Class Notes 13:

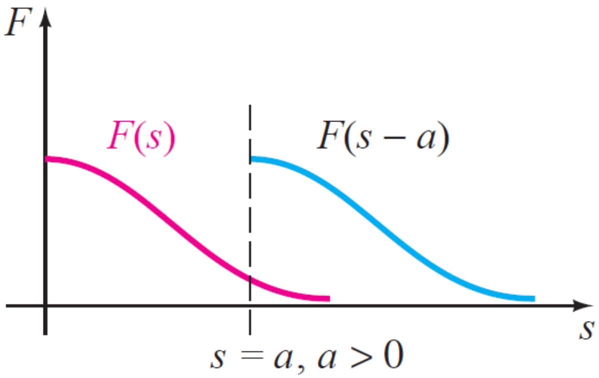
Laplace Transform (2/3)
Time and Frequency Shifts – Step Function

82 – Engineering Mathematics

Operation Properties – Translation on the S-Axis (Freq.) First Translation Theorem

$$L\{e^{at} f(t)\} = F(s-a) = L\{f(t)\}\big|_{s\to s-a}$$

$$L^{-1}\{F(s-a)\} = L^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at} f(t)$$



- Example

$$L\{e^{5t} t^3\} = L\{t^3\}\big|_{s\to s-5} = \left.\frac{3!}{s^n}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

Operation Properties – Translation on the t-Axis (Time)
Unite Step Function – Definition

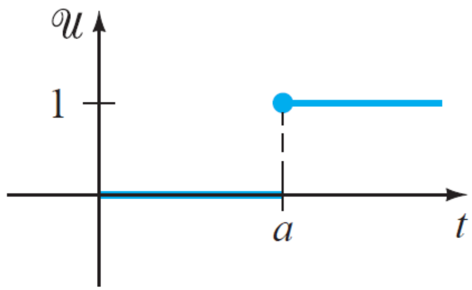
- Unit Step Function

OFF/ON
- Mechanical Force
- Electrical Voltage

Unit step function
Heaviside function

$$u(t-a)=\begin{cases}0 & 0\le t<a\\ 1 & t\ge a\end{cases}$$

Oliver Heaviside
(1850 - 1925)



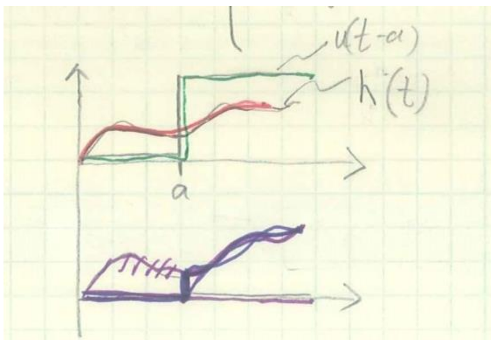
Only on the non-negative side of t-axis

$$L\{u(t-a)\}=\frac{e^{-as}}{s}$$

Single Piecewise Functions
Function Multiplied by Unite Step Function

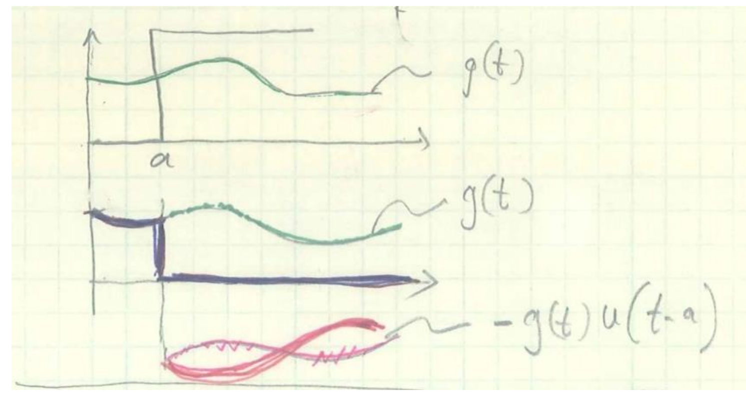
End of a function

$$u(t-a)h(t)=\begin{cases}0 & 0\le t<a\\ h(\mathrm{t}) & t\ge a\end{cases}$$



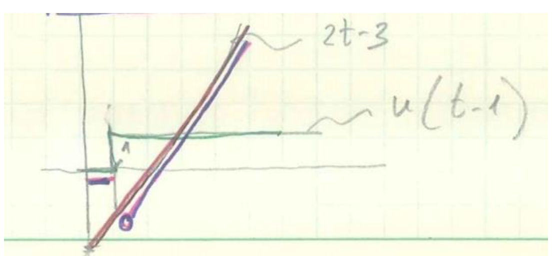
Beginning of a function

$$g(t)-g(t)u(t-a)=\begin{cases}g(\mathrm{t}) & 0\le t<a\\ 0 & t\ge a\end{cases}$$



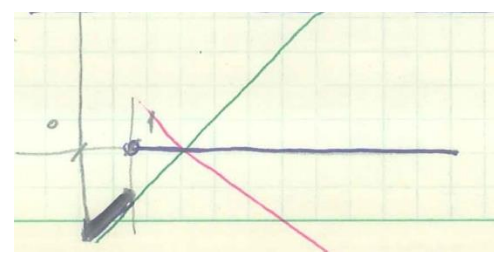
Example

$$u(t-1)(2t-3)$$



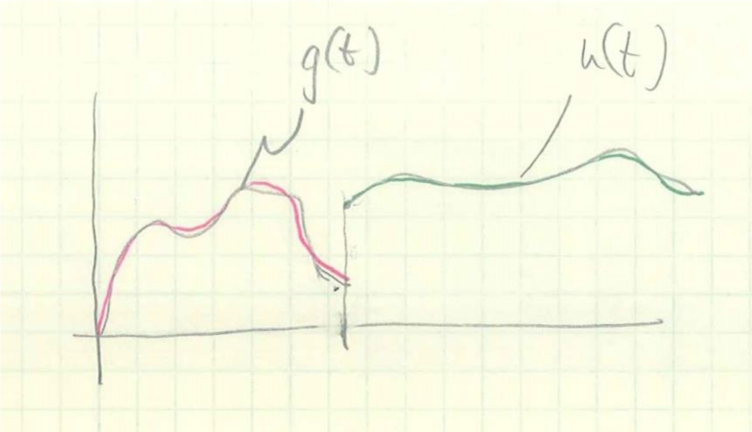
Example

$$(2t-3)-(2t-3)u(t-1)$$



Multiply Piecewise Functions - Example

$$f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

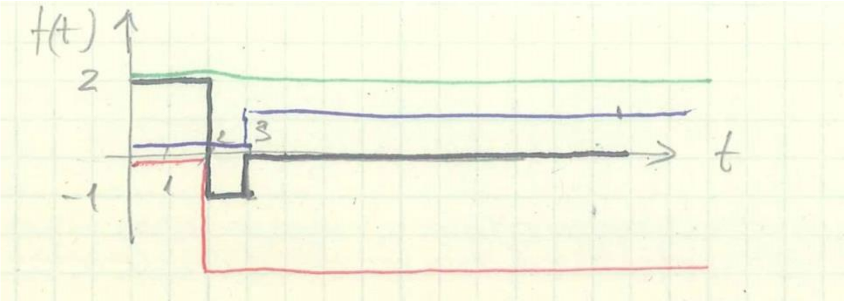


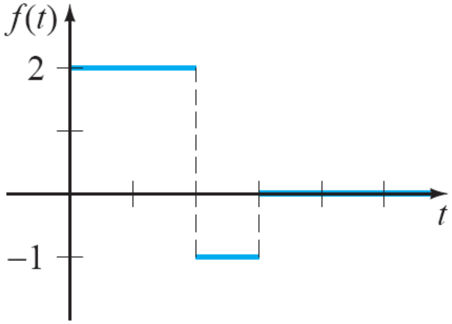
$$f(t) = \boxed{g(t) - g(t)u(t-a)} + \boxed{h(t)u(t-a)}$$

Multiply Piecewise Functions - Example

- Example

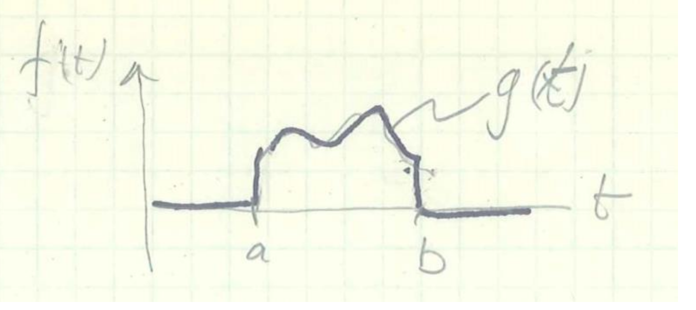
$$f(t)=\boxed{\boxed{2}-\boxed{3u(t-2)}+\boxed{u(t-3)}}$$



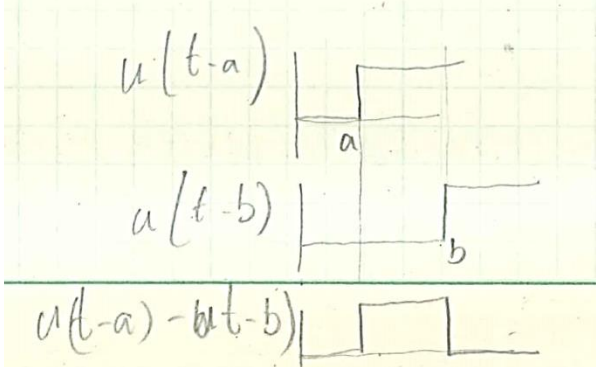


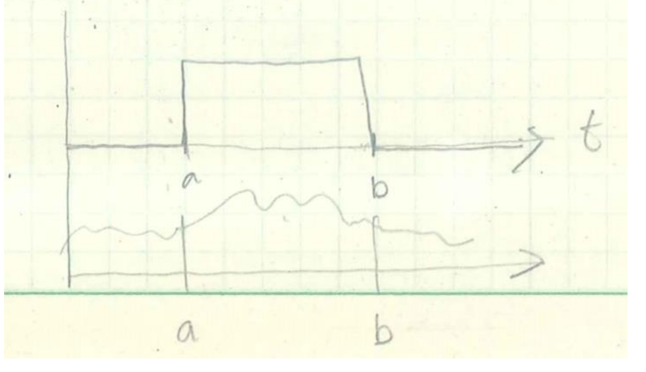
Multiply Piecewise Functions -
Functions Multiplied by Unite Step Functions

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ g(t) & a \le t < b \\ 0 & t \ge b \end{cases}$$



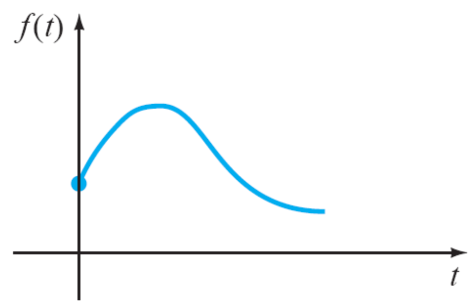
$$f(t) = g(t)\left[u(t-a) - u(t-b)\right]$$



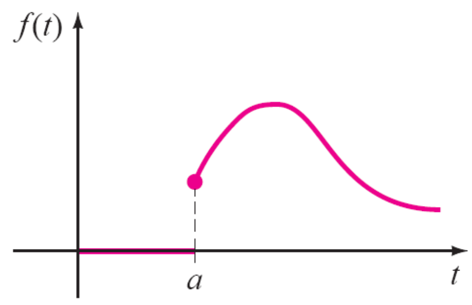


Operation Properties – Translation on the t-Axis (Time)
Second Translation Theorem

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$
$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$



(a) $f(t)$, $t \geq 0$



(b) $f(t-a)\mathcal{U}(t-a)$

Operation Properties – Translation on the t-Axis (Time)
Second Translation Theorem

Example

$$L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} \rightarrow \left\{\begin{array}{c} F(s)=1/(s-4) \\ a=2 \end{array}\right.$$

$$L^{-1}\left\{\frac{1}{s-4}\right\}\Bigg|_{t=t-2} = e^{4t}\Big|_{t=t-2}$$

$$L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}u(t-2)$$

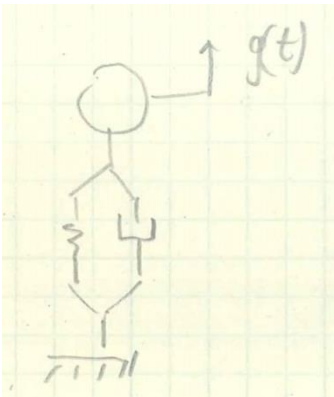
Example

$$L^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} \rightarrow \left\{\begin{array}{c} F(s)=\dfrac{s}{s^2+9} \\ a=\pi/2 \end{array}\right.$$

$$L^{-1}\left\{\frac{s}{s^2+9}\right\}\Bigg|_{3t=3t-\frac{\pi}{2}} = \cos(3t)\Big|_{3t=3t-\frac{\pi}{2}}$$

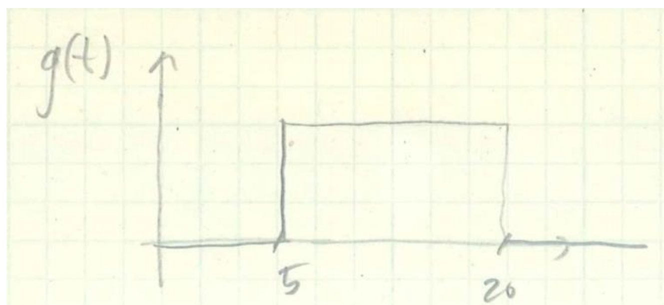
$$L^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos(3t-\frac{\pi}{2})u(t-\frac{\pi}{2})$$

Diff Eq. with Discontinuous Force Function



$$2y'' + y' + 2y = g(t) \qquad IC\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$g(t) = u(t-5) - u(t-20) = \begin{cases} 0 & 0 < t < 5 \\ 1 & 5 \le t < 20 \\ 0 & t \ge 20 \end{cases}$$



Laplace transform

$$\underbrace{2s^2Y(s) - 2sy(0) - 2y'(0)}_{L\{2y''\}} + \underbrace{sY(s) - y(0)}_{L\{y'\}} + \underbrace{2Y(s)}_{L\{2y\}} = \underbrace{(e^{-5s} - e^{-20s})/s}_{L\{g(t)\}}$$

Diff Eq. with Discontinuous Force Function

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = \underbrace{(e^{-5s} - e^{-20s})}_{\substack{\text{limit the function} \\ h(t)\,\text{between 5 to 20}}} \cdot \underbrace{\frac{1}{s(2s^2 + s + 2)}}_{H(s)}$$

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

If $h(t) = L^{-1}\{H(s)\}$ $\begin{cases} h(t)\big|_{t=t-5} \\ h(t)\big|_{t=t-20} \end{cases}$ then we have

$$y(t) = u(t-5)h(t-5) - u(t-20)h(t-20)$$

Determine h(t) by partial fraction expansion of H(s)

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}$$

$$a(2s^2 + s + 2) + s(bs + c) = 1$$

Diff Eq. with Discontinuous Force Function

$$2as^2 + as + 2a + bs^2 + cs = 1$$

$$(2a+b)s^2 + (a+c)s + 2a = 1$$

$$\left.\begin{aligned} 2a+b&=0 \\ a+c&=0 \\ 2a&=1 \end{aligned}\right\} \quad a = 1/2,\ b=-1,\ c=-1/2$$

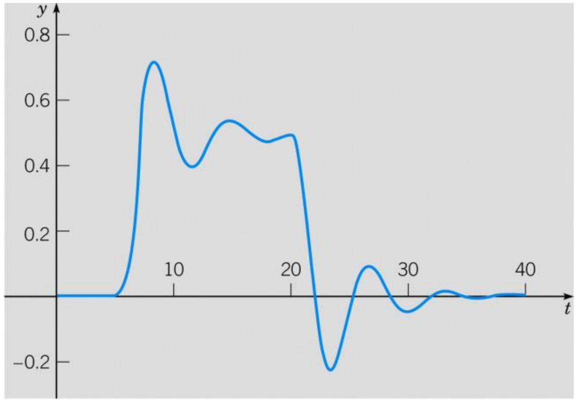
$$H(s) = \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} = \frac{1/2}{s} - \left(\frac{1}{2}\right)\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}$$

$$H(s) = \frac{1/2}{s} - \left(\frac{1}{2}\right)\left[\frac{s+1/4}{(s+1/4)^2+\left(\sqrt{15}/4\right)^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+1/4)^2+\left(\sqrt{15}/4\right)^2}\right]$$

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos\left(\sqrt{15}t/4\right) + \left(\sqrt{15}/15\right)e^{-t/4}\sin\left(\sqrt{15}t/4\right)\right]$$

Diff Eq. with Discontinuous Force Function

plot $$y(t) = u(t-5)h(t-5) - u(t-20)h(t-20)$$



Three distinct parts

$$0 < t < 5 \qquad 2y'' + y' + 2y = 0 \qquad IC\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

- No external force
- The initial conditions impart no energy

$$\rightarrow y = 0$$

Diff Eq. with Discontinuous Force Function

$5 \le t \le 20$

$$2y'' + y' + 2y = 1$$
$$y(5) = 0$$
$$y'(5) = 0$$

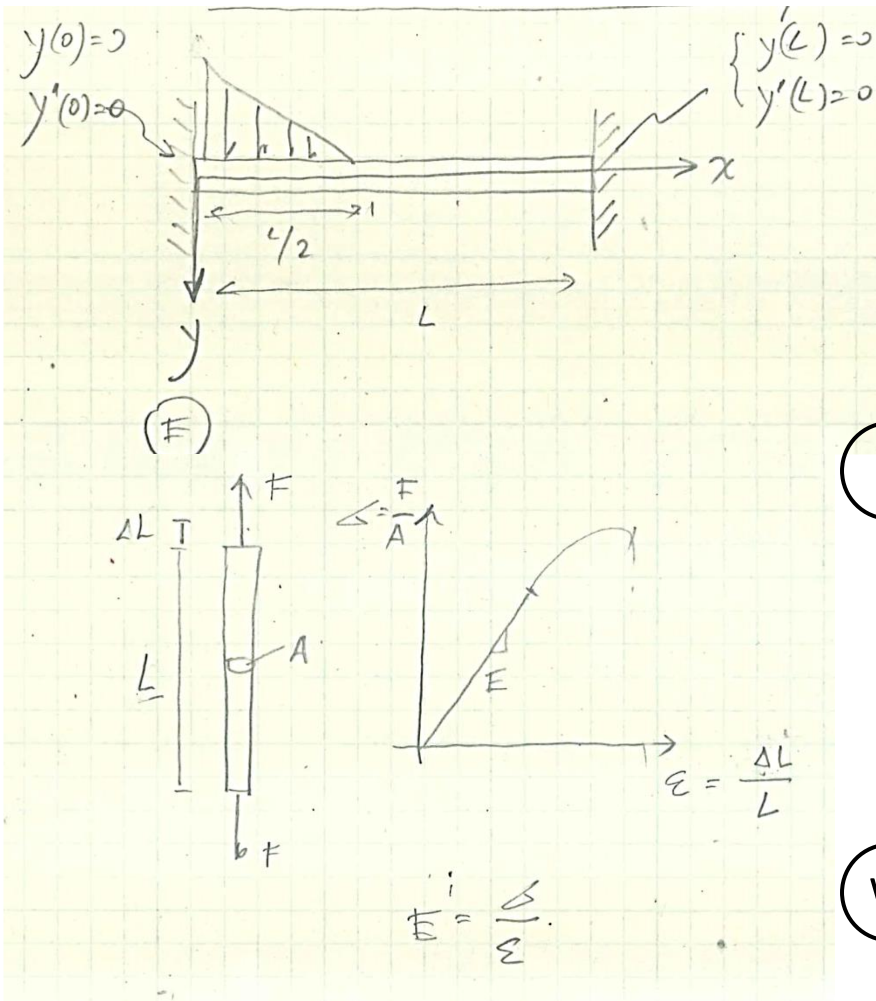
$t > 20$

$$2y'' + y' + 2y = 0$$

Find the initial conditions by evaluating the values of the solutions at t = 20

$$\begin{cases} y(20) \cong 0.50162 \\ y'(20) \cong 0.01125 \end{cases}$$

Boundary Value Problem – Example

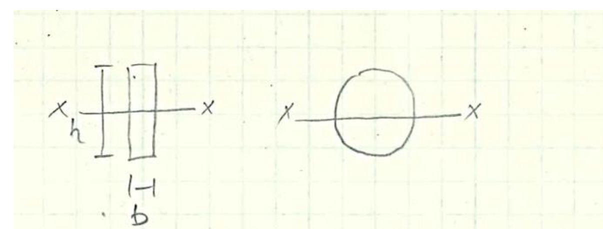


$$\text{EI}\ \frac{d^4y}{dx^4} = w(x)$$

E – Young Modulus

I – Moment of Inertia

I



$$I_{xx} = \frac{bh^3}{12} \qquad I = \frac{\pi}{4} r^4$$

W

$$w(x) = \begin{cases} w_0(1-2x/L) & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

Boundary Value Problem – Example

$$w(x) = w_0\left(1 - \frac{2}{L}x\right)\left[u(x) - u(x - \frac{L}{2})\right]$$

$$w(x) = w_0\left(1 - \frac{2}{L}x\right) - w_0\left(1 - \frac{2}{L}x\right)u\left(x - \frac{L}{2}\right)$$

$$w(x) = \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)u\left(x - \frac{L}{2}\right)\right]$$

$$\text{EI } y^{(4)} = w(x)$$

Laplace transform

$$\text{EI}\left(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right) = \frac{2w_0}{L}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

$$Let \begin{cases} c_1 = y''(0) \\ c_2 = y'''(0) \end{cases}$$

Boundary Value Problem – Example

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right]$$

$$y(x) = \frac{c_1}{2!}L^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{c_2}{3!}L^{-1}\left\{\frac{3!}{s^4}\right\} + \frac{2w_0}{EIL}\left[\frac{L/2}{4!}L^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{1}{5!}L^{-1}\left\{\frac{5!}{s^6}\right\} + \frac{1}{5!}L^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right]$$

$$= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 u\left(x - \frac{L}{2}\right)\right]$$

Applying the condition $y(L) = 0,\ y'(L) = 0$

$$\begin{cases} c_1\dfrac{L^2}{2} + c_2\dfrac{L^3}{6} + \dfrac{49w_0L^4}{1920EI} = 0 \\ c_1L + c_2\dfrac{L^2}{2} + \dfrac{85w_0L^3}{960EI} = 0 \end{cases} \quad c_1 = \frac{23w_0L^3}{960EI} \quad c_2 = \frac{-9w_0L}{40EI}$$

$$y(t) = \frac{23w_0L^2}{1920EI}x^2 - \frac{3w_0L}{80EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 u\left(x - \frac{L}{2}\right)\right]$$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s} F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all* s
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{df^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		