogeneous linear system **Ax** = **b** below using $\mathbf{A}^{-1}$:

$$\cancel{\begin{aligned} 0x_1 + x_2 + 2x_3 &= 2 \\ 1x_1 + 0x_2 + 3x_3 &= -2 \\ 4x_1 - 3x_2 + 8x_3 &= 0 \end{aligned}}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - \cancel{3}x_3 &= 4 \end{aligned}$$

- This system of equations can be written as **Ax** = **b**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

- Then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Example 1: Nonsingular Case (3 of 3)

- Alternatively, we could solve the nonhomogeneous linear system **Ax** = **b** below using row reduction.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned}$$

- To do so, form the augmented matrix (**A**|**b**) and reduce, using elementary row operations.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ x_2 - x_3 &= -2 \\ x_3 &= 1 \end{aligned} \to \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Singular Case

- If the coefficient matrix $\mathbf{A}$ is singular, then $\mathbf{A}^{-1}$ does not exist, and either a solution to $\mathbf{Ax} = \mathbf{b}$ does not exist, or there is more than one solution (not unique).
- Further, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has more than one solution. That is, in addition to the trivial solution $\mathbf{x} = \mathbf{0}$, there are infinitely many nontrivial solutions.
- The nonhomogeneous case $\mathbf{Ax} = \mathbf{b}$ has no solution unless $(\mathbf{b}, \mathbf{y}) = 0$, for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^{*}\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^{*}$ is the adjoint of $\mathbf{A}$.
- In this case, $\mathbf{Ax} = \mathbf{b}$ has solutions (infinitely many), each of the form $\mathbf{x} = \mathbf{x}^{(0)} + \xi$, where $\mathbf{x}^{(0)}$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$, and ξ is any solution of $\mathbf{Ax} = \mathbf{0}$.

Example 2: Singular Case (1 of 2)

- Solve the nonhomogeneous linear system $\mathbf{Ax} = \mathbf{b}$ below using row reduction. Observe that the coefficients are nearly the same as in the previous example

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ -x_1 + x_2 - 2x_3 &= b_2 \\ 2x_1 - x_2 + 3x_3 &= b_3 \end{aligned}$$

- We will form the augmented matrix $(\mathbf{A}|\mathbf{b})$ and use some of the steps in Example 1 to transform the matrix more quickly

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{pmatrix}$$

$$\to \begin{aligned} x_1 - 2x_2 + 3\,x_3 &= b_1 \\ x_2 - x_3 &= -b_1 - b_2 \\ 0 &= b_1 + 3b_2 + b_3 \end{aligned} \quad \to b_1 + 3b_2 + b_3 = 0$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ -x_1 + x_2 - 2x_3 &= b_2 \\ 2x_1 - x_2 + 3x_3 &= b_3 \end{aligned}$$

Example 2: Singular Case (2 of 2)

- From the previous slide, if $b_1 + 3b_2 + b_3 \neq 0$, there is no solution to the system of equations
- Requiring that $b_1 + 3b_2 + b_3 = 0$, assume, for example, that
$$b_1 = 2,\ b_2 = 1,\ b_3 = -5$$
- Then the reduced augmented matrix (**A**|**b**) becomes:

$$\begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{pmatrix} \rightarrow \begin{matrix} x_1 & -2x_2 & +3x_3 & = 2 \\ & x_2 & -\ x_3 & = -3 \\ & & 0 & = 0 \end{matrix} \rightarrow \mathbf{x} = \begin{pmatrix} -x_3 - 4 \\ x_3 - 3 \\ x_3 \end{pmatrix} \rightarrow \mathbf{x} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}$$

- It can be shown that the second term in **x** is a solution of the nonhomogeneous equation and that the first term is the most general solution of the homogeneous equation, letting $x_3 = \alpha$, where α is arbitrary

$$X = \alpha \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} -4 \\ -3 \\ 0 \end{Bmatrix}$$

Linear Dependence and Independence

- A set of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is **linearly dependent** if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

- If the only solution of

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

is $c_1 = c_2 = \ldots = c_n = 0$, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is **linearly independent.**

Example 3: Linear Dependence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

- We need to solve

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c\begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

Example 3: Linear Dependence (2 of 2)

- We can reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$, as before.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & 15 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} c_1 + 2c_2 - 4c_3 = 0 \\ \rightarrow \quad c_2 - 3c_3 = 0 \rightarrow \mathbf{c} = c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \text{ where } c_3 \text{ can be any number} \\ 0 = 0 \end{array}$$

- So, the vectors are linearly dependent: if $c_3 = -1$, $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0}$
- Alternatively, we could show that the following determinant is zero:

$$\det(x_{ij}) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix} = 0$$

Linear Independence and Invertibility

- Consider the previous two examples:
  - The first matrix was known to be nonsingular, and its column vectors were linearly independent.
  - The second matrix was known to be singular, and its column vectors were linearly dependent.
- This is true in general: the columns (or rows) of **A** are linearly independent iff **A** is nonsingular iff $\mathbf{A}^{-1}$ exists.
- Also, **A** is nonsingular iff det**A** $\neq 0$, hence columns (or rows) of **A** are linearly independent iff det**A** $\neq 0$.
- Further, if $\mathbf{A} = \mathbf{BC}$, then $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$. Thus if the columns (or rows) of **A** and **B** are linearly independent, then the columns (or rows) of **C** are also.

Linear Dependence & Vector Functions

- Now consider vector functions $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$, where

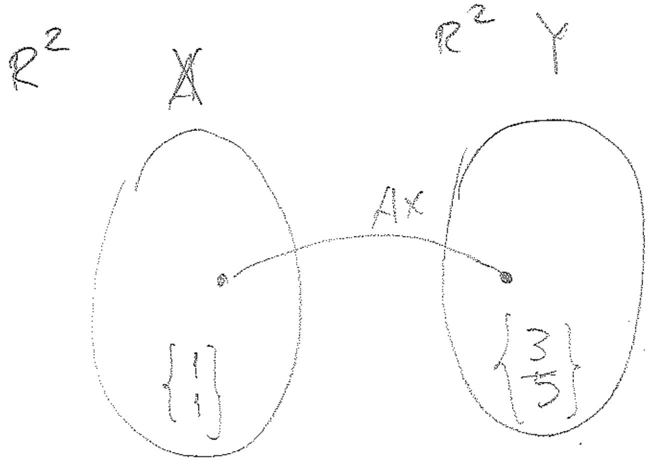
$$\mathbf{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_m^{(k)}(t) \end{pmatrix}, \quad k = 1, 2, \ldots, n, \qquad t \in I = (\alpha, \beta)$$

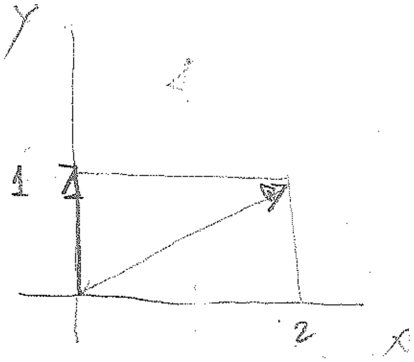
- As before, $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly dependent** on I if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) = \mathbf{0}, \text{ for all } t \in I$$

- Otherwise $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly independent** on I See text for more discussion on this.

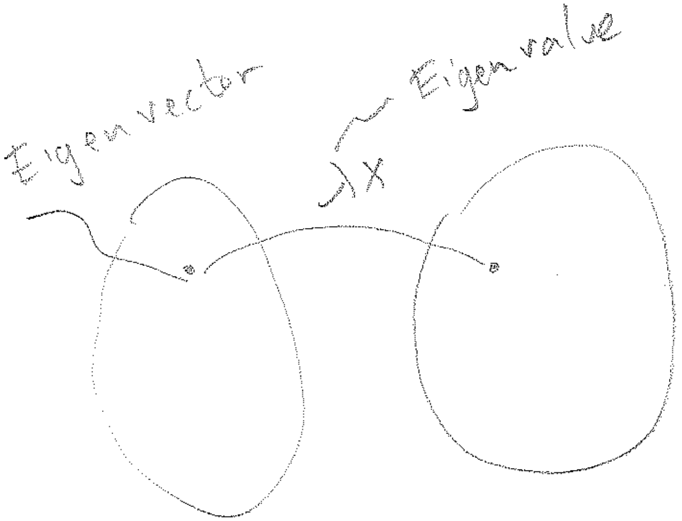
EIGEN VALUE / EIGEN VECTOR

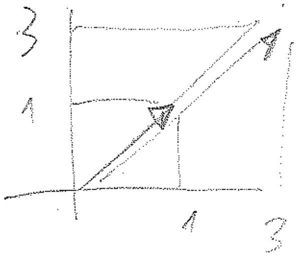




$$Y = AX$$

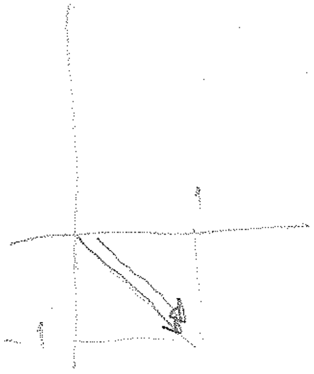
$$y = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} = \begin{Bmatrix}2\\1\end{Bmatrix}$$





$$y = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix} = \begin{Bmatrix}3\\3\end{Bmatrix}$$

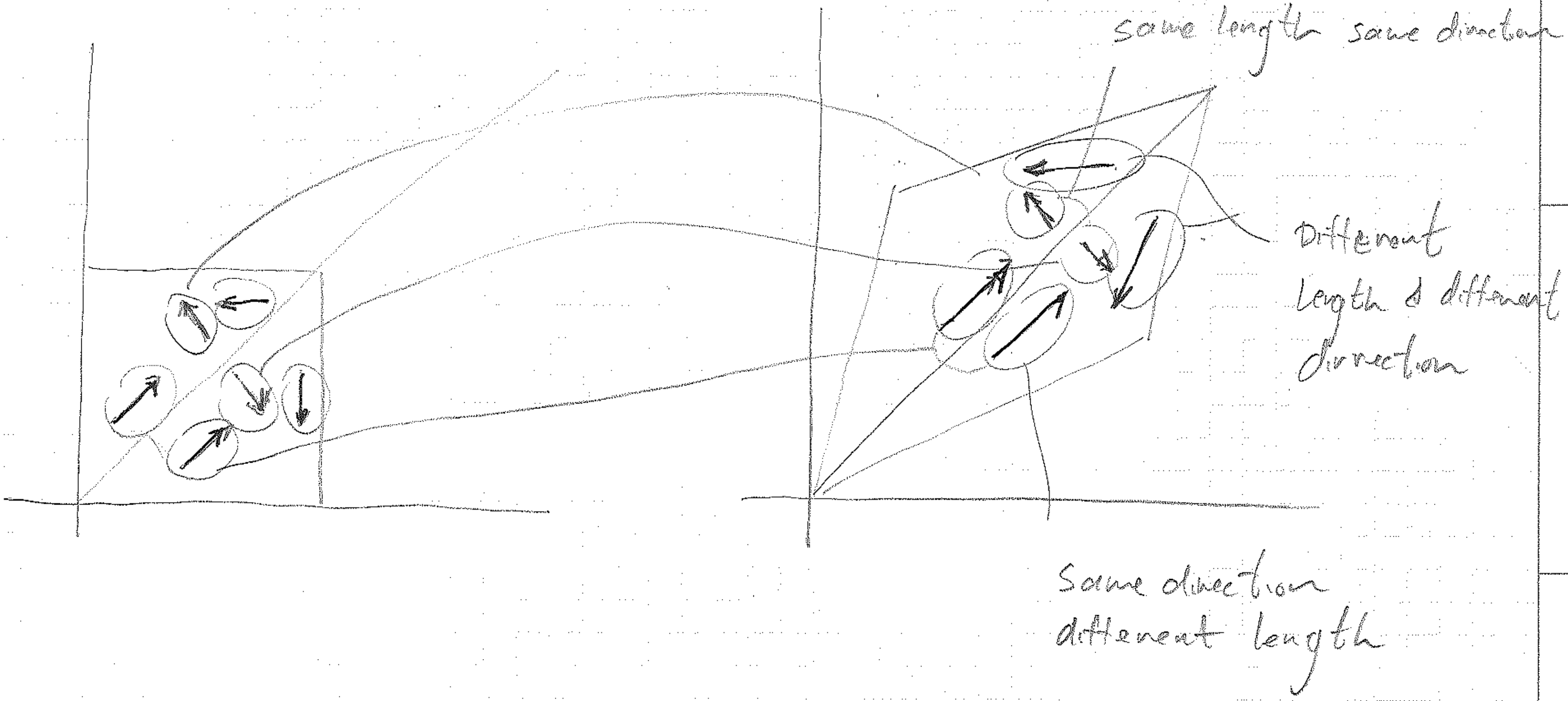
$$y = \lambda\begin{Bmatrix}1\\1\end{Bmatrix} = 3\begin{Bmatrix}1\\1\end{Bmatrix}$$



$$y = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{Bmatrix}1\\-1\end{Bmatrix} = \begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$y = \lambda\begin{Bmatrix}1\\-1\end{Bmatrix} = 1\begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y \end{bmatrix}$$



Eigenvalues and Eigenvectors

- The eqn. $\mathbf{Ax} = \mathbf{y}$ can be viewed as a linear transformation that maps (or transforms) $\mathbf{x}$ into a new vector $\mathbf{y}$.
- Nonzero vectors $\mathbf{x}$ that transform into multiples of themselves are important in many applications.
- Thus we solve $\mathbf{Ax} = \lambda\mathbf{x}$ or equivalently, $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- This equation has a nonzero solution if we choose λ such that $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Such values of λ are called **eigenvalues** of **A**, and the nonzero solutions $\mathbf{x}$ are called **eigenvectors**.

Example 4: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

- Solution: Choose λ such that det($\mathbf{A}$-$\lambda\mathbf{I}$) = 0, as follows.

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} \\ &= (3-\lambda)(-2-\lambda)-(-1)(4) \\ &= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \\ &\Rightarrow \lambda = 2,\ \lambda = -1 \end{aligned}$$

Example 4: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix **A**, we need to solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda = 2$ and $\lambda = -1$.
- Eigenvector for $\lambda = 2$: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_1 = x_2$. So

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 4: Second Eigenvector (3 of 3)

- Eigenvector for λ = -1: Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_2 = 4x_1$ So

$$\mathbf{x}^{(2)}=\begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix}=c\begin{pmatrix} 1 \\ 4 \end{pmatrix},\quad c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(2)}=\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Normalizing The Vectors

$$\|X\| = (1, 1) = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\|X\| = \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right) = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$$

↗ Normalized

$$\|X\| = (1, 4) = \sqrt{(1)^2 + (4)^2} = \sqrt{17}$$

$$\left(\frac{1}{\sqrt{17}} ; \frac{4}{\sqrt{17}}\right) = \sqrt{\left(\frac{1}{\sqrt{17}}\right)^2 + \left(\frac{4}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.
- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.
- For example, eigenvectors are sometimes normalized by choosing the constant so that $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an n x n matrix **A**, we solve det(**A**-λ**I**) = 0.
- Since this involves finding the determinant of an n x n matrix, the problem reduces to finding roots of an nth degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m.
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigevectors, $1 \leq q \leq m$, and q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an *n* x *n* matrix **A** is simple, then **A** has *n* distinct eigenvalues. It can be shown that the *n* eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more repeated eigenvalues, then there may be fewer than *n* linearly independent eigenvectors since for each repeated eigenvalue, we may have $q < m$. This may lead to complications in solving systems of differential equations.

Example 5: Eigenvalues (1 of 5)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda - 2)(\lambda + 1)^2$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = -1,\ \lambda_2 = -1$$

Example 5: First Eigenvector (2 of 5)

- Eigenvector for $\lambda = 2$: Solve $(\mathbf{A}\text{-}\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrrl} 1x_1 & & -1x_3 & = 0 \\ & 1x_2 & -1x_3 & = 0 \\ & & 0x_3 & = 0 \end{array}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 5: 2nd and 3rd Eigenvectors (3 of 5)

- Eigenvector for λ = -1: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{llll} 1x_1 & +1x_2 & +1x_3 & =0 \\ & 0x_2 & & =0 \\ & & 0x_3 & =0 \end{array}$$

$$x_1 + x_2 + x_3 = 0$$

$$x_3 = -x_1 - x_2$$

$$\to \mathbf{x}^{(2)} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_2, x_3 \text{ arbitrary}$$

$$\to \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$x^{([illegible])} = \begin{bmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{bmatrix} = c_1$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Example 5: Eigenvectors of A (4 of 5)

- Thus three eigenvectors of **A** are

$\lambda=-1$

$\lambda=-1$

$$\boxed{\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \boxed{\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$

$\lambda=2,$

where $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ correspond to the double eigenvalue λ = - 1.

- It can be shown that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are linearly independent.
- Hence **A** is a 3 x 3 **symmetric matrix** (**A** = **A**^{T}) with 3 real eigenvalues and 3 linearly independent eigenvectors.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 1(+1) - 1\overset{-2}{(-1-1)} + 0(\sim) = 1+2=3$$

Example 5: Eigenvectors of A (5 of 5)

- Note that we could have we had chosen

same

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$

$1 - 2 + 1 = 0$ ✓

- Then the eigenvectors are orthogonal, since

$$\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = 0, \quad \left(\mathbf{x}^{(1)}, \mathbf{x}^{(3)}\right) = 0, \quad \left(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}\right) = 0$$

- Thus **A** is a 3 x 3 symmetric matrix with 3 real eigenvalues and 3 linearly independent orthogonal eigenvectors.

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$, where we recall that $\mathbf{A}^* = \overline{\mathbf{A}}^T$.
- Thus for a Hermitian matrix, $a_{ij} = \overline{a}_{ji}$.
- Note that if **A** has real entries and is symmetric (see last example), then **A** is Hermitian.
- An *n* x *n* Hermitian matrix **A** has the following properties:
  - All eigenvalues of **A** are real.
  - There exists a full set of *n* linearly independent eigenvectors of **A**.
  - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of **A**, then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
  - Corresponding to an eigenvalue of algebraic multiplicity *m*, it is possible to choose *m* mutually orthogonal eigenvectors, and hence A has a full set of *n* linearly independent orthogonal eigenvectors.