Class Notes 14:

Laplace Transform (3/3)

82 – Engineering Mathematics

Operational Properties – Derivative of a transform

$$F(s) = L\{f(t)\} \text{ and } n = 1, 2, 3, \cdots, n$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

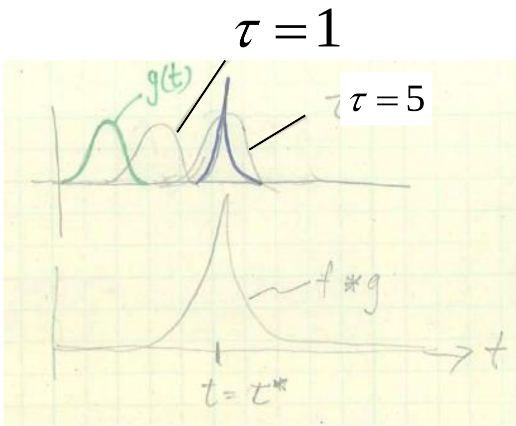
- Example

$$f(t) = \sin(kt) \qquad F(s) = L\{f(t)\} = L(\sin(kt)) = \frac{k}{s^2 + k^2}$$

$$L\{t^1 \sin(kt)\} = (-1)^1 \frac{d}{ds} L\{\sin(kt)\} = -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Operational Properties - Convolution

- Def : $f * g = \int_0^t f(\tau) g(t-\tau) d\tau$



$$f * g = g * f$$

$$L\{f * g\} = L\{f(t)\} \cdot L\{g(t)\} = F(s) \cdot G(s)$$

$$L^{-1}\{F(s) \cdot G(s)\} = f * g$$

- Example

$$e^{\tau} * \sin(t) = L\left\{\int_0^t e^{\tau} \sin(t-\tau) d\tau\right\} = L\{e^t\} \cdot L\{\sin(t)\} = \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$

Operational Properties - Transform of an integral

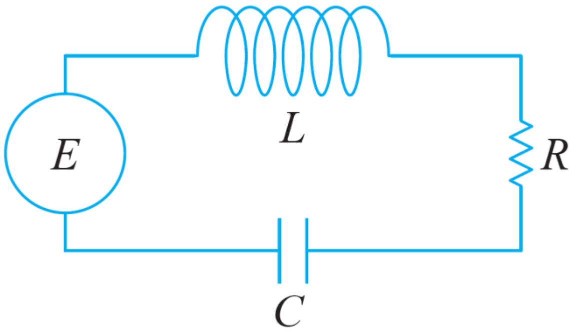
$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

- Volterra integral equation (Def.)

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

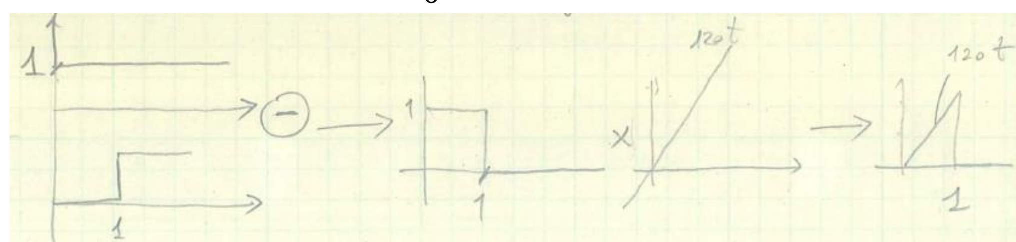
Transform of an integral - Example



$$\circlearrowright \sum V = 0$$

$$-E(i) + L\frac{di}{dt} + Ri(t) + \frac{1}{c}\int_0^t i(\tau)d\tau = 0$$

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$



$1-u(t-1)$ $\quad$ $[1-u(t-1)]\times 120t$ $\quad$ $120t[1-u(t-1)$

$$R = 2\Omega,\ C = 0.1f,\ L = 0.1h,\ i(t=0) = 0$$

$$\underbrace{0.1\frac{di}{dt} + 2i + 10\int_0^t i(\tau)dt}_{\text{I}} = \underbrace{120t}_{\text{II}} - \underbrace{120t\,u(t-1)}_{\text{III}}$$

Transform of an integral - Example

$$0.1\frac{di}{dt} + 2i + 10\underbrace{\int_0^t i(\tau)dt}_{\text{I}} = \underbrace{120t}_{\text{II}} - \underbrace{120t\,u(t-1)}_{\text{III}}$$

$$\text{I}: \quad \text{Table}\left[L\left\{\int_0^t i(\tau)d\tau\right\} = \frac{I(s)}{s}\right] \; where\; I(s) = L\{i(t)\}$$

$$\text{II}: \quad \text{Table}\left[L\left\{t^n\right\} = \frac{n!}{s^{n+1}}\right] \; L\{120t\} = 120L\{t\} = \frac{120}{s^2}$$

$$\text{III}: \quad \text{Table}\left[L\left\{f(t)u(t-a)\right\} = e^{-as}L\left\{f(t+a)\right\}\right]$$

$$120L\{t\,u(t-1)\} = 120e^{-s}L\{t+1\} = 120e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right]$$

$$s^2I(s) + 20sI(s) + 100I(s) = 1200\left[\frac{1}{s} - \frac{e^{-s}}{s} - e^{-s}\right]$$

Transform of an integral - Example

$$I(s)\left(\underbrace{s^2+20s+100}_{(s+10)^2}\right)=1200\left[\frac{1}{s}-\frac{e^{-s}}{s}-e^{-s}\right]$$

$$I(s)=1200\left[\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

$$\frac{1}{s(s+10)^2}=\frac{1/100}{s}-\frac{1/100}{s+10}-\frac{1/10}{(s+10)^2}\quad \text{(Partial fractions)}$$

$$I(s)=1200\left[\underbrace{\overset{①}{\frac{1/100}{s}}-\overset{③}{\frac{1/100}{s+10}}-\overset{⑤}{\frac{1/10}{(s+10)^2}}}_{\frac{1}{s(s+10)^2}}\underbrace{-\overset{②}{\frac{1/100}{s}}e^{-s}+\overset{④}{\frac{1/100}{s+10}}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}}_{-\frac{1}{s(s+10)^2}e^{-s}}-\frac{1}{(s+10)^2}e^{-s}\right]$$

$$-\frac{1080}{(s+10)^2}e^{-s}\ ⑥$$

Transform of an integral - Example

$$\text{Table}\left[\begin{array}{c} L^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u(t-a) \\ L^{-1}\left\{\dfrac{1}{(s-a)^n}\right\} = \dfrac{t^{n-1}e^{at}}{(n-1)!} \end{array}\right]$$

$$I(s) = 1200\left[\underbrace{\overset{①}{\frac{1/100}{s}} - \overset{③}{\frac{1/10}{s+10}} \cdot 0 + \dots}_{}\right]$$

$$I(s) = 1200\left[\underbrace{\overset{①}{\frac{1/100}{s}} - \overset{③}{\frac{1/100}{s+10}} - \overset{⑤}{\frac{1/10}{(s+10)^2}}}_{\frac{1}{s(s+10)^2}} \underbrace{- \overset{②}{\frac{1/100}{s}}e^{-s} + \overset{④}{\frac{1/100}{s+10}}e^{-s} + \frac{1/10}{(s+10)^2}e^{-s}}_{-\frac{1}{s(s+10)^2}e^{-s}} - \frac{1}{(s+10)^2}e^{-s}\right]$$

$$-\frac{1080}{(s+10)^2}e^{-s} \quad ⑥$$

$$i(t) = 12\left[1 - u(t-1)\right] + 12\left[e^{-10t} - e^{-10(t-1)}u(t-1)\right] - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}u(t-1)$$

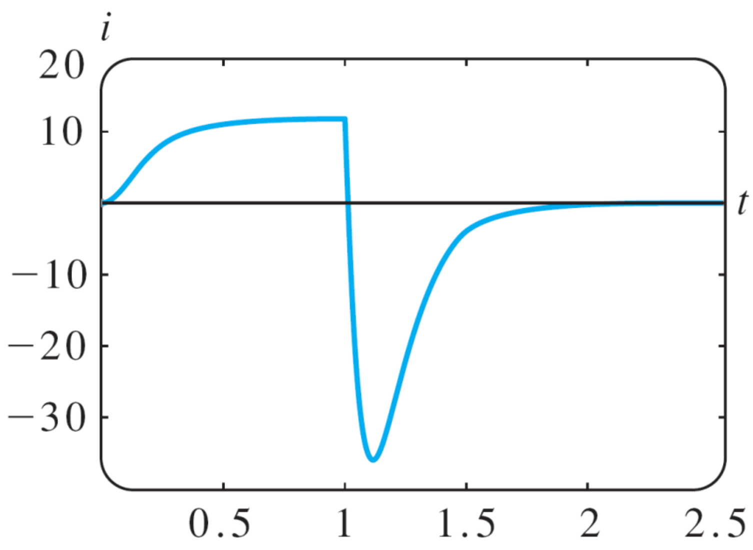
① ② ③ ④ ⑤ ⑥

Transform of an integral - Example

$$i(t) = 12[1 - u(t-1)] + 12[e^{-10t} - e^{-10(t-1)}u(t-1)] - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}u(t-1)$$

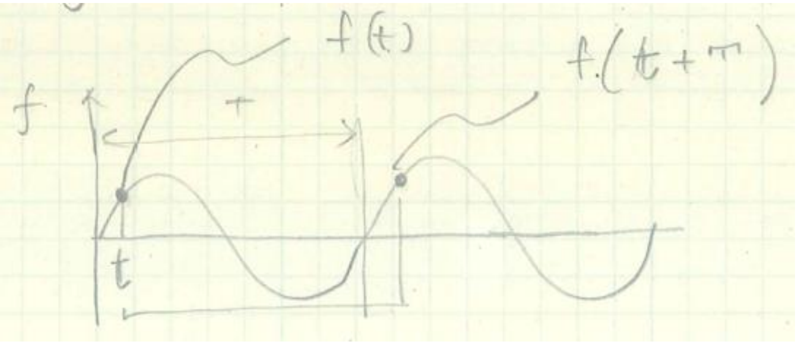
① ② ③ ④ ⑤ ⑥

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t} & 0 < t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)} & t \geq 1 \end{cases}$$



Operational Properties – Transform of a Periodic Function

- If a periodic function has period $T,\, T > 0$ then $f(t+T) = f(t)$

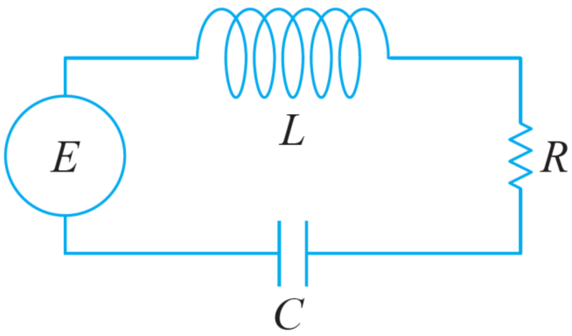


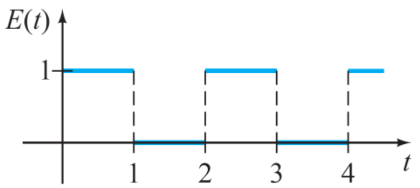
- If f(t) is
  - a piecewise continuous on $[0, \infty)$
  - Exponential order
  - Periodic with period T,
- Then

Note : integrate over period

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)dt$$

Transform of a Periodic Function - Example





$$\circlearrowright \sum V = -E(t) + L\frac{di}{dt} + Ri = 0$$

$$E(t) = L\frac{di}{dt} + Ri$$

$$E(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

$$L\{E(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)dt = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot 1dt + \underbrace{\int_1^2 e^{-st}\cdot 0dt}_{=0}\right]$$

$$= \frac{1}{\underbrace{1-e^{-2s}}_{(1+e^{-s})(1-e^{-s})}}\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Transform of a Periodic Function - Example

$$L\left\{L\frac{di}{dt}\right\}+RL\{i\}=L\{E(t)\}$$

$$L\cdot s\cdot I(s)+R\cdot I(s)=\frac{1}{s(1-e^{-s})}$$

$$I(s)(L\cdot s+R)=\frac{1}{s(1-e^{-s})}$$

$$I(s)=\underbrace{\frac{1/L}{s(s+R/L)}}_{\mathrm{II}}\cdot\underbrace{\frac{1}{1+e^{-s}}}_{\mathrm{I}}$$

- Term I

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots$$

$$\frac{1}{1+e^{-s}}=1-e^{-s}+e^{-2s}-e^{-3s}+\cdots$$

Transform of a Periodic Function - Example

- Term II

$$\frac{1}{s\left(s+R/L\right)}=\frac{L/R}{s}-\frac{L/R}{s+R/L}$$

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)\left(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots\right)$$

$$=\frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\cdots\right)-\frac{1}{R}\left(\frac{1}{s+R/L}-\frac{e^{-s}}{s+R/L}+\frac{e^{-2s}}{s+R/L}-\frac{e^{-3s}}{s+R/L}+\cdots\right)$$

$$i(t)=\frac{1}{R}\left(1-u(t-1)+u(t-2)-u(t-3)+\cdots\right)$$

$$-\frac{1}{R}\left(e^{-\frac{Rt}{L}}-e^{-\frac{R(t-1)}{L}}u(t-1)+e^{-\frac{R(t-2)}{L}}u(t-2)-e^{-\frac{R(t-3)}{L}}u(t-3)+\cdots\right)$$

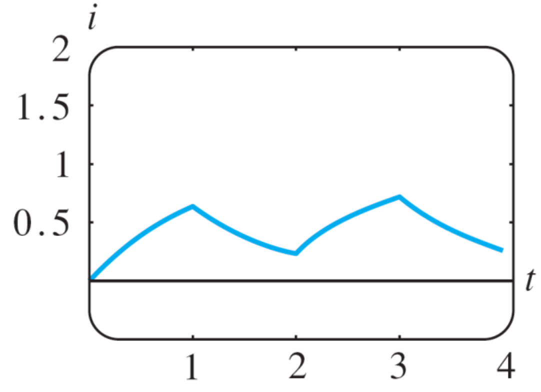
$$i(t)=\frac{1}{R}\left(1-e^{-\frac{Rt}{L}}\right)+\frac{1}{R}\sum_{n=1}^{\infty}(-1)^n\left(1-e^{-\frac{R(t-n)}{L}}\right)u(t-n)$$

Transform of a Periodic Function - Example

- For illustration assume R=1, L=1

$$i(t) = 1 - e^{-t} - \left(1 - e^{-(t-1)}\right)u(t-1) + \left(1 - e^{-(t-2)}\right)u(t-2) - \left(1 - e^{-(t-3)}\right)u(t-3) + \cdots$$

$$i(t) = \begin{cases} 1 - e^{-t} & 0 < t < 1 \\ -e^{-t} + e^{-(t-1)} & 1 \le t < 2 \\ 1 - e^{-t} + e^{-(t-1)} - e^{-(t-2)} & 2 \le t < 3 \\ -e^{-t} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)} & 3 \le t < 4 \\ \vdots & \end{cases}$$



Impulse Function

- Phenomena of impulsive nature
  - Voltage
  - Force
- Impulse
  - Large magnitude
  - Short time Interval

$$ay'' + by' + cy = g(t)$$

- g(t)
  - Large magnitude $\quad t_0 - \tau < t < t_0 + \tau$
  - Short time interval

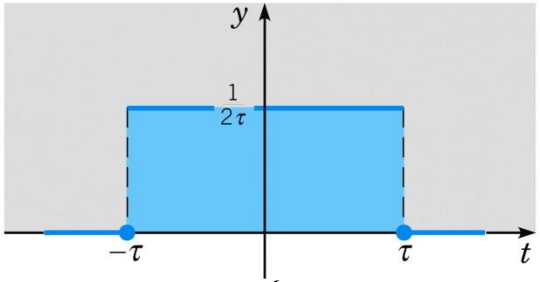
$$I(\tau) = \int_{t_0 - \tau}^{t_0 - \tau} g(t)\,dt$$

Impulse Function

- Since g(t)=0 outside of the interval (t_0-τ, t_0+τ)

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt$$

- Assume that t_0=0

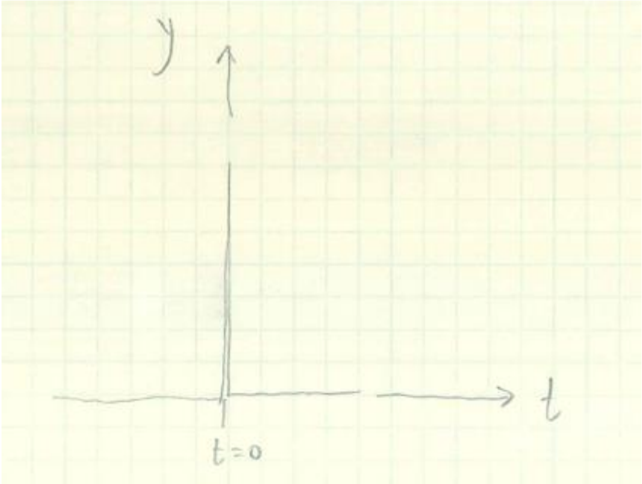


$$g(t) = d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & -\tau < t < \tau \\ 0 & t \le -\tau \text{ or } t \ge \tau \end{cases}$$

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau dt = 1$$ (independent of the value of τ)

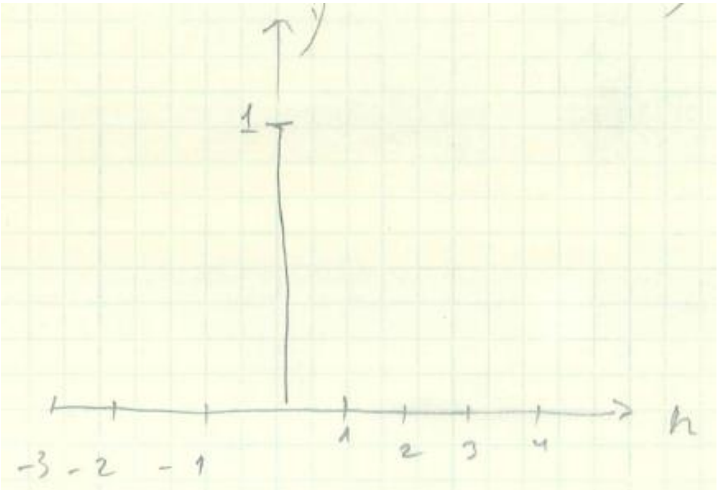
Delta Function

Mathematical (Dirac)



$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Engineering (Signal Processing) (Kronecker Impulse)

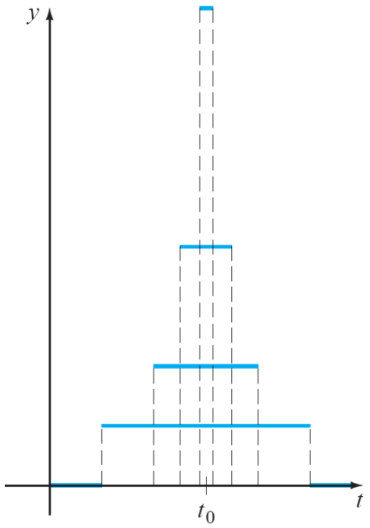


$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

$$r_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Impulse Function

$$\lim_{\tau \to 0^+} I(\tau) = 1$$



- Unit impulse function δ

(Dirac Delta Function)

$$t = 0 \qquad |\delta| = 1$$

$$t \neq 0 \qquad |\delta| = 0$$

$$\int_{-\infty}^{\infty} \delta(\tau) dt = 1$$

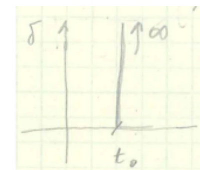
[Paul Dirac 1902 – 1989]
Nobel Prize – physics 1933
Quantum Mechanics
with Erwin Schrodinger

Transform of the Impulse Function

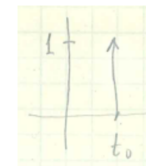
- Unit impulse at an arbitrary point t = t_0

$$\begin{cases} \delta(t-t_0)=0 & t\neq t_0 \\ \int_{-\infty}^{\infty}\delta(t-t_0)dt=1 \end{cases}$$

Dirac Impulse



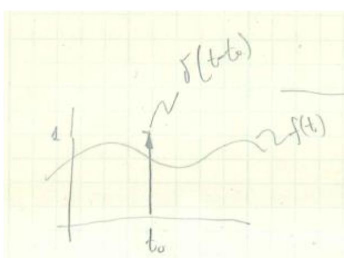
Kronecker Delta
Unit Impulse



- Laplace Transform

$$L\{\delta(t-t_0)\}=e^{-st_0}$$

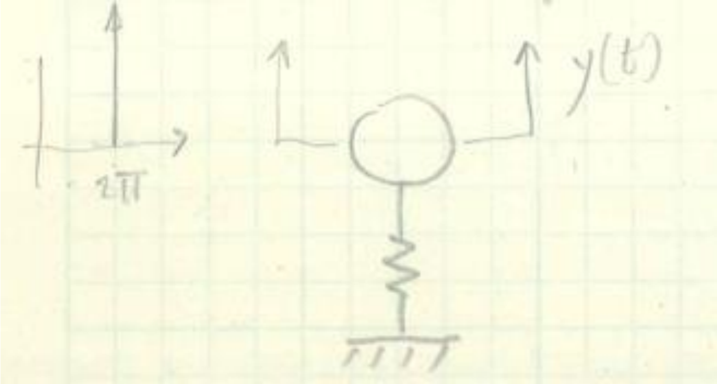
$$L\{\delta(t)\}\big|_{t=0}=1$$



$$\int_{-\infty}^{\infty} f(t-t_0)\delta(t-t_0)dt = \boxed{f(t-t_0)}$$

The value of the function at t=t_0

Transform of the Impulse Function - Example



$$y'' + y = 4\delta(t - 2\pi)$$

(a) $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$ (b) $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

Transform of the Impulse Function - Example

$$y'' + y = 4\delta(t - 2\pi) \qquad \text{(a)} \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$$s^2Y(s) - sy'(0) - y(0) + Y(s) = 4e^{-2\pi s}$$

$$s^2Y(s) - s + Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}$$

$$y(t) = \cos t + 4\sin(t - 2\pi)u(t - 2\pi)$$

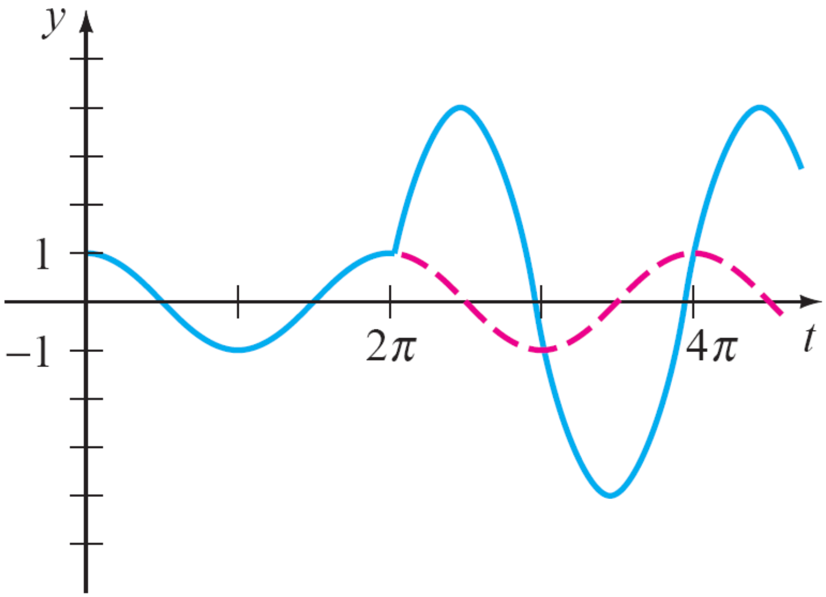
$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

Transform of the Impulse Function - Example

$$y'' + y = 4\delta(t - 2\pi) \qquad \text{(a)} \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$$y(t) = \cos t + 4\sin(t - 2\pi)u(t - 2\pi)$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$



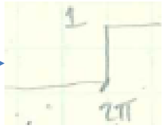
Transform of the Impulse Function - Example

$$y'' + y = 4\delta(t - 2\pi) \qquad \text{(b)} \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$s^2 Y(s) + Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1}$$

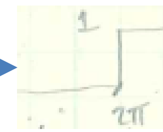
$$y(t) = 4\sin(t - 2\pi)u(t - 2\pi)$$



$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ 4\sin t & t \ge 2\pi \end{cases}$$

Transform of the Impulse Function - Example

$$y'' + y = 4\delta\left(t - 2\pi\right) \qquad \text{(b)} \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$



$$y(t) = 4\sin(t - 2\pi)u(t - 2\pi)$$

$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ 4\sin t & t \ge 2\pi \end{cases}$$