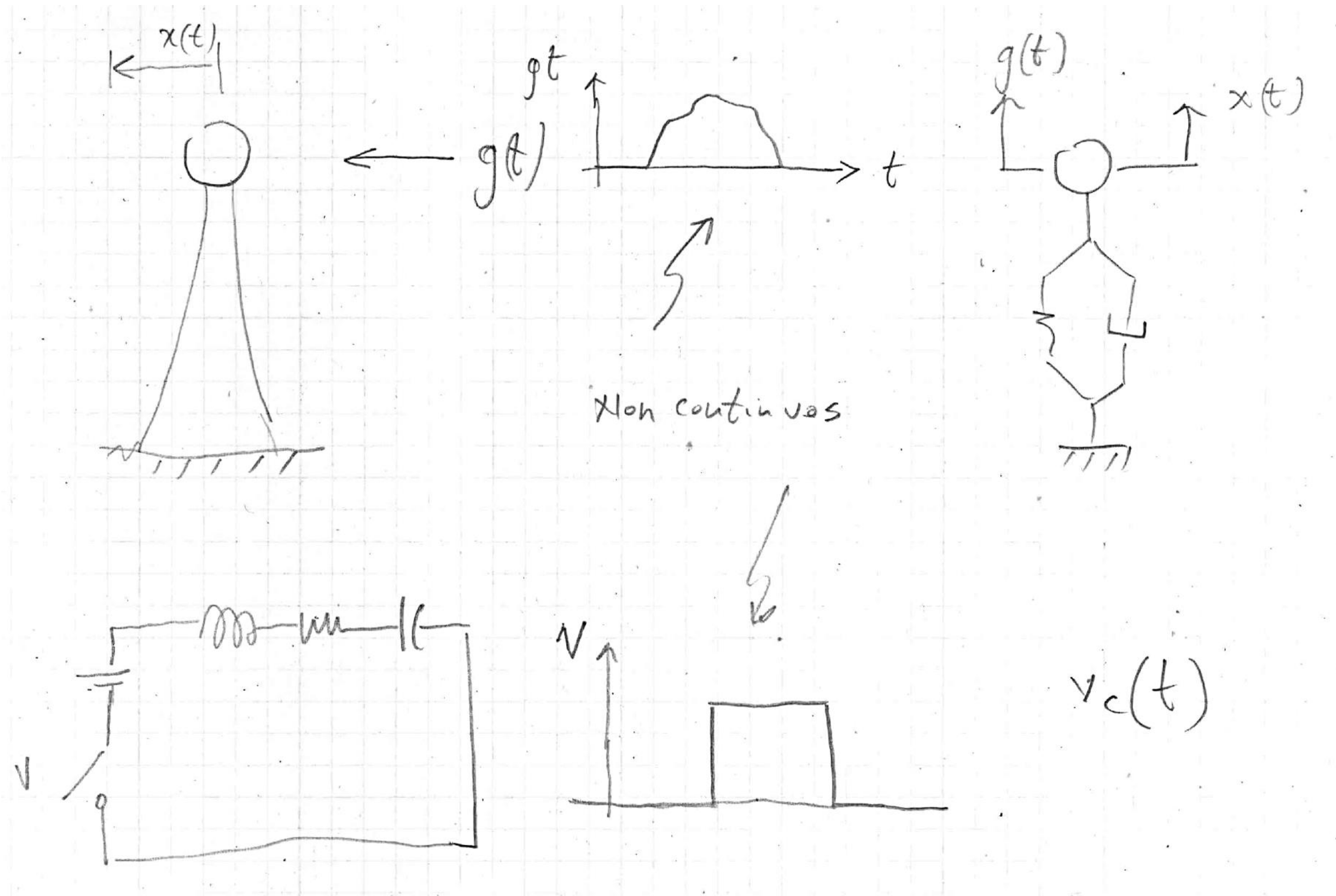
Class Notes 12:

Laplace Transform (1/3)
Intro & Differential Equations

82 – Engineering Mathematics

Laplace Transform – Introduction

Laplace Transform – Introduction



Pierre Simon Laplace (Marquis)

- French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy. He summarized and extended the work of his predecessors in his five volume Mécanique Céleste (Celestial Mechanics) (1799-1825). This seminal work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems.

- He formulated **Laplace's equation**, and invented the **Laplace transform.** In mathematics, the Laplace transform is one of the best known and most widely used **integral transforms**. It is commonly used to produce an easily solvable algebraic equation from an ordinary differential equation. It has many important applications in mathematics, physics, optics, **electrical engineering, control engineering**, signal processing, and probability theory.

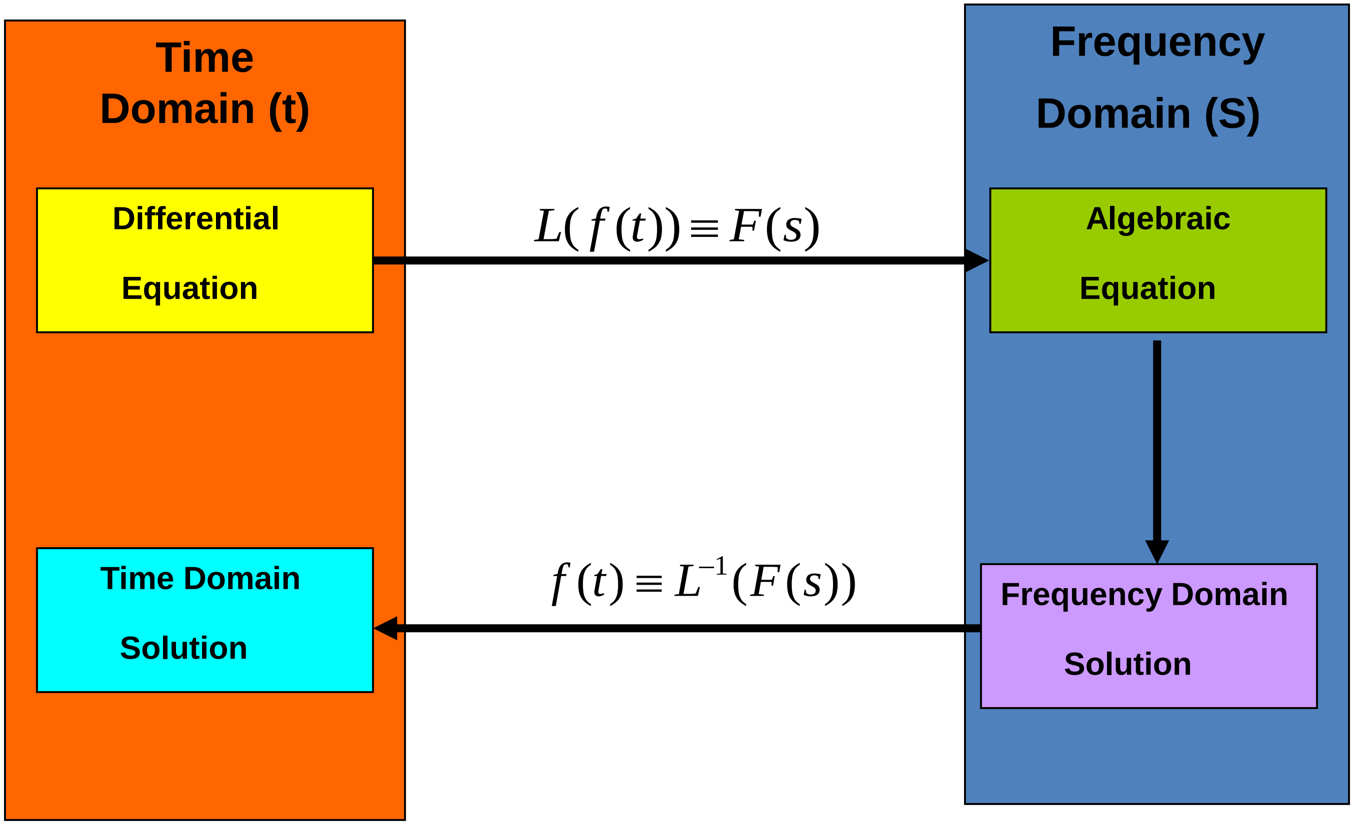
- He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

Pierre-Simon, marquis de Laplace
23 April 1749 - 5 March 1827

Video Clip

Laplace Transform – Motivation

- Transform between the time domain (t) to the frequency domain (s)
- Lossless Transform – No information is lost when the transform and the inverse transform is applied



Improper Integral – Definition

- An improper integral over an unbounded interval is defined as the limit of an integral over a finite interval

$$\int_a^{\infty} f(t)dt = \lim_{A\to\infty}\int_a^{A} f(t)dt$$

- where A is a positive real number.
- **If**
  - the integral from a to A exists for each A > a
  - the limit as A $\rightarrow \infty$ exists
- **then**
    - the improper integral is said to converge to that limiting value.
    - Otherwise, the integral is said to diverge or fail to exist.

Improper Integral – Example

$$\int_0^\infty e^{ct}dt = \lim_{A\to\infty}\int_0^A e^{ct}dt = \lim_{A\to\infty}\frac{e^{ct}}{c}\bigg|_0^A = \lim_{A\to\infty}\frac{1}{c}(e^{ct}-1)$$

$$= \begin{cases} c<0 & = -\dfrac{1}{c} \quad \text{converge} \\ c>0 & = \infty \quad \text{diverge} \\ c=0 & = 1 \end{cases}$$

Integral Transform - Laplace Transform – Definition

- Tool for solving linear diff. eq. – Integral transform

$$F(s) = \int_{\alpha}^{\beta} k(s,t) f(t) dt$$

k(s,t) - The kernel of the transformation

Given α, β ($\alpha = -\infty; \beta = +\infty$)

$$f \xrightarrow{\text{Transform}} F$$

- <u>Laplace transform</u>

Kernel $k(s,t) = e^{-st}$

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

whenever this improper integral converges

Laplace Transform – Theorem 6.1.2 – Sufficient Conditions for Existence

- Suppose that f is a function for which the following hold:
  (1) f is piecewise continuous on $[0, b]$ for all $b > 0$.
  (2) $|f(t)| \leq Me^{at}$ when $t \geq T$, with T, $M > 0$.

- Then the Laplace Transform of f exists for $s > a$.

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \ \text{ finite}$$

- Note: A function f that satisfies the conditions specified above is said to have **exponential order** as $t \to \infty$.

Laplace Transform – Theorem 6.1.2 – Condition No. 1 - Piecewise Continuous

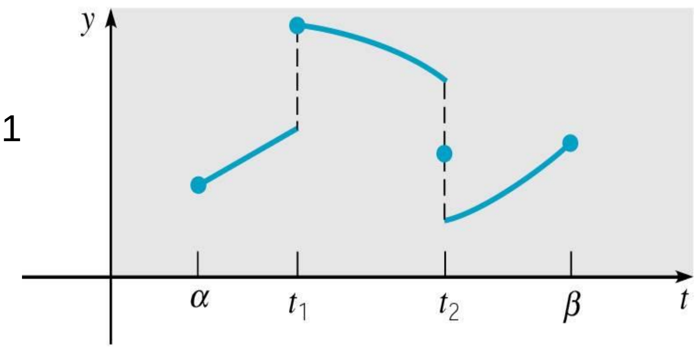
- A function *f* is **piecewise continuous** on an interval [*a*, *b*] if this interval can be partitioned by a finite number of points

  $a = t_0 < t_1 < \ldots < t_n = b$ such that

  (1) *f* is continuous on each $(t_k, t_{k+1}$

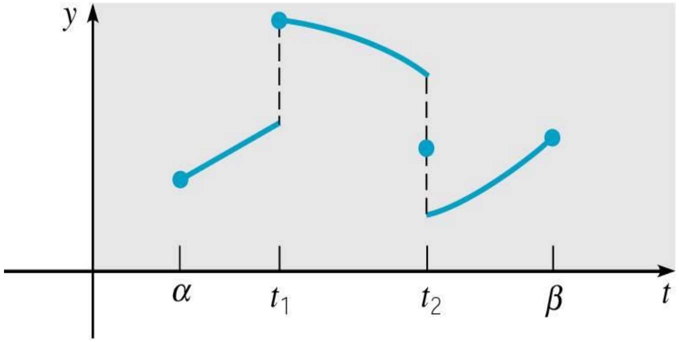
  $$(2)\ \left|\lim_{t \to t_k^+} f(t)\right| < \infty, \quad k = 0, \ldots, n-1$$

  $$(3)\ \left|\lim_{t \to t_{k+1}^-} f(t)\right| < \infty, \quad k = 1, \ldots, n$$



- In other words, *f* is piecewise continuous on [*a*, *b*] if it is continuous there except for a finite number of jump discontinuities.

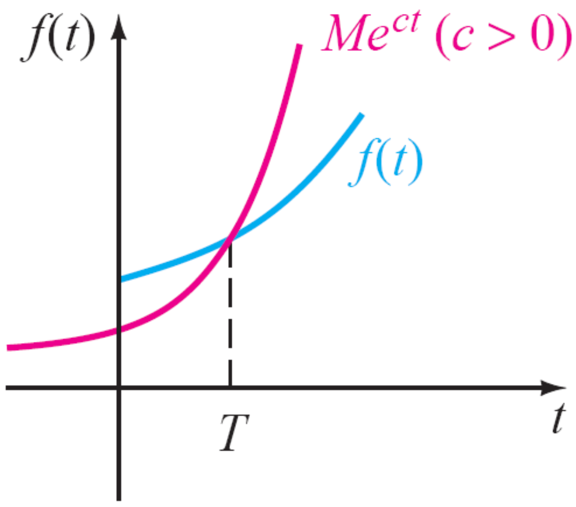
Laplace Transform – Theorem 6.1.2 – Condition No. 1 - Piecewise Continuous



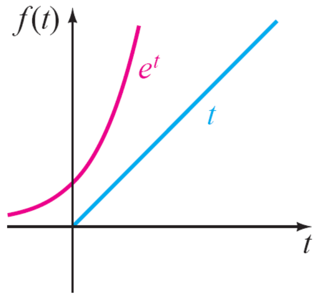
$$\int_{\alpha}^{\beta} f(t)dt = \int_{\alpha}^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \int_{t_2}^{\beta} f(t)dt$$

Laplace Transform – Theorem 6.1.2 – Condition No. 2 - Exponential Order

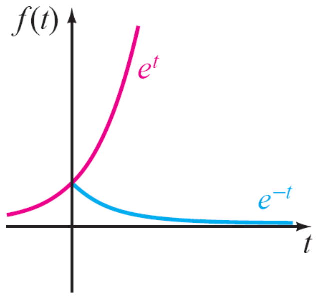
- If
  - f is an increasing function
- Then
  - $|\,f(t)\,| \leq Me^{at}$ when $t \geq T$, with T, $M > 0$.



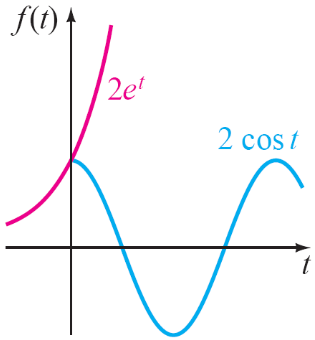
Laplace Transform – Theorem 6.1.2 – Condition No. 2 - Exponential Order



$$|t| \le e^t$$



$$\left|e^{-t}\right| \le e^t$$



$$|2\cos(t)| \le 2e^t$$

- A positive integral power of t is always of exponential order since for c>0

$$t^n \le Me^{ct}$$

$$\left|\frac{t^n}{e^{ct}}\right| \le M \quad for \;\; t > \tau$$

Laplace Transform (Function) – Example

$$L\{1\} = F(s) = \int_0^\infty e^{-st} 1 \quad dt$$

$$L\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \lim_{A \to \infty} \left. \frac{-e^{-st}}{s} \right|_0^A = \lim_{A \to \infty} \frac{-e^{-sA} + 1}{s} = \frac{1}{s}$$

Laplace Transform (Function) – Example

$$L\{e^{at}\} = F(s) = \int_0^\infty e^{-st} e^{at} \quad dt$$

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \lim_{A\to\infty} \left. \frac{-e^{-(s-a)t}}{s-a} \right|_0^A = \lim_{A\to\infty} \frac{-e^{-(s-a)A}+1}{s-a}$$

$$= \frac{1}{s-a}$$

Laplace Transform (Function) – Example

$$L\{t\} = F(s) = \int_0^\infty e^{-st} t \quad dt$$

$$\int uv' dt = uv - \int vu' dt$$

$$L\{t\} = \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{t}_{u} dt = \left.\frac{-te^{-st}}{s}\right|_0^\infty - \int_0^\infty (\underbrace{1}_{u'}) \left(\underbrace{-\frac{e^{-st}}{s}}_{v} \right) dt = 0 + \frac{1}{s}\int_0^\infty 1e^{-st}$$

$$= \frac{1}{s} L\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Laplace Transform (Function) – Example

$$L\{\sin(at)\} = F(s) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{\sin(at)}_{v'} dt$$

$$= \lim_{A\to\infty}\left[-\frac{e^{-st}\cos(at)}{a}\Bigg|_0^A - \frac{-s}{-a}\int_0^A \overbrace{e^{-st}}^{u'} \overbrace{\cos(at)}^{v} dt \right]$$

$$F(s) = \frac{1}{a} - \frac{s}{a}\int_0^A \underbrace{e^{-st}}_{u} \underbrace{\cos(at)}_{v'} dt = \frac{1}{a} - \frac{s^2}{a^2}\underbrace{\int_0^\infty e^{-st}\sin(at)dt}_{F(s)}$$

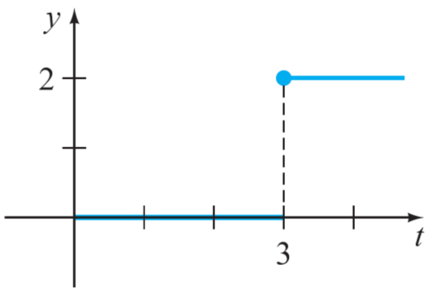
second integration by parts

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

$$F(s) = \frac{a}{s^2 + a^2}$$

Laplace Transform – Discontinues Function - Example

- Transformation of piecewise continuous function



$$f(t)=\begin{cases}0 & 0\le t<3\\ 2 & t>3\end{cases}$$

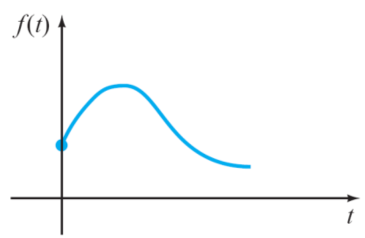
$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^3 e^{-st}(0)dt+\int_3^\infty e^{-st}2dt$$

$$=0+\left.\frac{2e^{-st}}{-s}\right|_0^\infty=\frac{2e^{-3s}}{s}\qquad s>0$$

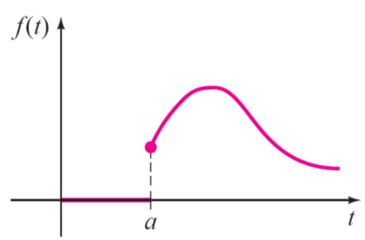
Operation Properties – Translation on the t-Axis (Time)
Second Translation Theorem

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$L^{-1}\{e^{-as}F(s)\} = L^{-1}\{F(s)\}\Big|_{t\to t-a} u(t-a) = f(t-a)u(t-a)$$



(a) $f(t),\ t \geq 0$

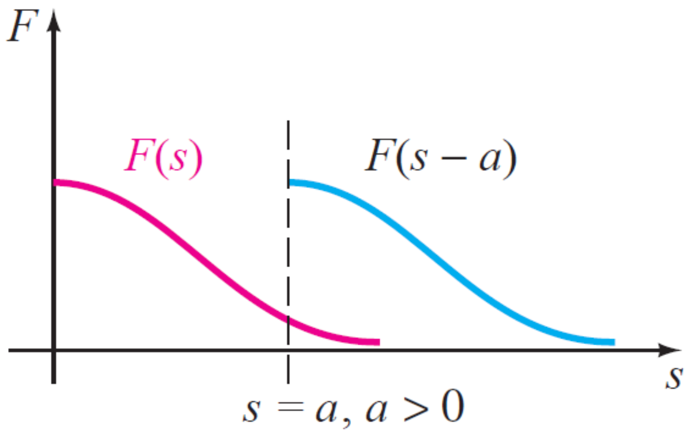


(b) $f(t-a)\mathcal{U}(t-a)$

Operation Properties – Translation on the S-Axis (Freq.) First Translation Theorem

$$L\{e^{at} f(t)\} = F(s-a) = L\{f(t)\}\Big|_{s \to s-a}$$

$$L^{-1}\{F(s-a)\} = L^{-1}\left\{F(s)\Big|_{s-a \to s}\right\} = e^{at} f(t)$$



- Example: $$L\{e^{5t} t^3\} = L\{t^3\}\Big|_{s \to s-5} = \frac{3!}{s^n}\bigg|_{s \to s-5} = \frac{6}{(s-5)^4}$$

Laplace Transform – Linearity

- Suppose *f* and *g* are functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively.
- Then, for *s* greater than the maximum of a_1 and a_2, the Laplace transform of $c_1 f(t) + c_2 g(t)$ exists. That is,

$$L\left\{c_1 f(t) + c_2 g(t)\right\} = \int_0^\infty e^{-st}\left[c_1 f(t) + c_2 g(t)\right]dt \quad \text{is finite}$$

with

$$\begin{aligned} L\left\{c_1 f(t) + c_2 g(t)\right\} &= c_1\int_0^\infty e^{-st} f(t)dt + c_2\int_0^\infty e^{-st} g(t)dt \\ &= c_1 L\left\{f(t)\right\} + c_2 L\left\{g(t)\right\} \end{aligned}$$

Laplace Transform – Linearity – Example

$$L\left\{5e^{-2t} - 3\sin(4t)\right\} = 5L\left\{e^{-2t}\right\} - 3L\left\{\sin(4t)\right\}$$

$$= \frac{5}{s+2} - \frac{12}{s^2+16}$$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2+\omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1+at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s}F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all s*
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2f}{df^2}$	$s^2F(s) - sf(0) - f'(0)$
$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		

Inverse Laplace Transform – Example

$$L^{-1}\left\{\frac{1}{S^5}\right\} = \frac{1}{4!} L^{-1}\left\{\frac{4!}{S^5}\right\} = \frac{1}{24} t^4$$

t^n	$\frac{n!}{s^{n+1}}$	$n = 1,2,3..$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$	$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$\frac{1}{2}t^2e^{-at}$	$\frac{1}{(s+a)^3}$	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
e^{at}	$\frac{1}{s-a} \quad s > a$	$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s-a)^2}$	$\cosh\omega t$	$\frac{s}{s^2-\omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\frac{1}{a^2}[1-e^{-at}(1+at)]$	$\frac{1}{s(s+a)^2}$	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
t^ne^{at}	$\frac{n!}{(s-a)^{n+1}} \quad s > a$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}} \quad s > a$	$1-e^{-at}$	$\frac{a}{s(s+a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t-t_1)$	$e^{-t_1 s}F(s)$
$g(t)\cdot p(t)$	$G(s)\cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all s*
$\frac{df}{dt}$	$sF(s)-f(0)$	$\frac{d^2f}{df^2}$	$s^2F(s)-sf(0)-f'(0)$
$\frac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{n-1}(0)$		

Inverse Laplace Transform – Example

$$L^{-1}\left\{\frac{1}{S^2+7}\right\}=\frac{1}{\sqrt{7}}L^{-1}\left\{\frac{\sqrt{7}}{S^2+7}\right\}=\frac{1}{\sqrt{7}}\sin\sqrt{7}t$$

$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

$Laplace\ transforms - Table$			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
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e^{-at}	$\frac{1}{s + a}$	$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s + a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$\frac{1}{2}t^2 e^{-at}$	$\frac{1}{(s + a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s + a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s - a} \qquad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \qquad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s - a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \qquad s > \lvert\omega\rvert$
$\frac{1}{b - a}(e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s + a)^2}$	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \qquad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}} \qquad s > a$	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s + a)^{n+1}} \qquad s > a$	$1 - e^{-at}$	$\frac{a}{s(s + a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s + a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \qquad s > 0$	$f(t - t_1)$	$e^{-t_1 s}F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)\ unit\ impulse$	$1 \qquad all\ s$
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{df^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)$		

Inverse Laplace Transform – Example

$$L^{-1}\left\{\frac{-2S+6}{S^2+4}\right\} = L^{-1}\left\{\frac{-2S}{S^2+4} + \frac{6}{S^2+4}\right\}$$

$$= -2L^{-1}\left\{\frac{S}{S^2+4}\right\} + \frac{6}{2}L^{-1}\left\{\frac{2}{S^2+4}\right\} \quad \text{(linearity)}$$

$$= -2\cos 2t + 3\sin 2t$$

Inverse Laplace Transform – Partial Fraction

$$L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

Partial Fraction:

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$=\frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$s^2+6s+9=A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

Option 1: 3 equations with 3 unknown variables A, B, C

$$s^2+6s+9=f_2(A,B,C)s^2+f_1(A,B,C)s+f_0(A,B,C)$$

$$f_2(A,B,C)=1$$

$$f_1(A,B,C)=6$$

$$f_0(A,B,C)=9$$

Inverse Laplace Transform – Partial Fraction

$$s^2+6s+9 = A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

Option 2:

Plug (s=1) 16=A(-1)(5) $\Longrightarrow$ A=-16/5
Plug (s=2) 25=B(1)(6) $\Longrightarrow$ B=25/6
Plug (s=-4) 1=C(-5)(-6) $\Longrightarrow$ C=1/30

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1}+\frac{25/6}{s-2}+\frac{1/30}{s+4}$$

$$L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}L^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

Laplace Transform of a Derivative (First)

$$L\{f'(t)\} = \boxed{\int_0^\infty \underbrace{e^{-st}}_{u}\underbrace{f'(t)}_{v'}dt} = \underbrace{e^{-st}}_{u}\underbrace{f(t)}_{v}\Bigg|_0^\infty - \int \underbrace{(-s)e^{-st}}_{u'}\underbrace{f(t)}_{v}dt$$

$$= e^{-st} f(t)\Big|_0^\infty - (-s)\int e^{-st} f(t)dt$$

$$\boxed{L\{f'(t)\} = sL\{f(t)\} - f(0) + = sF(s) - f(0)}$$

Laplace Transform of a Derivative (Second)

$$L\{f''(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f''(t)}_{v'} dt = \underbrace{e^{-st}}_{u} \underbrace{f'(t)}_{v} \Big|_0^\infty + s \boxed{\int_0^\infty \underbrace{e^{-st}}_{u'} \underbrace{f'(t)}_{v} dt}$$

$$L\{f''(t)\} = -f'(0) + sL\{f'(t)\} = s[sF(s) - f(0)] - f'(0)$$

$$\boxed{L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)}$$

Laplace Transform of a Derivative

first derivative

Initial Conditions

$$L\{f'(t)\} = sF(s) - f(0)$$

second derivative

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

third derivative

$$L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$\vdots$

n-th derivative

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Laplace Transform – Solving Linear ODEs

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots a_0 y = g(t)$$

- Initial Conditions: $y(0) = y_0; y'(0) = y_1; \cdots y^{(n-1)}(0) = y_{n-1}$

  $a_i \quad i = 0,1,\cdots,n$ - constants

  $y_0; y_1; \cdots y_{n-1}$ (I.C.s) - constants

Apply Laplace Transform

$$\begin{aligned}
&a_n\left[s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)\right] + \\
&a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0)\right] + \\
&\vdots \\
&+ a_0\left[Y(s)\right] = G(s)
\end{aligned}$$

Laplace Transform – Solving Linear ODEs

- The Laplace Transform of a linear differential equation with constant coefficients become an algebraic equation in Y(s)

Time Domain
Differential Equation

Laplace Transform →

Frequency Domain
Algebraic Equation

$$\underbrace{P(s)}_{\text{Polynimial}}\, Y(s) = \underbrace{G(s)}_{\substack{\text{Non-homo}\\\text{input}}} + \underbrace{Q(s)}_{\text{I.C.}}$$

$$Y(s) = \frac{G(s)}{P(s)} + \frac{Q(s)}{P(s)}$$

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{G(s)}{P(s)} + \frac{Q(s)}{P(s)}\right\}$$

Laplace Transform – Example First Order Linear ODEs

- Example: $\dfrac{dy}{dt} + 3y = 13\sin 2t \qquad y(0) = 6$

$$L\{y' + 3y\} = L\{13\sin 2t\}$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4}$$

$$(s+3)Y(s) = \frac{26}{s^2+4} + 6$$

$$Y(s) = \frac{26}{(s^2+4)(s+3)} + \frac{6}{s+3} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$As^2 + 4A + Bs^2 + 3Bs + Cs + 3C$$

$$= (A+B)s^2 + (3B+C)s + (4A+3C)$$

Laplace Transform – Example First Order Linear ODEs

$$\begin{cases} A + B = 6 \\ 3B + C = 0 \\ 4A + 3C = 50 \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = -2 \\ C = 6 \end{cases}$$

$$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4} = \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

$$y(t) = 8L^{-1}\left\{\frac{1}{s+3}\right\} - 2L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

Laplace Transform – Example Second Order Linear ODEs

- Example: $y'' - 3y' + 2y = e^{-4t} \qquad y(0) = 1;\ y'(0) = 5$

$$L\{y'' - 3y' + 2y\} = L\{e^{-4t}\}$$

$$\underbrace{s^2 Y(s) - sy(0) - y'(0)}_{L\{y''\}} - \underbrace{3[sY(s) - y(0)]}_{L\{3y'\}} + \underbrace{2Y(s)}_{L\{2y\}} = \underbrace{\frac{1}{s+4}}_{L\{e^{-4t}\}}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s+4)(s^2 - 3s + 2)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

(See previous example) $\downarrow$

$$y(t) = L^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$