

Class Notes 13:

Laplace Transform (2/3)
Time and Frequency Shifts – Step Function

82 – Engineering Mathematics

Operation Properties – Translation on the t-Axis (Time)

Unit Step Function – Definition

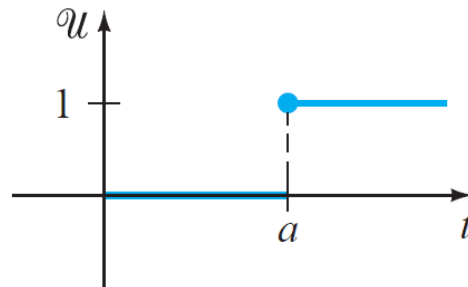
- Unit Step Function

OFF/ON { Mechanical Force
Electrical Voltage

Unit step function }
Heaviside function }

$$u_a = u(t - a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$

↑
Oliver Heaviside
(1850 - 1925)



Only on the non-negative side of t-axis

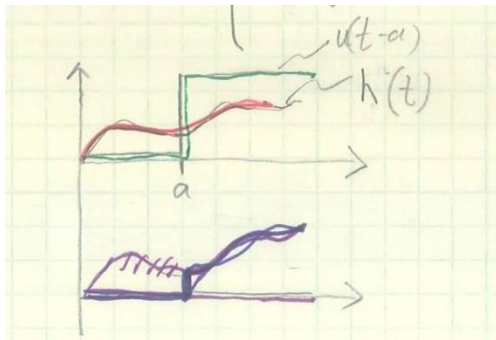
$$L\{u(t - a)\} = \frac{e^{-as}}{s}$$

Single Piecewise Functions

Function Multiplied by Unite Step Function

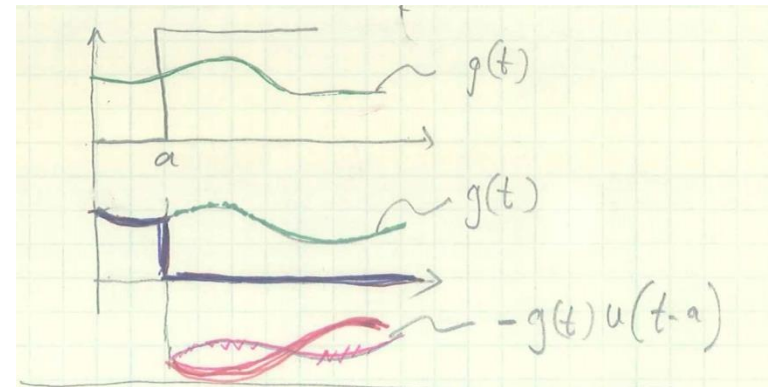
End of a function

$$u(t-a)h(t) = \begin{cases} 0 & 0 \leq t < a \\ h(t) & t \geq a \end{cases}$$



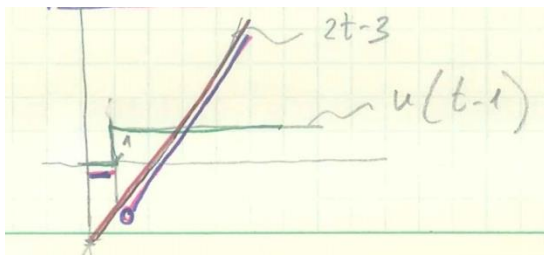
Beginning of a function

$$g(t) - g(t)u(t-a) = \begin{cases} g(t) & 0 \leq t < a \\ 0 & t \geq a \end{cases}$$



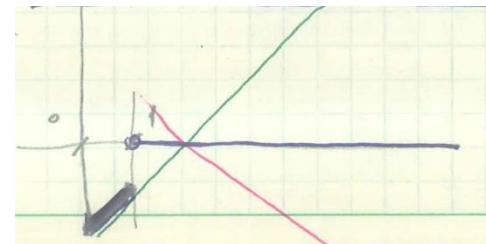
Example

$$u(t-1)(2t-3)$$



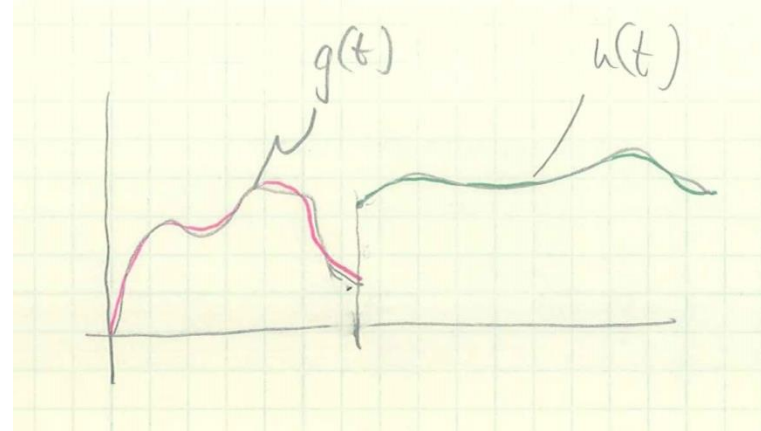
Example

$$(2t-3) - (2t-3)u(t-1)$$



Multiply Piecewise Functions - Example

$$f(t) = \begin{cases} g(t) & 0 \leq t < a \\ h(t) & t \geq a \end{cases}$$

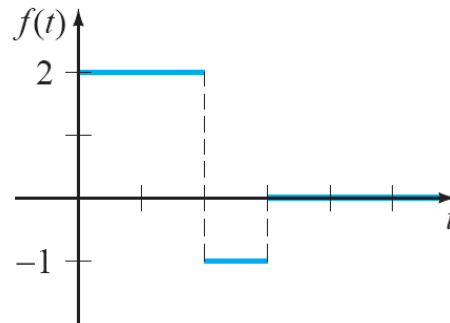
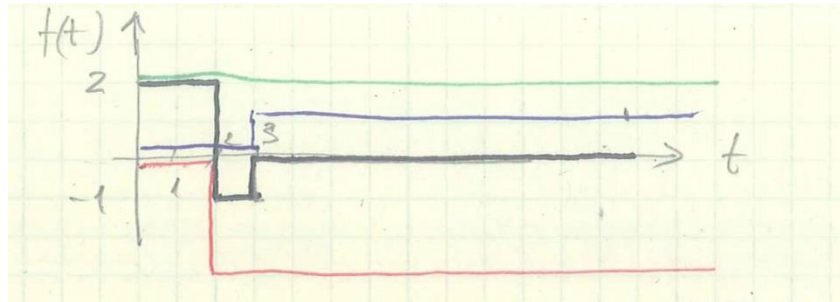


$$f(t) = \boxed{g(t) - g(t)u(t-a)} + \boxed{h(t)u(t-a)}$$

Multiply Piecewise Functions - Example

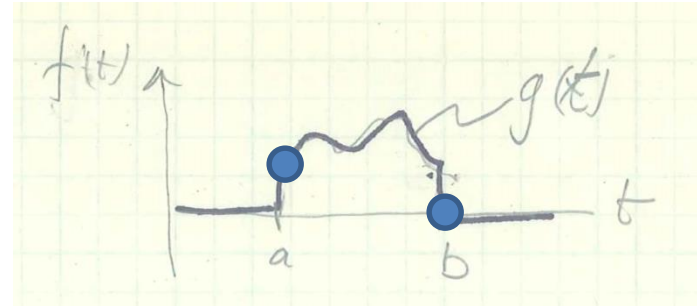
- Example

$$f(t) = 2 - 3u(t-2) + u(t-3)$$

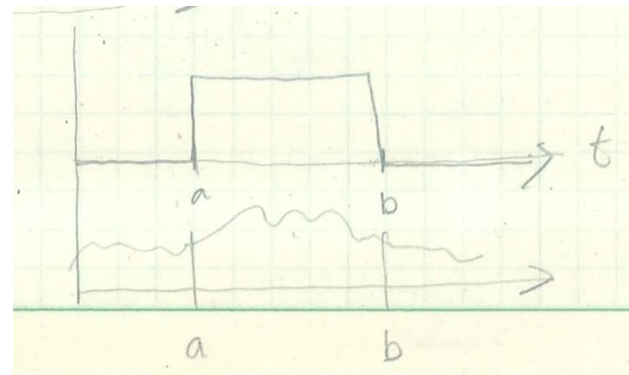
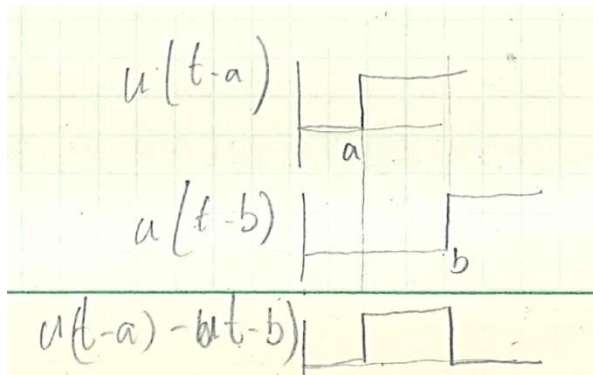


Multiply Piecewise Functions - Functions Multiplied by Unite Step Functions

$$f(t) = \begin{cases} 0 & 0 \leq t < a \\ g(t) & a \leq t < b \\ 0 & t \geq b \end{cases}$$



$$f(t) = g(t)[u(t-a) - u(t-b)]$$

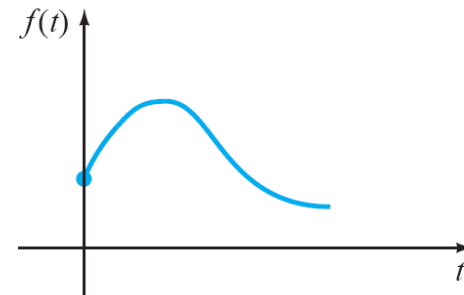


Operation Properties – Translation on the t-Axis (Time)

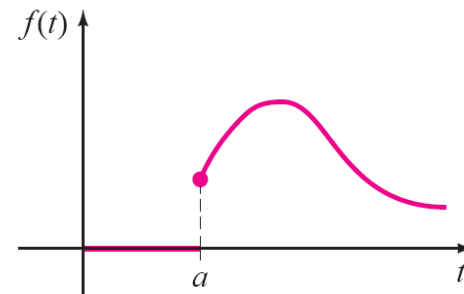
Second Translation Theorem

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$



(a) $f(t), t \geq 0$



(b) $f(t-a)u(t-a)$

Operation Properties – Translation on the t-Axis (Time)

Second Translation Theorem

Example $L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} \rightarrow \begin{cases} F(s) = 1/(s-4) \\ a = 2 \end{cases}$

$$L^{-1}\left\{\frac{1}{s-4}\right\}\Big|_{t=t-2} = e^{4t}\Big|_{t=t-2}$$

$$L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}u(t-2)$$

Example $L^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} \rightarrow \begin{cases} F(s) = \frac{s}{s^2+9} \\ a = \pi/2 \end{cases}$

$$L^{-1}\left\{\frac{s}{s^2+9}\right\}\Big|_{t=t-\frac{\pi}{2}} = \cos(3t)\Big|_{t=t-\frac{\pi}{2}}$$

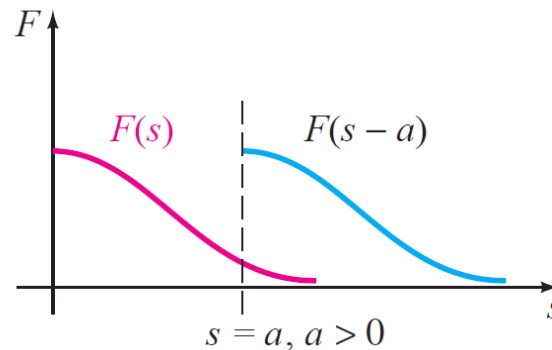
$$L^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3\left(t - \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)$$

Operation Properties – Translation on the S-Axis (Freq.)

First Translation Theorem

$$L\{e^{at} f(t)\} = F(s-a) = L\{f(t)\}\Big|_{s \rightarrow s-a}$$

$$L^{-1}\{F(s-a)\} = L^{-1}\{F(s)\Big|_{s-a \rightarrow s}\} = e^{at} f(t)$$

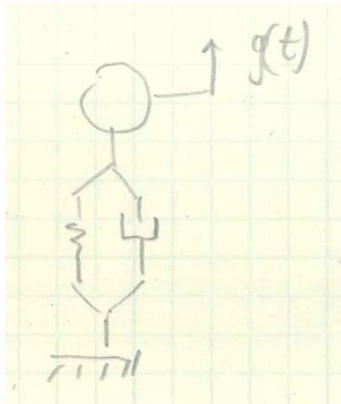


• Example:

$$L\{e^{5t} t^3\} = L\{t^3\}\Big|_{s \rightarrow s-5} = \frac{3!}{s^{n+1}}\Big|_{s \rightarrow s-5} = \frac{6}{(s-5)^4}$$

$$L^{-1}\left\{\frac{6}{(s-5)^4}\right\} = L^{-1}\left\{\frac{3!}{(s-5)^4}\right\} = L^{-1}\left\{\frac{3!}{s^{3+1}}\Big|_{s-5 \rightarrow s}\right\} = e^{5t} t^3$$

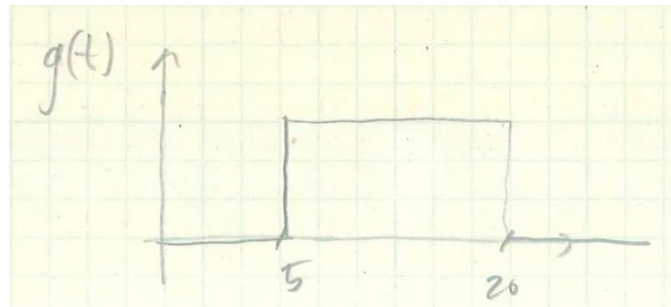
Diff Eq. with Discontinuous Force Function



$$2y'' + y' + 2y = g(t)$$

$$IC \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$g(t) = u(t-5) - u(t-20) = \begin{cases} 0 & 0 < t < 5 \\ 1 & 5 \leq t < 20 \\ 0 & t \geq 20 \end{cases}$$



Laplace transform

$$\underbrace{2s^2Y(s) - 2sy(0) - 2y'(0)}_{L\{2y''\}} + \underbrace{sY(s) - y(0)}_{L\{y'\}} + \underbrace{2Y(s)}_{L\{2y\}} = \underbrace{(e^{-5s} - e^{-20s}) / s}_{L\{g(t)\}}$$

Diff Eq. with Discontinuous Force Function

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)} = \underbrace{(e^{-5s} - e^{-20s})}_{\substack{\text{limit the function} \\ h(t) \text{ between } 5 \text{ to } 20}} \cdot \underbrace{\frac{1}{s(2s^2 + s + 2)}}_{H(s)}$$

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

If $h(t) = L^{-1}\{H(s)\} \begin{cases} h(t)|_{t=t-5} \\ h(t)|_{t=t-20} \end{cases}$ then we have

$$y(t) = u(t-5)h(t-5) - u(t-20)h(t-20)$$

Determine $h(t)$ by partial fraction expansion of $H(s)$

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}$$

$$a(2s^2 + s + 2) + s(bs + c) = 1$$

Diff Eq. with Discontinuous Force Function

$$2as^2 + as + 2a + bs^2 + cs = 1$$

$$(2a + b)s^2 + (a + c)s + 2a = 1$$

$$\left. \begin{array}{l} 2a + b = 0 \\ a + c = 0 \\ 2a = 1 \end{array} \right\} a = 1/2, b = -1, c = -1/2$$

$$H(s) = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2} = \frac{1/2}{s} - \left(\frac{1}{2}\right) \frac{(s + 1/4) + 1/4}{(s + 1/4)^2 + 15/16}$$

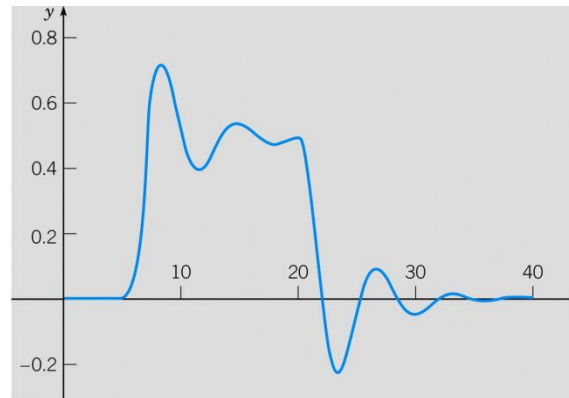
$$H(s) = \frac{1/2}{s} - \left(\frac{1}{2}\right) \left[\frac{s + 1/4}{(s + 1/4)^2 + (\sqrt{15}/4)^2} + \frac{1}{\sqrt{15}} \frac{\sqrt{15}/4}{(s + 1/4)^2 + (\sqrt{15}/4)^2} \right]$$

$$h(t) = \frac{1}{2} - \frac{1}{2} \left[e^{-t/4} \cos(\sqrt{15}t/4) + (\sqrt{15}/15) e^{-t/4} \sin(\sqrt{15}t/4) \right]$$

Diff Eq. with Discontinuous Force Function

plot

$$y(t) = u(t-5)h(t-5) - u(t-20)h(t-20)$$



Three distinct parts

$$0 < t < 5 \quad 2y'' + y' + 2y = 0 \quad IC \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

- No external force
- The initial conditions impart no energy

$$\rightarrow y = 0$$

Diff Eq. with Discontinuous Force Function

$$5 \leq t \leq 20 \quad 2y'' + y' + 2y = 1$$

$$y(5) = 0$$

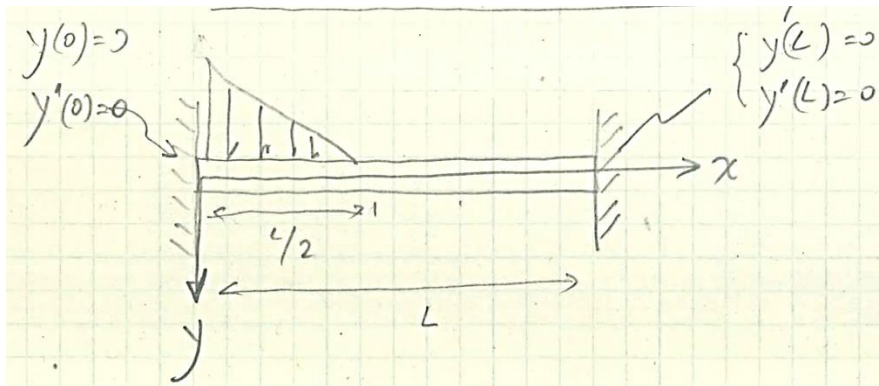
$$y'(5) = 0$$

$$t > 20 \quad 2y'' + y' + 2y = 0$$

Find the initial conditions by evaluating the values of the solutions at $t = 20$

$$\begin{cases} y(20) \cong 0.50162 \\ y'(20) \cong 0.01125 \end{cases}$$

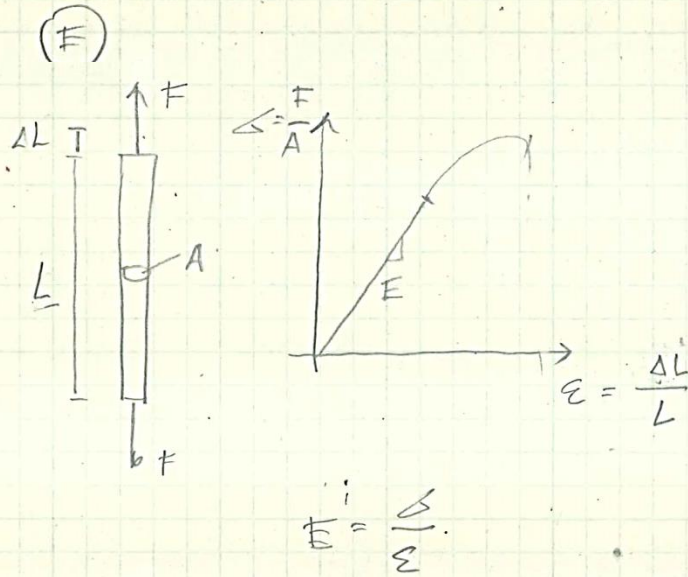
Boundary Value Problem – Example



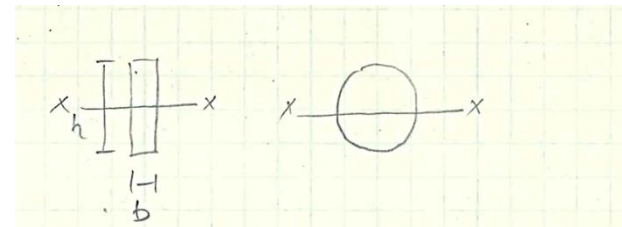
$$EI \frac{d^4 y}{dx^4} = w(x)$$

E – Young Modulus

I – Moment of Inertia



I



$$I_{xx} = \frac{bh^3}{12} \quad I = \frac{\pi}{4} r^4$$

W

$$w(x) = \begin{cases} w_0(1-2x/L) & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

Boundary Value Problem – Example

$$w(x) = w_0 \left(1 - \frac{2}{L} x \right) \left[u(x) - u\left(x - \frac{L}{2}\right) \right]$$

$$w(x) = w_0 \left(1 - \frac{2}{L} x \right) - w_0 \left(1 - \frac{2}{L} x \right) u\left(x - \frac{L}{2}\right)$$

$$w(x) = \frac{2w_0}{L} \left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right) u\left(x - \frac{L}{2}\right) \right]$$

$$EI y^{(4)} = w(x)$$

Laplace transform

$$EI \left(s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) \right) = \frac{2w_0}{L} \left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2} e^{-Ls/2} \right]$$

$$s^4 Y(s) - s y''(0) - y'''(0) = \frac{2w_0}{EIL} \left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2} e^{-Ls/2} \right]$$

$$\text{Let } \begin{cases} c_1 = y''(0) \\ c_2 = y'''(0) \end{cases}$$

Boundary Value Problem – Example

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL} \left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6} e^{-Ls/2} \right]$$

$$\begin{aligned} y(x) &= \frac{c_1}{2!} L^{-1} \left\{ \frac{2!}{s^3} \right\} + \frac{c_2}{3!} L^{-1} \left\{ \frac{3!}{s^4} \right\} + \frac{2w_0}{EIL} \left[\frac{L/2}{4!} L^{-1} \left\{ \frac{4!}{s^5} \right\} - \frac{1}{5!} L^{-1} \left\{ \frac{5!}{s^6} \right\} + \frac{1}{5!} L^{-1} \left\{ \frac{5!}{s^6} e^{-Ls/2} \right\} \right] \\ &= \frac{c_1}{2} x^2 + \frac{c_2}{6} x^3 + \frac{w_0}{60EIL} \left[\frac{5L}{2} x^4 - x^5 + \left(x - \frac{L}{2} \right)^5 u \left(x - \frac{L}{2} \right) \right] \end{aligned}$$

Applying the condition $y(L) = 0$, $y'(L) = 0$

$$\begin{cases} c_1 \frac{L^2}{2} + c_2 \frac{L^3}{6} + \frac{49w_0 L^4}{1920EI} = 0 \\ c_1 L + c_2 \frac{L^2}{2} + \frac{85w_0 L^3}{960EI} = 0 \end{cases} \quad \begin{cases} c_1 = \frac{23w_0 L^3}{960EI} \\ c_2 = \frac{-9w_0 L}{40EI} \end{cases}$$

$$y(t) = \frac{23w_0 L^2}{1920EI} x^2 - \frac{3w_0 L}{80EI} x^3 + \frac{w_0}{60EIL} \left[\frac{5L}{2} x^4 - x^5 + \left(x - \frac{L}{2} \right)^5 u \left(x - \frac{L}{2} \right) \right]$$

<i>Laplace transforms – Table</i>			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \omega $
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \omega $
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1, 2, 3, \dots$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
\sqrt{t}	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s} F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t) dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ unit impulse	1 all s
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$		