Class Notes 11:

Power Series (3/3)
Series Solution Singular Point

182A – Engineering Mathematics

Series Solution Ordinary & Singular Point

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad \text{At} \quad x_0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \qquad \text{At} \quad x_0$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{At} \quad \boxed{x_0}$$

- x_0 Ordinary Point (OP)
- x_0 Singular Point (SP)
  - x_0 Regular Singular Point (RSP)
  - x_0 Irregular Singular Point (ISP)

Series Solution about Ordinary Point (OP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{At} \quad x_0$$

- Definition $\quad P(x) \neq 0$

- Solution (Typically 2) $\quad y\ (x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$

- Example $\quad y'' + xy = 0 \quad \text{At} \quad x_0 = 0$

Series Solution about Regular Singular Point (RSP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{At} \quad x_0$$

- Definition

$$AND \begin{cases} P(x) = 0 \\ \tilde{p} = \lim_{x \to x_0}\left[(x - x_0)\,p(x)\right] = \lim_{x \to x_0}\left[x - x_0 \frac{Q(x)}{P(x)}\right] & \text{Bounded} \\ \tilde{q} = \lim_{x \to x_0}\left[(x - x_0)^2 q(x)\right] = \lim_{x \to x_0}\left[(x - x_0)^2 \frac{R(x)}{P(x)}\right] & \text{Bounded} \end{cases}$$

- Solution (Typically 2) $y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$

  $y_2(x)$ See Cases (1,2,3)

- Example $xy'' + y = 0 \qquad \text{At} \quad x_0 = 0$

RSP - Second solution

Assume $r_1 > r_2$ case 1, 2; $r_1 = r_2$ case 3

Case 1 If $r_1,\ r_2$ are distinct

$(r_1 - r_2)$ is positive non integer

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} & c_0 \neq 0 \\ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} & b_0 \neq 0 \end{cases}$$

Case 2 If $r_1,\ r_2$ are distinct

$(r_1 - r_2)$ is a positive integer

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} & c_0 \neq 0 \\ y_2(x) = c y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{cases}$$

constant that can be zero

RSP - Second solution (Continue)

<u>Case 3</u> $r_1 = r_2$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_1}$$

Series Solution about Non Regular Singular Point (NSP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{At} \quad x_0$$

- Definition

$$AND \quad \begin{cases} P(x) = 0 \\ \text{One of the follwoing limits is unbounded} \\ \tilde{p} = \lim_{x \to x_0}\left[(x - x_0)p(x)\right] = \lim_{x \to x_0}\left[x - x_0 \frac{Q(x)}{P(x)}\right] \quad \text{Bounded} \\ \tilde{q} = \lim_{x \to x_0}\left[(x - x_0)^2 q(x)\right] = \lim_{x \to x_0}\left[(x - x_0)^2 \frac{R(x)}{P(x)}\right] \quad \text{Bounded} \end{cases}$$

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Classification of Singular Points – Example

$$\left(x^2-4\right)y''+3\left(x-2\right)y'+5y=0$$

$$y''+\frac{3\left(x-2\right)}{\underbrace{x^2-4}_{=\left(x-2\right)^2\left(x+2\right)^2}}y'+\frac{5}{x^2-4}y=0$$

$$p(x)=\frac{3\cancel{\left(x-2\right)}}{\left(x-2\right)^{\cancel{2}}\left(x+2\right)^2}\qquad q(x)=\frac{5}{\left(x-2\right)^2\left(x+2\right)^2}$$

$$\left.\begin{array}{l} x=2\rightarrow \text{Singular} \\ x=-2\rightarrow \text{Singular} \end{array}\right\}$$

which type of singular point (Regular/Irregular)

Classification of Singular Points – Example

- For $x_0=2$
  Factor (x-2)

$$\begin{cases} \tilde{p}(x) = (x-2)p(x) = \cancel{(x-2)}\,\dfrac{3\cancel{(x-2)}}{\cancel{(x-2)^2}(x+2)^2} \\ \tilde{q}(x) = (x-2)^2 q(x) = \cancel{(x-2)^2}\,\dfrac{5}{\cancel{(x-2)^2}(x+2)^2} \end{cases}$$

Both functions are analytic at x=0

$\Rightarrow$ Regular Singular Point (RSP)

- For $x_0=-2$
  Factor (x-(-2))
  =x+2

$$\tilde{p}(x) = (x+2)p(x) = \cancel{(x+2)}\,\frac{3\cancel{(x-2)}}{(x-2)^{\cancel{2}}(x+2)^{\cancel{2}}}$$

$\Rightarrow$ Non-analytic at $x_0=-2$

Irregular Singular Point (IRS)

Frobenius' Theorem

- If x= x_0 is a regular singular point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

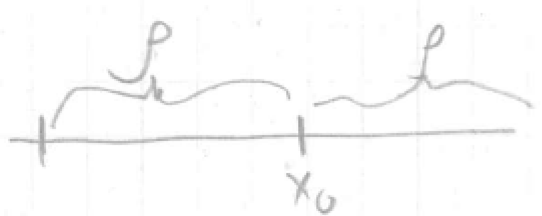
- Then there exists at least one solution of the form

$$y = \boxed{(x-x_0)}^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n\boxed{+r}}$$

where the number r is a constant to be determined.

The series will converge at least at some interval

$$0 < |x - x_0| < \rho$$



Solution Protocol

(1) Substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ into the given differential equation

(2) Determine the unknown coefficients c_n using reduction relations

(3) Find the unknown exponent r. (non-negative integer)

(4) For the sake of simplicity in solving the differential equation, the regular singular point is $x_0=0$

for any other regular singular point x_0

$$x \rightarrow (x - x_0)$$

RSP (Case 1) - Example (Two Series Solutions)

$$3xy'' + y' - y = 0$$

$$y'' + \frac{1}{3x} y' - \frac{1}{3x} y = 0$$

- What type of point x_0=0 is of the ideational equation?
  - Regular Point (RP)
  - Regular Singular Point (RSP)
  - Irregular Singular Point (IRS)

$$\left.\begin{aligned} \tilde{p}(x) &= \lim_{x\to 0}(x - x_0)\, p(x) = \lim_{x\to 0}(x - x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\left(\frac{1}{3x}\right) = \frac{1}{3} \\ \tilde{q}(x) &= \lim_{x\to 0}(x - x_0)^2 q(x) = \lim_{x\to 0}(x - x_0)\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\left(-\frac{1}{3x}\right) = \lim_{x\to 0} -\frac{x}{3} = 0 \end{aligned}\right\}$$

- Conclusion - Regular Singular Point (RSP)

RSP (Case 1) - Example (Two Series Solutions)

- Assuming a solution of the form

$$\begin{cases} y = \sum_{n=0}^{\infty} c_n x^{n+r} \\ y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\ y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \end{cases}$$

- Plug the solution into the differential equation

$$3xy'' + y' - y = 0$$

RSP (Case 1) - Example (Two Series Solutions)

$$3xy''+y'-y=\underbrace{3\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}}_{3xy''}+\underbrace{\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}}_{y'}-\underbrace{\sum_{n=0}^{\infty}c_nx^{n+r}}_{y}$$

$$=\sum_{n=0}^{\infty}(n+r)(3n+3r-2)c_nx^{n+r-1}-\sum_{n=0}^{\infty}c_nx^{n+r}$$

$$=x^r\left[\underbrace{r(3r-2)c_0x^{-1}}_{n=0}+\underbrace{\sum_{n=1}^{\infty}(n+r)(3n+3r-2)c_nx^{n-1}}_{\substack{k=n-1\\ n=k+1}}-\underbrace{\sum_{n=0}^{\infty}c_nx^{n}}_{k=n}\right]$$

$$=x^r\left[\underbrace{r(3r-2)c_0x^{-1}}_{=0}+\underbrace{\sum_{k=0}^{\infty}\left[(k+1+r)(3k+3r+1)c_{k+1}-c_k\right]x^{k}}_{=0}\right]=0$$

$$\Rightarrow\begin{cases} r(3r-2)c_0=0\\ (k+r+1)(3k+3r+1)c_{k+1}-c_k=0\end{cases}$$

RSP (Case 1) - Example (Two Series Solutions)

Because nothing is gained by taking c_0=0, we must then have

$$r(3r-2)=0 \Rightarrow \begin{cases} r_1 = \dfrac{2}{3} \\ r_2 = 0 \end{cases} \qquad (r_1 - r_2) = \frac{2}{3}$$

Case 1 – Positive Non-Integer

(*) $$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)} \qquad k = 0, 1, 2 \ldots$$

Plugging r_1, r_2 into (*)

(o) $$r_1 = \frac{2}{3} \Rightarrow c_{k+1} = \frac{c_k}{(3k+5)(k+1)} \qquad k = 0, 1, 2, 3, \ldots$$

(+) $$r_1 = 0 \Rightarrow c_{k+1} = \frac{c_k}{(k+1)(3k+1)} \qquad k = 0, 1, 2, 3, \ldots$$

RSP (Case 1) - Example (Two Series Solutions)

For (r_1=2/3) Eq (o)	For (r_2=0) Eq (+)
$c_1 = \frac{c_0}{5 \cdot 1}$	$c_1 = \frac{c_0}{1 \cdot 1}$
$c_2 = \frac{c_1}{8 \cdot 2} = \frac{c_0}{2! 5 \cdot 8}$	$c_2 = \frac{c_1}{2 \cdot 4} = \frac{c_0}{2! 1 \cdot 4}$
$c_3 = \frac{c_2}{11 \cdot 3} = \frac{c_0}{3! 5 \cdot 8 \cdot 11}$	$c_3 = \frac{c_2}{3 \cdot 7} = \frac{c_0}{3! 1 \cdot 4 \cdot 7}$
$c_4 = \frac{c_3}{14 \cdot 4} = \frac{c_0}{4! 5 \cdot 8 \cdot 11 \cdot 14}$	$c_4 = \frac{c_3}{4 \cdot 10} = \frac{c_0}{4! 1 \cdot 4 \cdot 7 \cdot 10}$
$\vdots$	$\vdots$
$c_n = \frac{\boxed{c_0}}{n! 5 \cdot 8 \cdot 11 \cdot 14 \cdots (3n+2)}$	$c_n = \frac{\boxed{c_0}}{n! 1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}$

- We encounter something that didn't happen when we obtained solutions about an ordinary point. The two sets of coefficients contains the same multiple c_0. We can omit the term c_0

RSP (Case 1) - Example (Two Series Solutions)

r=2/3 $$y_1(x) = x^{\frac{2}{3}}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,5\cdot 8\cdot 11\cdots(3n+2)}x^n\right]$$

r=0 $$y_2(x) = x^{0}\left[1+\sum_{n=1}^{\infty}\frac{1}{n!\,5\cdot 8\cdot 11\cdots(3n+2)}x^n\right]$$

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Using the ratio test both $y_1(x)$ and $y_2(x)$ converge for all values of x
  $|x| < \infty$ (except the origin)

- $y_1(x)$ and $y_2(x)$ are linearly independent for all $|x| < \infty$

RSP - Indicial Equation

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad r(3r-2) = 0 \quad \begin{cases} r_1 = 2/3 \\ r_2 = 0 \end{cases}$$

- Indicial equation
- $r_1,\ r_2$ Indicial roots or exponents

$\Big\}$ of the singularity $x_0 = 0$

- The indicial equation is typically quadratic equation
- It is possible to obtain the indicial equation in advance of substituting

$y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the diff. eq.

RSP - Indicial Equation

Obtain the indicial equation in advance of substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

into the diff. eq.

- If x = 0 is a regular singular point of $y'' + p(x)y' + q(x)y = 0$

- Than $\left.\begin{aligned} \tilde{p}(x) &= xp(x) \\ \tilde{q}(x) &= x^2 q(x) \end{aligned}\right\}$ are analytic at x = 0

- That is the power series expansions

$$\tilde{p}(x) = xp(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$\tilde{q}(x) = x^2 q(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

are valid or intervals that have a positive radius at convergence ρ

RSP - Indicial Equation

- By multiplying $y'' + p(x)y' + q(x)y = 0$ by x^2 we get

$$x^2 y'' + x\underbrace{\left[xp(x)\right]}_{\tilde{p}(x)}y' + \underbrace{\left[x^2 q(x)\right]}_{\tilde{q}(x)}y = 0$$

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and the two series for $\tilde{p}(x), \tilde{q}(x)$

we find that the general indicial equation to be

$$r(r-1) + \boxed{a_0} r + \boxed{b_0} = 0$$

where a_0, b_0 are defined by $\tilde{p}(x), \tilde{q}(x)$

$$\tilde{p}(x) = xp(x) = \boxed{a_0} + a_1 x + a_2 x^2 + \cdots$$

$$\tilde{q}(x) = x^2 q(x) = \boxed{b_0} + b_1 x + b_2 x^2 + \cdots$$

RSP (Case 2)- Example
Indicial Equation (Direct Derivation)

$$xy'' + y = 0 \qquad x_0 = 0$$

$$y'' + \underbrace{0}_{p(x)} y' + \underbrace{\frac{1}{x}}_{q(x)} y = 0$$

$$\begin{cases} \tilde{p}(x) = xp(x) = x \cdot 0 = 0 \quad = \overbrace{0}^{a_0} + \overbrace{0}^{a_1}x + \overbrace{0}^{a_2}x^2 + \cdots \\ \tilde{q}(x) = x^2 q(x) = x^2 \dfrac{1}{x} = x \ = \underbrace{0}_{b_0} + \underbrace{1}_{b_1}x + \underbrace{0}_{b_2}x^2 + \cdots \end{cases}$$

$$a_0 = 0; \quad b_0 = 0$$

The indicial equation is

$$r(r-1) = 0 \quad \begin{matrix} \nearrow & r_2 = 0 \\ \searrow & r_1 = 1 \end{matrix} \qquad (r_1 - r_2) = 1$$

Case 2 – Positive Integer

RSP (Case 2) - Example (only one series solution)

$$xy'' + y = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$x^r\left[r(r-1)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n}\right]$$

$$\begin{cases} k = n-1 \\ n = k+1 \end{cases} \qquad n = r$$

$$x^r\left[r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}(k+1+r)(k+r)c_{k+1} x^{k} + \sum_{k=0}^{\infty} c_k x^{k}\right]$$

$$x^r\left[r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty}\left[(k+r+1)(k+r)c_{k+1} + c_k\right]x^{k}\right]$$

$$\begin{cases} r(r-1) = 0 \\ (k+r+1)(k+r)c_{k+1} + c_k = 0 \end{cases} \qquad c_{k+1} = -\frac{c_k}{(k+r+1)(k+r)}$$

RSP (Case 2) - Example (only one series solution)

$$r = 1$$

$$c_{k+1} = -\frac{c_k}{(k+2)(k+1)} \qquad k = 0, 1, 2, 3 \cdots$$

$$k = 0 \quad c_1 = -\frac{c_0}{2 \cdot 1}$$

$$k = 1 \quad c_2 = -\frac{c_1}{3 \cdot 2} = +\frac{c_0}{3 \cdot 2 \cdot 2 \cdot 1}$$

$$k = 2 \quad c_3 = -\frac{c_2}{4 \cdot 3} = -\frac{c_0}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

$$k = 3 \quad c_4 = -\frac{c_3}{5 \cdot 4} = +\frac{c_0}{5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

$$\vdots$$

$$k = n \quad c_n = \frac{(-1)^n}{n!(n+1)!}$$

$$r = 0$$

$$c_{k+1} = -\frac{c_k}{(k+1)(k)} \qquad k = 0, 1, 2, 3 \cdots$$

$$c_1 = -\frac{c_0}{1^2 \cdot 0} = \text{undefined}$$

$$c_2 = -\frac{c_1}{2 \cdot 1}$$

$$c_3 = -\frac{c_2}{3 \cdot 2} = +\frac{c_0}{3 \cdot 2 \cdot 2 \cdot 1}$$

$$c_4 = -\frac{c_3}{4 \cdot 3} = -\frac{c_0}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

RSP (Case 2) - Example (only one series solution)

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$r = 1$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$c_0 x^1 + c_1 x^2 + c_2 x^3$$

$$r = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3$$

$$c_0 = c_1$$

$$c_1 = c_2$$

$$c_2 = c_3$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2} x^2 + \frac{1}{12} x^3 - \frac{1}{144} x^4 + \cdots$$

RSP (Case 2) - Second Solution

Finding a second solution ($r_1 - r_2$ positive integer case II)

- We may or may not be able to find a second solution depends on
  - Indicial roots
  - Recurrence relation of c_n

- c may be equal to 0 if

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \\ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{cases}$$

We may use the following term for the second solution

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1{}^2 x} dx$$

RSP (Case 2) - Second Solution

$$xy'' + y = 0$$

$$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

Construct a second solution $y_2(x)$ using the formula

$$y_2(x) = y_1 \int \frac{e^{\int 0\,dx}}{(y_1)^2} = y_1 \int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right]^2}$$

$$= y_1 \int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots\right]}$$

$$= y_1 \int \left[\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x\right] dx$$

Example (Finding a second solution)

$$= y_1\left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right]$$

$$= y_1 \ln x + y_1\left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right]$$