Class Notes 15:

Laplace Transform (4)
Systems Analysis

82 – Engineering Mathematics

System Analysis (Stability)

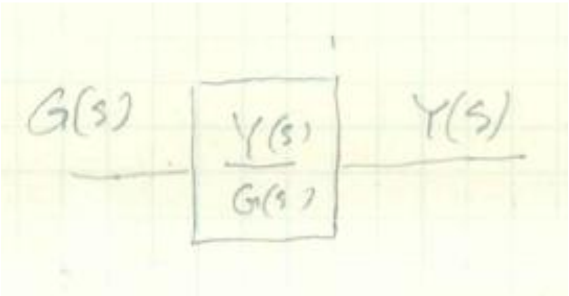
- First Order System $y' + ay = g(t)$

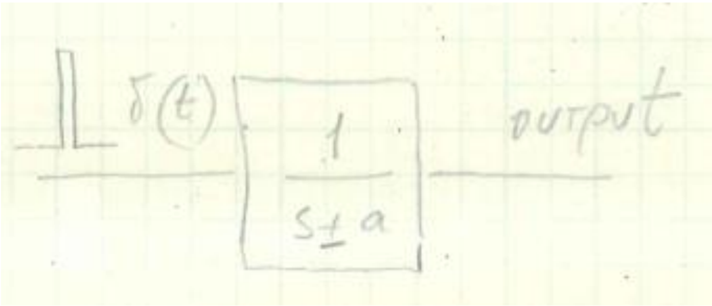
  Initial Condition: $y(0) = 0$

  Laplace $sY(s) - \cancel{y(0)} + aY(s) = G(s)$

  $Y(s)(s+a) = G(s)$

$$\frac{\text{Output}}{\text{Input}} \quad \frac{Y(s)}{G(s)} = \frac{1}{s+a}$$





$y = e^{\pm at}$

Second Order System (No Damping)

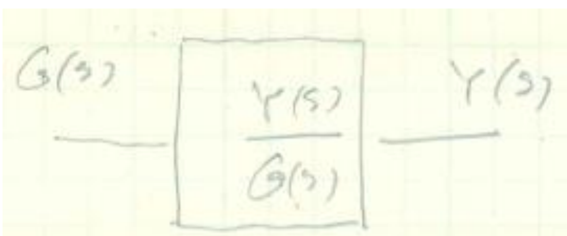
$$y'' + \omega_n^2 y = g(t)$$

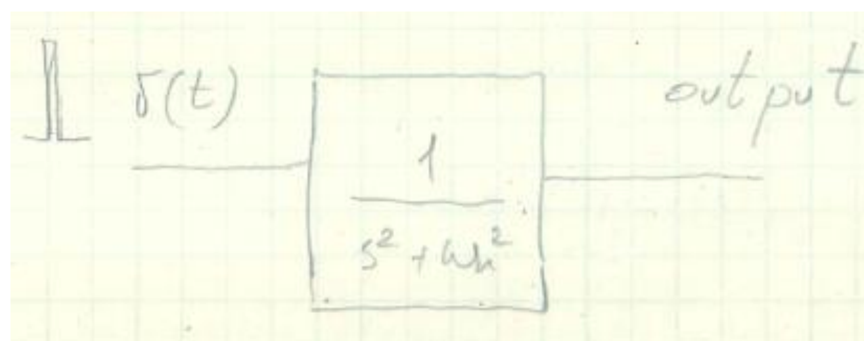
I.C. $$y(0) = 0;\ y'(0) = 0$$

Laplace $$s^2 Y(s) - \cancel{sy'(0)} - \cancel{y(0)} + \omega_n^2 Y(s) = G(s)$$

$$Y(s)\left(s^2 + \omega_n^2\right) = G(s)$$

$$\frac{Y(s)}{G(s)} = \frac{1}{s^2 + \omega_n^2}$$





$$y(t) = \frac{\sin \omega_n t}{\omega_n^2}$$

Second Order System (with Damping)

$$y'' + 2\varsigma\omega_n y' + \omega_n^2 y = g(t)$$

I.C. $$y(0) = 0; y'(0) = 0$$

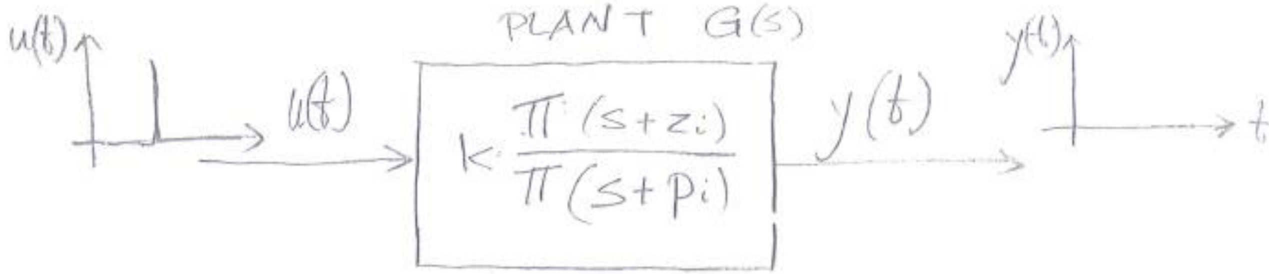
Laplace $$s^2 Y(s) - \cancel{sy'(0)} - \cancel{y(0)} + 2\varsigma\omega_n sY(s) - \cancel{2\varsigma\omega_n y(0)} + \omega_n^2 Y(s) = G(s)$$

$$Y(s)\left(s^2 + 2\varsigma\omega_n + \omega_n^2\right) = G(s)$$

$$\frac{Y(s)}{G(s)} = \frac{1}{s^2 + 2\varsigma\omega_n + \omega_n^2}$$

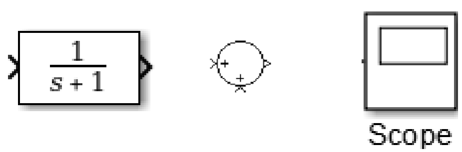
$$\frac{1}{s^2 + 2\varsigma\omega_n + \omega_n^2} = \begin{cases} \dfrac{1}{(s-a)(s-b)} \rightarrow y = \dfrac{e^{bt} - e^{at}}{b-a} \\ \dfrac{1}{(s-a)^2} \rightarrow y = te^{at} \end{cases}$$

Matlab Simulink

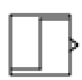


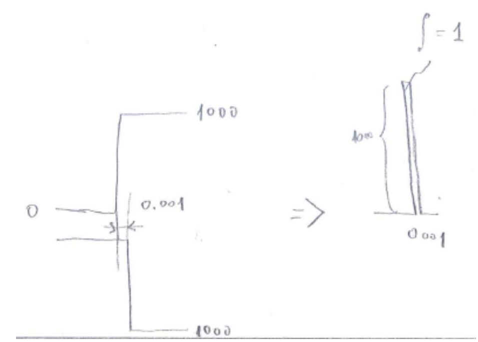
- Matlab $as^2 + bs + c \rightarrow [a \quad b \quad c]$
- Simulink: commonly used block

continuous

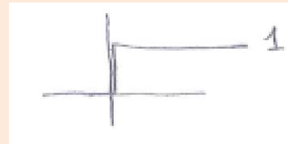
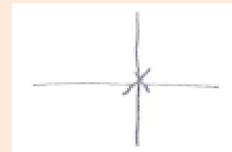
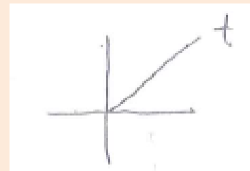
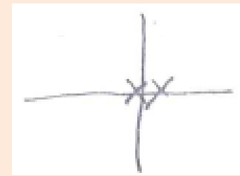
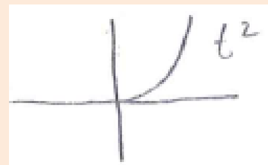
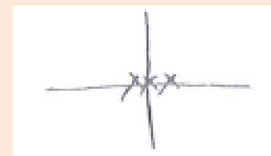
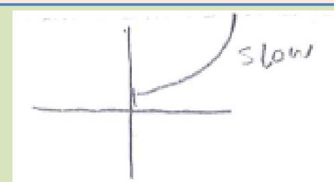
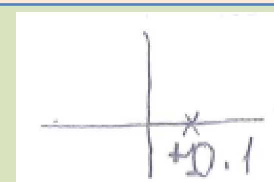
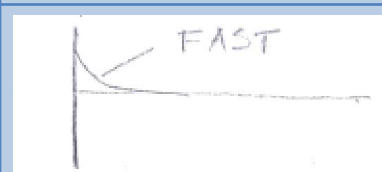
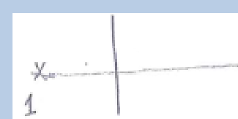


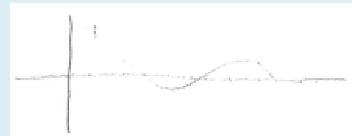
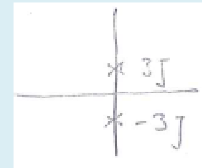
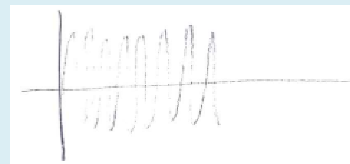
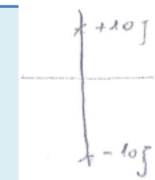
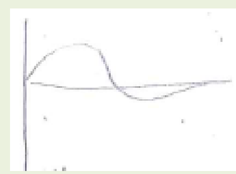
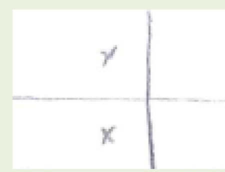
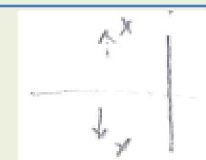
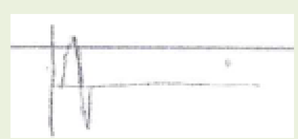
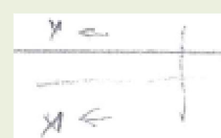
sources

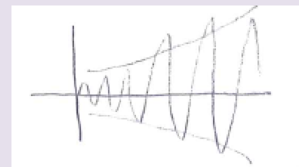


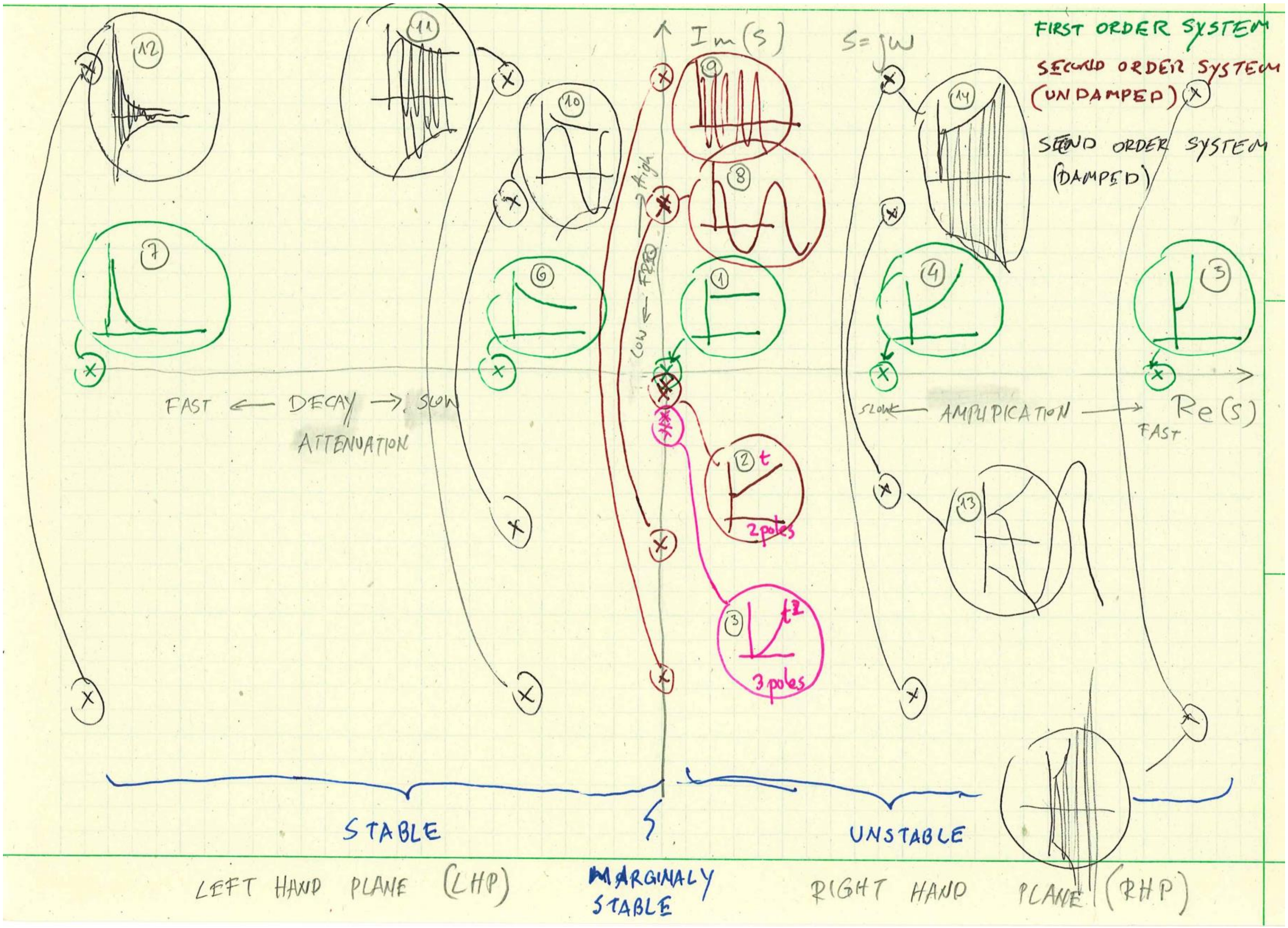


Order	Time domain	
1	$\dot{y} + ay = u(t)$	$sY(s) + aY(s) = U(s)$ $\frac{Y(s)}{U(s)} = \frac{1}{s+a}$
2	$\ddot{y} + 2\varsigma\omega_n\dot{y} + \omega_n^2 = u(t)$	$s^2Y(s) + 2\varsigma\omega_n sY(s) + \omega_n^2Y(s) = U(s)$ With damping ζ≠0: $\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$ No damping ζ=0: $\frac{Y(s)}{U(s)} = \frac{1}{s^2 + \omega_n^2}$ Case 1: ζ>1 Overdamping $\left.\begin{matrix} s_1 \\ s_2 \end{matrix}\right\} = \left(-\varsigma \pm \sqrt{\varsigma^2 - 1}\right)\omega_n$ Real, negative, distinct Case 2: ζ=1 Critical Damping $s_1 = s_2 = -\omega_n$ Real, negative, equal Case 3: ζ<1 Underdamping $\left.\begin{matrix} s_1 \\ s_2 \end{matrix}\right\} = \left(-\varsigma \pm j\sqrt{1 - \varsigma^2}\right)\omega_n$ Complex conjugates (negative real part)

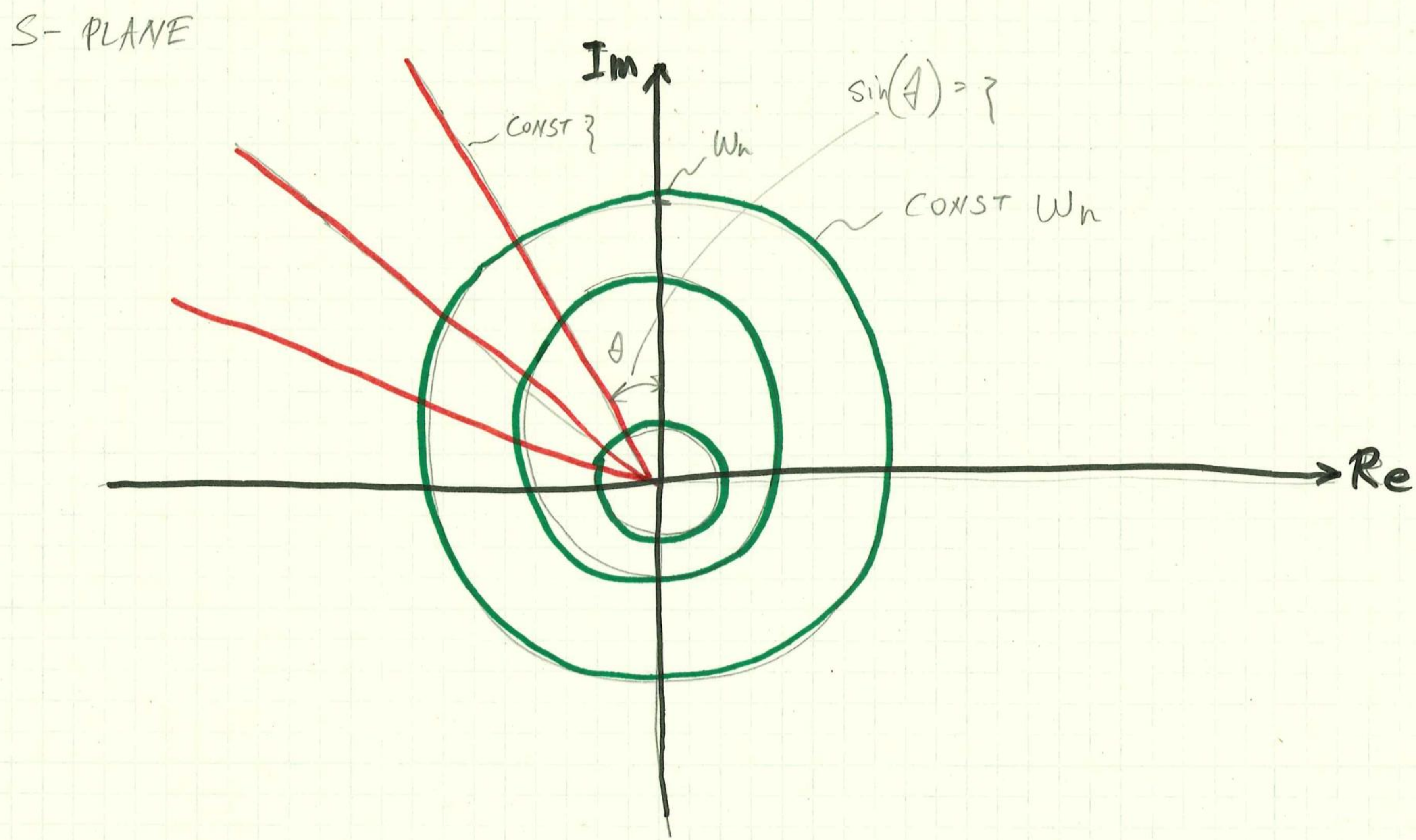
1	$\frac{1}{s}$	$\frac{[1]}{[1 \quad 0]}$	 	Origin
2	$\frac{1}{s^2}$	$\frac{[1]}{[1 \quad 0 \quad 0]}$	 	Origin
3	$\frac{1}{s^3}$	$\frac{[1]}{[1 \quad 0 \quad 0 \quad 0]}$	 	Origin
4	$\frac{1}{s-0.1}$	$\frac{[1]}{[1 \quad -0.1]}$	 	(+) Real
5	$\frac{1}{s-1}$	$\frac{[1]}{[1 \quad -1]}$	 	(+) Real
6	$\frac{1}{s+0.1}$	$\frac{[1]}{[1 \quad 0.1]}$	 	(-) Real
7	$\frac{1}{s+1}$	$\frac{[1]}{[1 \quad 1]}$	 	(-) Real

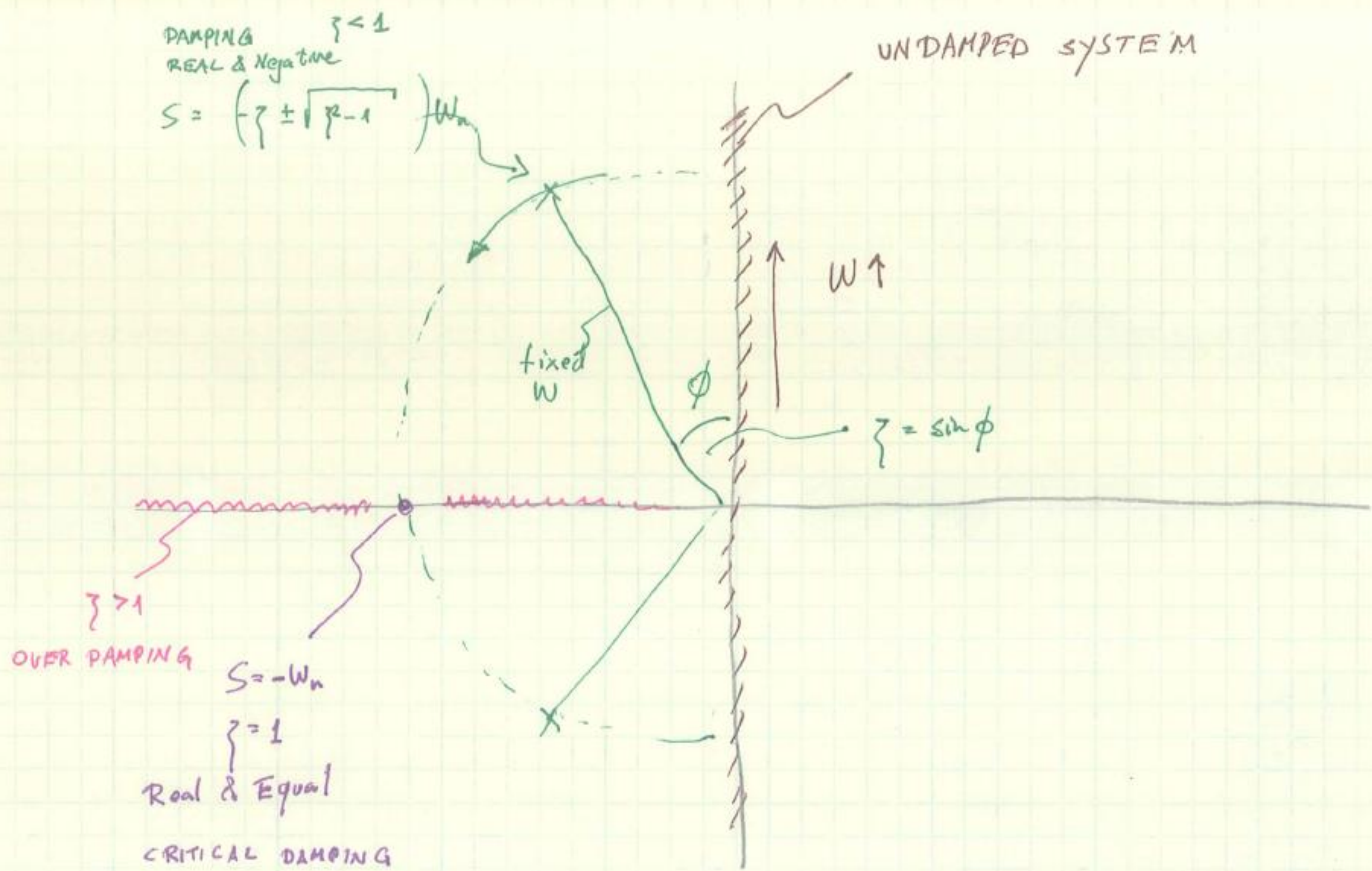
$\frac{1}{(s+j\omega)(s-j\omega)}=\frac{1}{s^2-sj\omega+sj\omega+\omega^2}=\frac{1}{s^2+\omega^2};s_{1,2}=\pm j\omega$				
8	$\frac{1}{s^2+9};s=\pm 3j$	$\frac{[1]}{[1\quad 0\quad 9]}$	 	Imaginary
9	$\frac{1}{s^2+100};s=\pm 10j$	$\frac{[1]}{[1\quad 0\quad 100]}$	 	Imaginary
$\frac{1}{(s+\sigma+j\omega)(s+\sigma-j\omega)}=\frac{1}{s^2+(\sigma+j\omega)s+(\sigma-j\omega)s+(\sigma+j\omega)(\sigma-j\omega)}=\frac{1}{s^2+2\sigma s+\sigma^2+\omega^2};s_{1,2}=-\sigma\pm j\omega$				
10	$\frac{1}{s^2+2s+2};s=-1\pm j$	$\frac{[1]}{[1\quad 2\quad 2]}$	 	Complex In LHP
11	$\frac{1}{s^2+2s+101};s=-1\pm 10j$	$\frac{[1]}{[1\quad 2\quad 101]}$	 	Complex In LHP
12	$\frac{1}{s^2+20s+200};s=-10\pm 10j$	$\frac{[1]}{[1\quad 20\quad 200]}$	 	Complex In LHP

$\frac{1}{(s-\sigma+j\omega)(s-\sigma-j\omega)}=\frac{1}{s^2+(-\sigma+j\omega)s+(-\sigma-j\omega)s+(-\sigma+j\omega)(-\sigma-j\omega)}=\frac{1}{s^2-2\sigma s+\sigma^2+\omega^2};s_{1,2}=\sigma\pm j\omega$				
13	$\frac{1}{s^2-2s+2};s=1\pm j$	$\frac{[1]}{[1\quad -2\quad 2]}$		Complex In RHP
14	$\frac{1}{s^2-2s+101};s=1\pm 10j$	$\frac{[1]}{[1\quad -2\quad 101]}$		Complex In RHP



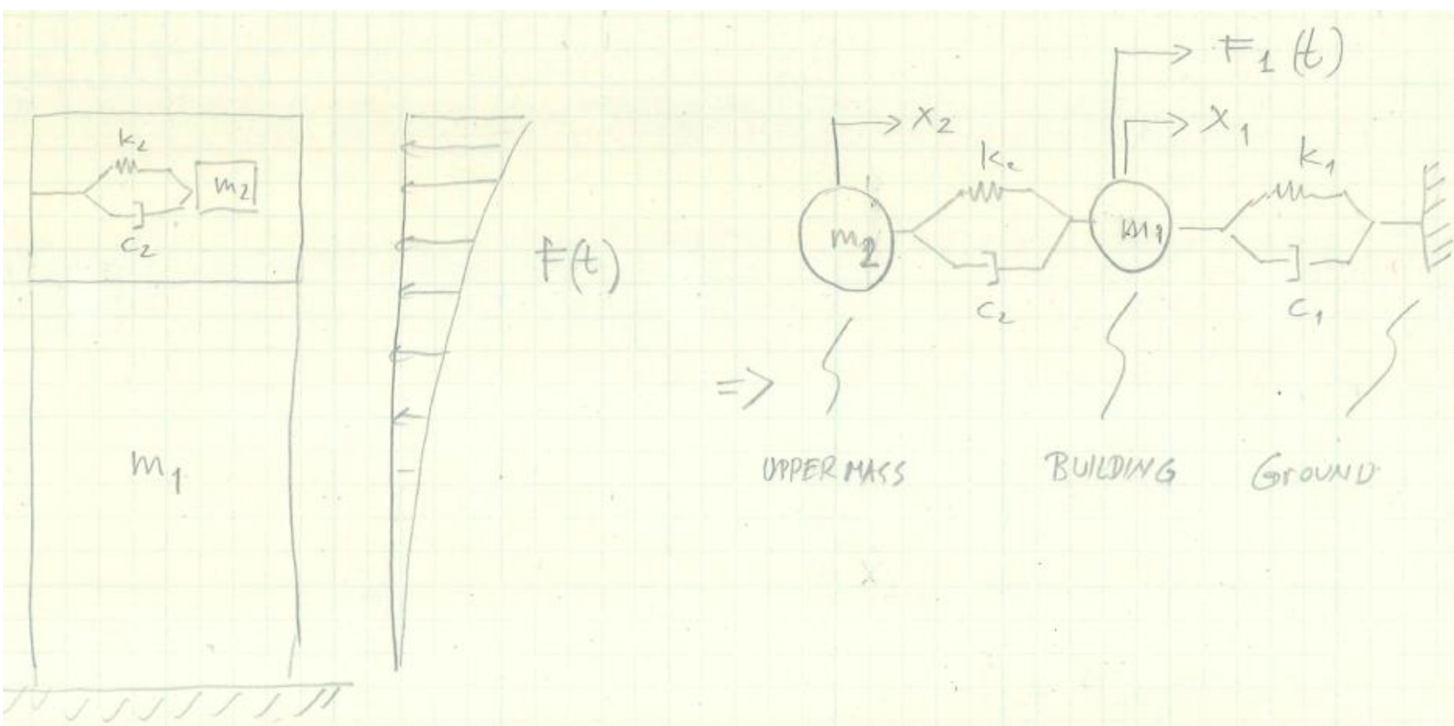
Im(s)
s=jw
FIRST ORDER SYSTEM
SECOND ORDER SYSTEM (UNDAMPED)
SECOND ORDER SYSTEM (DAMPED)
high
LOW
FREQ
FAST ← DECAY → SLOW
ATTENUATION
SLOW ← AMPLIPICATION → FAST
Re(s)
t
2 poles
t²
3 poles
STABLE
UNSTABLE
LEFT HAND PLANE (LHP)
MARGINALY STABLE
RIGHT HAND PLANE (RHP)


S- PLANE
Im
Re
CONST ζ
sin(θ) = ζ
ωn
CONST ωn
θ


DAMPING ζ < 1
REAL & Negative
S = (-ζ ± √(ζ²-1)) Wn
UNDAMPED SYSTEM
W ↑
fixed W
ϕ
ζ = sin ϕ
ζ > 1
OVER DAMPING
S = -Wn
ζ = 1
Real & Equal
CRITICAL DAMPING

System of Linear Differential Equation

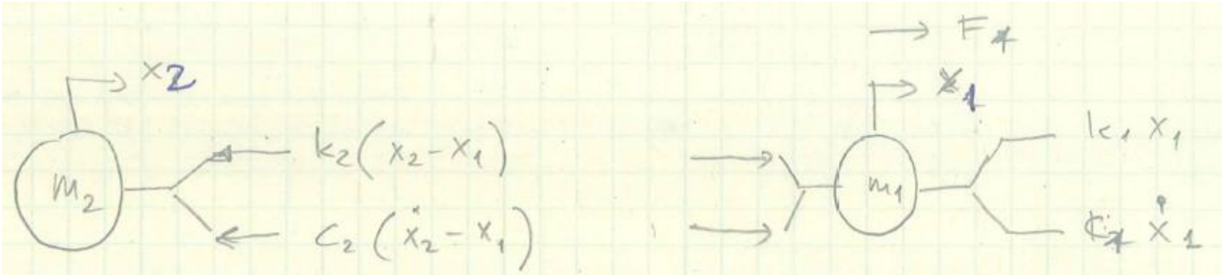
Differential Equation → Laplace Transform → Algebraic Equation



System of Linear Differential Equation

- Assumption $\begin{cases} c_1 = c_2 = 0 \\ x_1(0) = x_2(0) = x_1'(0) = x_2'(0) = 0 \end{cases}$ No dampingIC

$x_2 > x_1$



$$+\rightarrow \sum F = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = m_2\ddot{x}_2$$

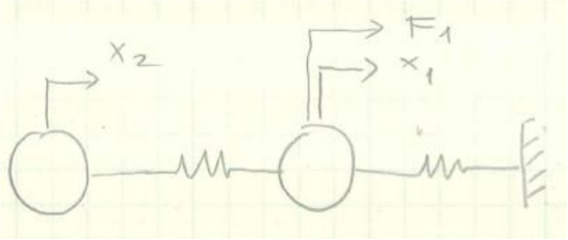
$$+\rightarrow \sum F = k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - c_1\dot{x}_1 + F_1 = m_1\ddot{x}_1$$

set $c_1 = c_2 = 0$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t)$$

$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

System of Linear Differential Equation



$$\begin{cases} m_1 x_1'' = -(k_1 + k_2) x_1 + k_2 x_2 + F_1 \\ m_2 x_2'' = + \qquad k_2 x_1 - k_2 x_2 \end{cases}$$

$$\begin{cases} x_1'' = -\dfrac{(k_1 + k_2)}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 + \dfrac{F_1}{m_1} \\ x_2'' = + \qquad \dfrac{k_2}{m_2} x_1 - \dfrac{k_2}{m_2} x_2 \end{cases}$$

Four first order linear Diff. Eq.

$$\begin{cases} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -\dfrac{(k_1 + k_2)}{m_1} x_1 + \dfrac{k_2}{m_1} x_2 + \dfrac{F_1}{m_1} \\ x_4' = + \qquad \dfrac{k_2}{m_2} x_1 - \dfrac{k_2}{m_2} x_2 \end{cases}$$

System of Linear Differential Equation

$$x' = P(t)x + g(t)$$

$$\underbrace{\begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{Bmatrix}}_{\substack{\uparrow \\ x'}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & 0 \end{bmatrix}}_{P(t)} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}}_{x} + \underbrace{\begin{Bmatrix} 0 \\ 0 \\ F_1/m_1 \\ 0 \end{Bmatrix}}_{g(t)}$$

$$\begin{cases} \text{Homogeneous} & g(t) = 0 \\ \text{Nonhomogeneous} & g(t) \neq 0 \end{cases}$$

System of Linear Differential Equation

Laplace

$$m_1 s^2 X_1(s) + (k_1 + k_2) X_1(s) - k_2 X_2(s) = F_1(s)$$
$$m_2 s^2 X_2(s) - k_2 X_1(s) + k_2 X_2(s) = 0$$

$$\begin{bmatrix} k_1 + k_2 + m_1 s^2 & -k_2 \\ -k_2 & k_2 + m_2 s^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(s) \\ 0 \end{Bmatrix}$$

$$X_1(s) = \frac{F_1(m_2 s^2 + k_2)}{(m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$$
$$X_2(s) = \frac{+k_2 F_1}{(m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_2) - k_2^2}$$

System of Linear Differential Equation

Check

$$\left[1-\left(\frac{w}{w_a}\right)^2\right]X_{ST}=\left[1-w^2\frac{m_2}{k_2}\right]\frac{F_1}{k_1}=\frac{F_1}{k_1}-w^2\frac{m_2F_1}{k_1k_2}$$

$$=\left(\frac{F_1}{k_1}-w^2\frac{m_2F_1}{k_1k_2}\right)k_1k_2$$

$$=F_1k_2-w^2m_2F_1$$

$$\left.\begin{array}{l} X_{ST}=\dfrac{F_1}{k_1}\left(k_1k_2\right)=F_1k_2 \\ (k_1k_2)=(w_n^2m_1)(w_a^2m_2) \end{array}\right]\text{NUM}$$

$$\left.\left\{\left[1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2\right]\left[1-\left(\frac{w}{w_n}\right)^2\right]-\mu\left(\frac{w_a}{w_n}\right)^2\right\}k_1k_2\quad\right]\text{DEN}$$

System of Linear Differential Equation

$$s = jw, \quad s^2 = (jw)^2 = -w^2$$

$$|X_1| = \frac{(k_2 - w^2 m_2)F_1}{(k_1 + k_2 - w^2 m_1)(k_2 - w^2 m_2) - k_2^2}$$

$$|X_2| = \frac{k_2 F_1}{(k_1 + k_2 - w^2 m_1)(k_2 - w^2 m_2) - k_2^2}$$

$$* \frac{\overbrace{k_1 k_2}^{extract\ from\ Num}}{\underbrace{k_1 k_2}_{divide\ by}}$$

$$w_n = \sqrt{\frac{k_1}{m_1}}, \quad w_a = \sqrt{\frac{k_2}{m_2}}, \quad X_{st} = \frac{F_1}{k_1}, \quad \mu = \frac{m_2}{m_1}$$

$$|X_1| = \frac{\left[1 - \left(\frac{w}{w_a}\right)^2\right] X_{ST}}{\left[1 + \mu\left(\frac{w_a}{w_n}\right)^2 - \left(\frac{w}{w_n}\right)^2\right]\left[1 - \left(\frac{w}{w_a}\right)^2\right] - \mu\left(\frac{w_a}{w_n}\right)^2}$$

$$|X_2| = \frac{X_{ST}}{\left[1 + \mu\left(\frac{w_a}{w_n}\right)^2 - \left(\frac{w}{w_n}\right)^2\right]\left[1 - \left(\frac{w}{w_a}\right)^2\right] - \mu\left(\frac{w_a}{w_n}\right)^2}$$

System of Linear Differential Equation

$$\mu\left(\frac{w_a}{w_n}\right)^2 k_1 k_2 = \frac{m_2}{m_1}\frac{k_2}{m_2}\frac{m_1}{k_1}k_1 k_2 = k_2^2$$

$$\left[1-\left(\frac{w}{w_a}\right)^2\right]k_2 = k_2 - \left(\frac{w^2}{k_2}m_2\right)k_2 = k_2 w^2 m_2$$

$$\left[1+\mu\left(\frac{w_a}{w_n}\right)^2 - \frac{w}{w_n}\right]k_1 = \left[k_1 + \frac{m_2}{m_1}\frac{k_2}{m_2}\frac{m_1}{k_1}k_1 - w^2\frac{m_1}{k_1}k_1\right]$$

$$= k_1 + k_2 - w^2 m_1$$

Den

System of Linear Differential Equation

From Eq for $\left|X_1\right| = \dfrac{\left[1-\left(\dfrac{w}{w_a}\right)^2\right]X_s}{\Delta}$

when $w = w_a \rightarrow x_1 = 0$

The amplitude of the main mass reduces to zero

Hence the absorber can indeed perform the task for which it is designed, namely to eliminate the vibration of the main mass, provided the natural frequency of the absorber is the same as the frequency of the external exitation.

From Eq for x_2 $w = w_a$

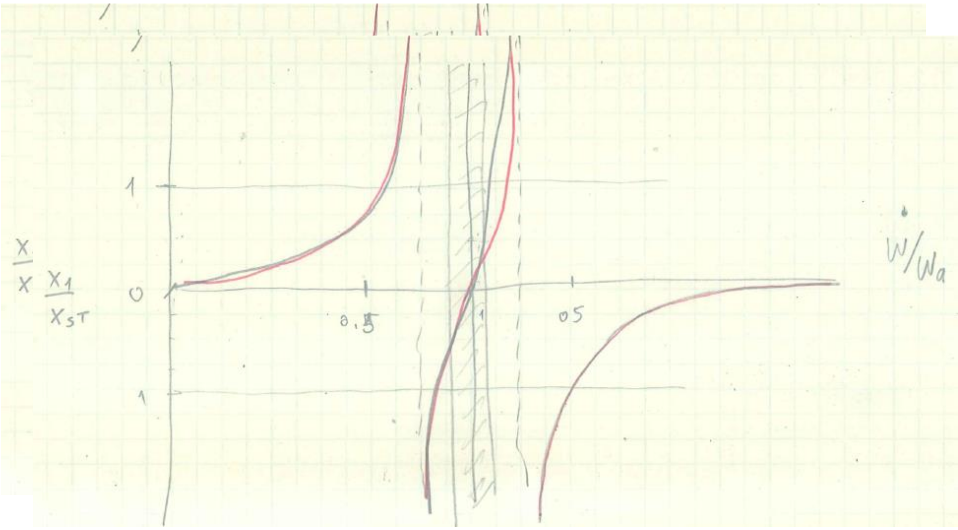
$$\left|X_2\right| = \frac{X_{ST}}{\left[1+\mu\left(\dfrac{w_a}{w_n}\right)^2 - \left(\dfrac{w_a}{w_n}\right)\right]\underbrace{\left[1-\left(\dfrac{w_a}{w_a}\right)^2\right]}_{=0} - \mu\left(\dfrac{w_a}{w_n}\right)^2} = -\frac{X_{ST}}{\mu}\left(\frac{w_n}{w_a}\right)^2$$

System of Linear Differential Equation

$$|X_2| = -\underbrace{\frac{F_1}{k_1}}_{X_{ST}}\underbrace{\frac{m_1}{m_2}}_{\frac{1}{\mu}}\underbrace{\frac{k_1}{m_1}}_{w_n^2}\underbrace{\frac{m_2}{k_2}}_{\frac{1}{w_a^2}} = -\frac{F_1}{k_2}$$

$$F_1 = |X_2| k_2$$ max force generated by the absorber

$$For\ \ \mu = 0.2\left(=\frac{m_2}{m_1}\right)$$



System of Linear Differential Equation

$$\frac{X_1}{X_{ST}} = \frac{\left[1-\left(\frac{w}{w_a}\right)^2\right]}{\underbrace{\left[1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2\right]\left[1-\left(\frac{w}{w_a}\right)^2\right]-\mu\left(\frac{w_a}{w_n}\right)^2}_{=\Delta}}$$

Resonance $\Delta = 0$

$$1+\mu\left(\frac{w_a}{w_n}\right)^2-\left(\frac{w}{w_n}\right)^2-\left(\frac{w}{w_a}\right)^2-\mu\left(\frac{w_a}{w_n}\right)^2\left(\frac{w}{w_a}\right)^2+\left(\frac{w}{w_n}\right)^2\left(\frac{w}{w_a}\right)^2-\mu\left(\frac{w_a}{w_n}\right)^2=0$$

$$1-\frac{w^2}{w_n^2}-\frac{w^2}{w_a^2}-\mu\frac{w_a^2}{w_n^2}\frac{w^2}{w_a^2}+\frac{w^4}{w_n^2w_a^2}=0$$

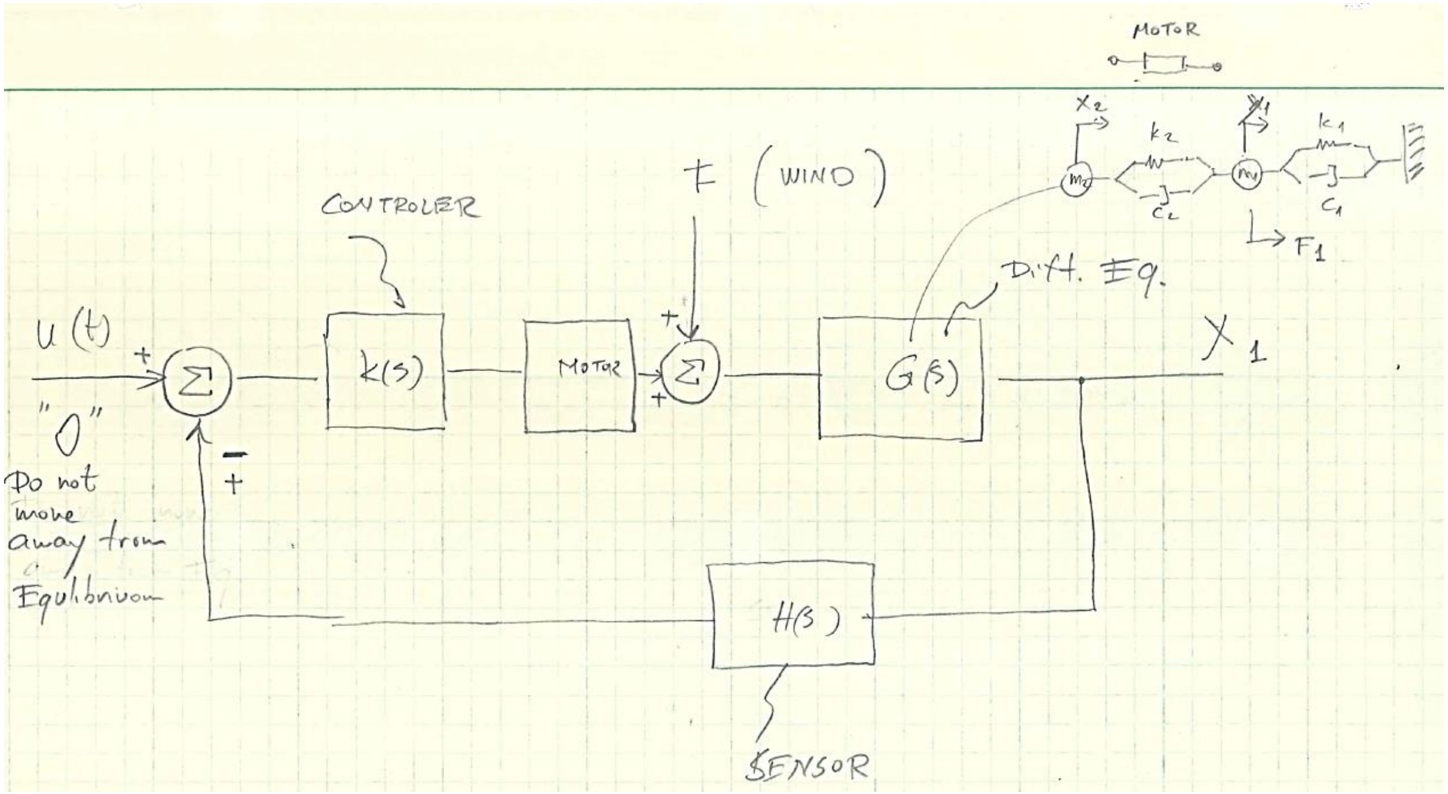
$$\frac{w_n^2w_a^2-w^2w_a^2-w^2w_n^2-\mu w^2w_a^2+w^4}{w_n^2w_a^2}=0$$

$$w^4+w^2\left(\underbrace{-w_a^2-w_n^2-\mu w_a^2}_{b}\right)+\underbrace{w_n^2w_a^2}_{c}=0$$

$$w=\sqrt{\frac{-b\pm\sqrt{b^2-4c}}{2}}$$

$$=\sqrt{\frac{w_a^2+w_n^2-\mu w_a^2\pm\sqrt{\left(w_a(1+\mu)-w_n\right)^2-4w_n^4w_a^2}}{2}}$$

System of Linear Differential Equation



$$\frac{Y(s)}{U(s)} = \frac{K(s)G(s)}{1 + K(s)G(s)H(s)}$$