

Class Notes 11:

Power Series (3/3)
Series Solution Singular Point

182A – Engineering Mathematics

Series Solution Ordinary & Singular Point

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \text{At } x_0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \quad \text{At } x_0$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{At } x_0$$

x_0 Ordinary Point (OP)

x_0 Singular Point (SP)

x_0 Regular Singular Point (RSP)

x_0 Irregular Singular Point (ISP)

Series Solution about Ordinary Point (OP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{At } x_0$$

- Definition

$$P(x) \neq 0$$

- Solution (Typically two solutions)

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

- Example

$$y'' + xy = 0 \quad \text{At } x_0 = 0$$

Series Solution about Regular Singular Point (RSP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{At } x_0$$

- Definition

$$\text{AND } \left\{ \begin{array}{l} P(x) = 0 \\ \tilde{p} = \lim_{x \rightarrow x_0} [(x - x_0)p(x)] = \lim_{x \rightarrow x_0} \left[x - x_0 \frac{Q(x)}{P(x)} \right] \quad \text{Bounded} \\ \tilde{q} = \lim_{x \rightarrow x_0} [(x - x_0)^2 q(x)] = \lim_{x \rightarrow x_0} \left[(x - x_0)^2 \frac{R(x)}{P(x)} \right] \quad \text{Bounded} \end{array} \right.$$

- Solution (Typically 2) $y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$
 $y_2(x)$ See Cases (1,2,3)

- Example $xy'' + y = 0 \quad \text{At } x_0 = 0$

RSP - Second solution

Assume $r_1 > r_2$ case 1, 2; $r_1 = r_2$ case 3

Case 1 If r_1, r_2 are distinct

$(r_1 - r_2)$ is positive non integer

$$\left\{ \begin{array}{l} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad c_0 \neq 0 \\ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \quad b_0 \neq 0 \end{array} \right.$$

Case 2 If r_1, r_2 are distinct

$(r_1 - r_2)$ is a positive integer

$$\left\{ \begin{array}{l} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad c_0 \neq 0 \\ y_2(x) = \underset{\substack{\uparrow \\ \text{constant that can be zero}}}{c} y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{array} \right.$$

RSP - Second solution (Continue)

Case 3

$$r_1 = r_2$$

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$$

$$y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_1}$$

Series Solution about Non Regular Singular Point (NSP) x_0

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{At } x_0$$

- **Definition**

$$\text{AND } \left\{ \begin{array}{l}
 P(x) = 0 \\
 \text{One of the following limits is unbounded} \\
 \tilde{p} = \lim_{x \rightarrow x_0} [(x - x_0)p(x)] = \lim_{x \rightarrow x_0} \left[x - x_0 \frac{Q(x)}{P(x)} \right] \quad \text{Unbounded} \\
 \tilde{q} = \lim_{x \rightarrow x_0} [(x - x_0)^2 q(x)] = \lim_{x \rightarrow x_0} \left[(x - x_0)^2 \frac{R(x)}{P(x)} \right] \quad \text{Unbounded}
 \end{array} \right.$$

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Classification of Singular Points – Example

$$(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0$$

$$y'' + \frac{3(x-2)}{\underbrace{(x^2-4)^2}_{=(x-2)^2(x+2)^2}} y' + \frac{5}{(x^2-4)^2} y = 0$$

$$p(x) = \frac{3\cancel{(x-2)}}{(x-2)\cancel{(x-2)}(x+2)^2}$$

$$q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

$x = 2 \rightarrow$ Singular } which type of singular point
 $x = -2 \rightarrow$ Singular } (Regular/Irregular)

Classification of Singular Points – Example

- For $x_0=2$
Factor $(x-2)$

$$\left\{ \begin{array}{l} \tilde{p}(x) = \lim_{x_0 \rightarrow 2} (x-2)p(x) = \lim_{x_0 \rightarrow 2} \cancel{(x-2)} \frac{3\cancel{(x-2)}}{\cancel{(x-2)}^2 (x+2)^2} = \frac{3}{16} \\ \tilde{q}(x) = \lim_{x_0 \rightarrow 2} (x-2)^2 q(x) = \lim_{x_0 \rightarrow 2} \cancel{(x-2)}^2 \frac{5}{\cancel{(x-2)}^2 (x+2)^2} = \frac{5}{16} \end{array} \right.$$

Both functions are analytic at $x=0$

\Rightarrow Regular Singular Point (RSP)

- For $x_0=-2$
Factor $(x-(-2))$
 $=x+2$

$$\tilde{p}(x) = \lim_{x_0 \rightarrow -2} (x+2)p(x) = \lim_{x_0 \rightarrow -2} \cancel{(x+2)} \frac{3\cancel{(x-2)}}{(x-2)^2 (x+2)^2} = \frac{3}{4 \cdot 0}$$

\Rightarrow Non-analytic at $x_0=-2$

Irregular Singular Point (IRS)

Frobenius' Theorem

- If $x = x_0$ is a regular singular point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

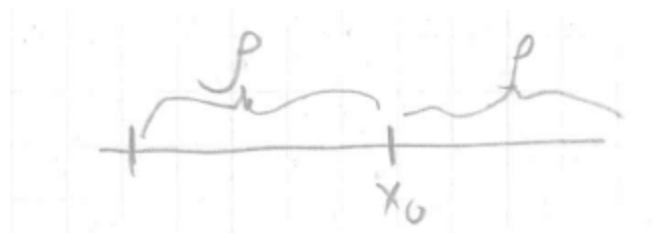
- Then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant to be determined.

The series will converge at least at some interval

$$0 < |x - x_0| < \rho$$



Solution Protocol

- (1) Substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ into the given differential equation
- (2) Determine the unknown coefficients c_n using reduction relations
- (3) Find the unknown exponent r . (non-negative integer)
- (4) For the sake of simplicity in solving the differential equation, the regular singular point is $x_0=0$
for any other regular singular point x_0
$$x \rightarrow (x - x_0)$$

RSP (Case 1) - Example (Two Series Solutions)

$$3xy'' + y' - y = 0$$
$$y'' + \frac{1}{3x} y' - \frac{1}{3x} y = 0$$

- What type of point $x_0=0$ is of the ideational equation?
 - Regular Point (RP)
 - Regular Singular Point (RSP)
 - Irregular Singular Point (IRS)

$$\left. \begin{aligned} \tilde{p}(x) &= \lim_{x \rightarrow x_0} (x - x_0) p(x) = \lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 0} x \left(\frac{1}{3x} \right) = \frac{1}{3} \\ \tilde{q}(x) &= \lim_{x \rightarrow x_0} (x - x_0)^2 q(x) = \lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \rightarrow 0} x^2 \left(-\frac{1}{3x} \right) = \lim_{x \rightarrow 0} -\frac{x}{3} = 0 \end{aligned} \right\}$$

- Conclusion - Regular Singular Point (RSP)

RSP (Case 1) - Example (Two Series Solutions)

- Assuming a solution of the form

$$\left\{ \begin{array}{l} y = \sum_{n=0}^{\infty} c_n x^{n+r} \\ y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \\ y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \end{array} \right.$$

- Plug the solution into the differential equation

$$3xy'' + y' - y = 0$$

RSP (Case 1) - Example (Two Series Solutions)

$$\begin{aligned}
 3xy'' + y' - y &= \underbrace{3 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1}}_{3xy''} + \underbrace{\sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}}_{y'} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_y \\
 &= \sum_{n=0}^{\infty} (n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} \\
 &= x^r \left[\underbrace{r(3r-2)c_0 x^{-1}}_{n=0} + \underbrace{\sum_{n=1}^{\infty} (n+r)(3n+3r-2)c_n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \right] \\
 &= x^r \left[\underbrace{r(3r-2)c_0 x^{-1}}_{=0} + \underbrace{\sum_{k=0}^{\infty} [(k+1+r)(3k+3r+1)c_{k+1} - c_k] x^k}_{=0} \right] = 0 \\
 \Rightarrow &\begin{cases} r(3r-2)c_0 = 0 \\ (k+r+1)(3k+3r+1)c_{k+1} - c_k = 0 \end{cases}
 \end{aligned}$$

RSP (Case 1) - Example (Two Series Solutions)

Because nothing is gained by taking $c_0=0$, we must then have

$$r(3r - 2) = 0 \Rightarrow \begin{cases} r_1 = \frac{2}{3} \\ r_2 = 0 \end{cases} \quad (r_1 - r_2) = \frac{2}{3} \quad \text{Case 1 - Positive Non-Integer}$$

$$(*) \quad c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)} \quad k = 0, 1, 2, \dots$$

Plugging r_1, r_2 into (*)

$$(o) \quad r_1 = \frac{2}{3} \Rightarrow c_{k+1} = \frac{c_k}{(3k+5)(k+1)} \quad k = 0, 1, 2, 3, \dots$$

$$(+) \quad r_2 = 0 \Rightarrow c_{k+1} = \frac{c_k}{(k+1)(3k+1)} \quad k = 0, 1, 2, 3, \dots$$

RSP (Case 1) - Example (Two Series Solutions)

For ($r_1=2/3$) Eq (o)	For ($r_2=0$) Eq (+)
$c_1 = \frac{c_0}{5 \cdot 1}$	$c_1 = \frac{c_0}{1 \cdot 1}$
$c_2 = \frac{c_1}{8 \cdot 2} = \frac{c_0}{2! \cdot 5 \cdot 8}$	$c_2 = \frac{c_1}{2 \cdot 4} = \frac{c_0}{2! \cdot 1 \cdot 4}$
$c_3 = \frac{c_2}{11 \cdot 3} = \frac{c_0}{3! \cdot 5 \cdot 8 \cdot 11}$	$c_3 = \frac{c_2}{3 \cdot 7} = \frac{c_0}{3! \cdot 1 \cdot 4 \cdot 7}$
$c_4 = \frac{c_3}{14 \cdot 4} = \frac{c_0}{4! \cdot 5 \cdot 8 \cdot 11 \cdot 14}$	$c_4 = \frac{c_3}{4 \cdot 10} = \frac{c_0}{4! \cdot 1 \cdot 4 \cdot 7 \cdot 10}$
\vdots	\vdots
$c_n = \frac{c_0}{n! \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdots (3n+2)}$	$c_n = \frac{c_0}{n! \cdot 1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}$

- We encounter something that didn't happen when we obtained solutions about an ordinary point. The two sets of coefficients contains the same multiple c_0 . We can omit the term c_0 (or alternatively set $c_0=1$)

RSP (Case 1) - Example (Two Series Solutions)

$$r=2/3 \quad y_1(x) = c_0 x^{\frac{2}{3}} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n \right]$$

$$r=0 \quad y_2(x) = c_0 x^0 \left[1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)} x^n \right]$$

- set $c_0=1$ $y = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r} + \dots$
- The two solutions $y = c_1 y_1(x) + c_2 y_2(x)$
- Using the ratio test both $y_1(x)$ and $y_2(x)$ converge for all values of x
 $|x| < \infty$ (except the origin)
- $y_1(x)$ and $y_2(x)$ are linearly independent for all $|x| < \infty$

RSP - Indicial Equation

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$r(3r-2) = 0 \begin{cases} \rightarrow r_1 = 2/3 \\ \rightarrow r_2 = 0 \end{cases}$$

- Indicial equation
 - r_1, r_2 Indicial roots or exponents
 - The indicial equation is typically quadratic equation
 - It is possible to obtain the indicial equation in advance of substituting
- } of the singularity $x_0 = 0$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \text{ into the diff. eq.}$$

RSP - Indicial Equation

Obtain the indicial equation in advance of substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the diff. eq.

- If $x = 0$ is a regular singular point of $y'' + p(x)y' + q(x)y = 0$

- Than $\left. \begin{array}{l} \tilde{p}(x) = xp(x) \\ \tilde{q}(x) = x^2q(x) \end{array} \right\}$ are analytic at $x = 0$

- That is the power series expansions

$$\tilde{p}(x) = xp(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$\tilde{q}(x) = x^2q(x) = b_0 + b_1x + b_2x^2 + \dots$$

are valid on intervals that have a positive radius at convergence ρ

RSP - Indicial Equation

- By multiplying $y'' + p(x)y' + q(x)y = 0$ by x^2 we get

$$x^2 y'' + x \underbrace{[xp(x)]}_{\tilde{p}(x)} y' + \underbrace{[x^2 q(x)]}_{\tilde{q}(x)} y = 0$$

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and the two series for $\tilde{p}(x), \tilde{q}(x)$

we find that the general indicial equation to be

$$r(r-1) + a_0 r + b_0 = 0$$

where a_0, b_0 are defined by $\tilde{p}(x), \tilde{q}(x)$

$$\tilde{p}(x) = xp(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\tilde{q}(x) = x^2 q(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

RSP (Case 2)- Example Indicial Equation (Direct Derivation)

$$xy'' + y = 0 \quad x_0 = 0$$

$$y'' + \underbrace{0}_{p(x)} y' + \frac{1}{\underbrace{x}_{q(x)}} y = 0$$

$$\left\{ \begin{array}{l} \tilde{p}(x) = xp(x) = x \cdot 0 = 0 = \underbrace{0}_{a_0} + \underbrace{0}_{a_1}x + \underbrace{0}_{a_2}x^2 + \dots \\ \tilde{q}(x) = x^2q(x) = x^2 \frac{1}{x} = x = \underbrace{0}_{b_0} + \underbrace{1}_{b_1}x + \underbrace{0}_{b_2}x^2 + \dots \end{array} \right.$$

$$a_0 = 0; \quad b_0 = 0$$

The indicial equation is

$$r(r-1) = 0 \quad \begin{cases} r_2 = 0 \\ r_1 = 1 \end{cases} \quad (r_1 - r_2) = 1 \quad \text{Case 2 - Positive Integer}$$

RSP (Case 2) - Example (only one series solution)

$$xy'' + y = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$x^r \left[r(r-1)c_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right]$$

$$\begin{cases} k = n-1 \\ n = k+1 \end{cases}$$

$$x^r \left[r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} (k+1+r)(k+r)c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k \right]$$

$$x^r \left[r(r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(k+r)c_{k+1} + c_k] x^k \right]$$

$$\begin{cases} r(r-1) = 0 \\ (k+r+1)(k+r)c_{k+1} + c_k = 0 \end{cases}$$

$$c_{k+1} = -\frac{c_k}{(k+r+1)(k+r)}$$

RSP (Case 2) - Example (only one series solution)

$r = 1$

$$c_{k+1} = -\frac{c_k}{(k+2)(k+1)} \quad k = 0, 1, 2, 3, \dots$$

$$k = 0 \quad c_1 = -\frac{c_0}{2 \cdot 1}$$

$$k = 1 \quad c_2 = -\frac{c_1}{3 \cdot 2} = +\frac{c_0}{3 \cdot 2 \cdot 2 \cdot 1}$$

$$k = 2 \quad c_3 = -\frac{c_2}{4 \cdot 3} = -\frac{c_0}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

$$k = 3 \quad c_4 = -\frac{c_3}{5 \cdot 4} = +\frac{c_0}{5 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$

\vdots

$$k = n \quad c_n = \frac{(-1)^n}{n!(n+1)!}$$

$r = 0$

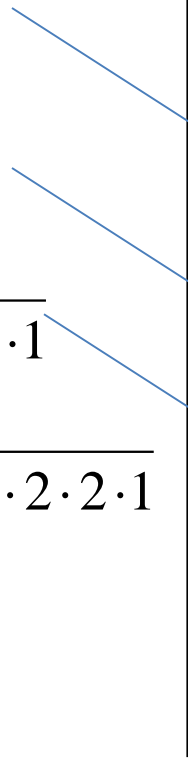
$$c_{k+1} = -\frac{c_k}{(k+1)(k)} \quad k = 0, 1, 2, 3, \dots$$

$$c_1 = -\frac{c_0}{1^2 \cdot 0} = \text{undefined}$$

$$c_2 = -\frac{c_1}{2 \cdot 1}$$

$$c_3 = -\frac{c_2}{3 \cdot 2} = +\frac{c_0}{3 \cdot 2 \cdot 2 \cdot 1}$$

$$c_4 = -\frac{c_3}{4 \cdot 3} = -\frac{c_0}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1}$$



RSP (Case 2) - Example (only one series solution)

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$r = 1$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$c_0 x^1 + c_1 x^2 + c_2 x^3$$

$$r = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3$$

$$c_0 = c_1$$

$$c_1 = c_2$$

$$c_2 = c_3$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2} x^2 + \frac{1}{12} x^3 - \frac{1}{144} x^4 + \dots$$

RSP (Case 2) - Second Solution

Finding a second solution ($r_1 - r_2$ positive integer case II)

- We may or may not be able to find a second solution depends on
 - Indicial roots
 - Recurrence relation of c_n
- c may be equal to 0 if

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \\ y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{cases}$$

We may use the following term for the second solution

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{y_1(x)^2} dx$$

RSP (Case 2) - Second Solution

$$xy'' + y = 0$$

$$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots$$

Construct a second solution $y_2(x)$ using the formula

$$y_2(x) = y_1 \int \frac{e^{\int 0 dx}}{(y_1)^2} = y_1 \int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots \right]^2}$$

$$= y_1 \int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \dots \right]}$$

After Squaring

$$= y_1 \int \left[\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x \right] dx$$

After Long Division

Example (Finding a second solution)

$$= y_1 \left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \dots \right]$$

After Integration

$$= y_1 \ln x + y_1 \left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \dots \right]$$

$$= y_1 \ln x + \left[-1 - \frac{1}{2}x + \frac{1}{2}x^2 + \dots \right]$$

After Multiplication by y_1