

Class Notes 10:

Power Series (2/3)
Series Solution Near and Ordinary Point

182A – Engineering Mathematics

Series Solution – Ordinary Point and Singular Point – Introduction

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0$$

- Assume P , Q , R are polynomials with no common factors, and that we want to solve the equation in a neighborhood of a point of interest x_0
- If there is a common factor we divided it out before proceeding

Series Solution – Ordinary Point – Definition

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

- **Ordinary Point (Definition)** - The point x_0 is called an **ordinary point** if $P(x_0) \neq 0$.
- Since P is continuous, $P(x) \neq 0$ for all x in some interval about x_0 . For x in this interval, divide the differential equation by P to get

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0,$$

$$\text{where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- Since $P(x_0)$ is continuous, there is an interval about x_0 in which $P(x)$ is never zero
- Solve the equation in some neighborhood of a point of interest x_0
- Since p and q are continuous, Theorem 3.2.1 indicates that there is a unique solution, given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$

Ordinary Point – Examples

$$y'' + y = 0$$

$$y'' + 3y' + 2y = 0$$

$$y'' + e^x y' + (\sin(x))y = 0$$

- Every point in $[-\infty, \infty]$ is an ordinary point

Series Solution – Singular Point – Definition

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0$$

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = 0,$$

$$\text{where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- **Singular Point (Definition)** - The point x_0 is called a **singular point** if $P(x_0) = 0$.
- Since P , Q , R are polynomials with no common factors, it follows that $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, or both.
- Then at least one of p or q becomes unbounded as $x \rightarrow x_0$, and therefore Theorem 3.2.1 does not apply in this situation.
- Sections 5.4 through 5.8 deal with finding solutions in the neighborhood of a singular point.

Singular Point – Example 1

$$y'' + \underbrace{x}_{\frac{Q(x)}{P(x)}} y' + \underbrace{\ln(x)}_{\frac{R(x)}{P(x)}} y = 0$$

$\frac{Q}{P} = x \rightarrow$ analytic at every real number

$\frac{R}{P} = \ln(x) \rightarrow$ analytic at every positive real number

discontinues at $x = 0$

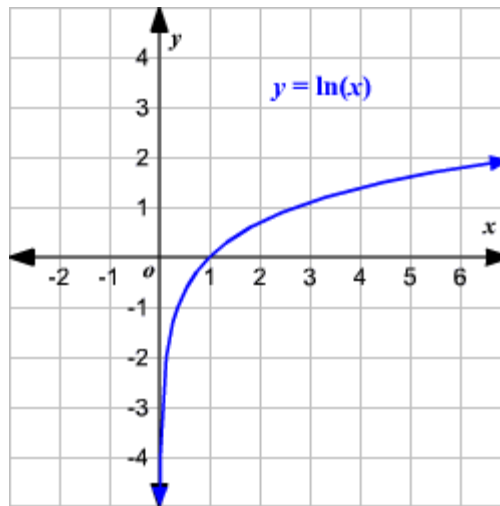
$\frac{R}{P} = \ln(x) -$ can't be represented by a power series in x

at $x_0 = 0 \rightarrow x = 0$ is a singular point of the diff. eq

Singular Point – Example 1

$\frac{R}{P} = \ln(x)$ – can't be represented by a power series in x

at $x_0 = 0 \rightarrow x = 0$ is a singular point of the diff. eq



Singular Point – Example 2

$$xy'' + y' + xy = 0$$

$$y'' + \frac{1}{x}y' + y = 0$$

$$\frac{Q(x)}{P(x)} = \frac{1}{x} \quad \text{Fail to be analytic at } x = 0$$

$x = 0$ is a singular point of the equation

Singular Point – Example 3

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

$$y'' + \left(\frac{2x}{x^2 - 1}\right)y' + \left(\frac{6}{x^2 - 1}\right)y = 0$$

$x^2 - 1 = 0 \rightarrow x = \pm 1$ singular point ; all the rest are ordinary points

Series Solution – Ordinary Point

- In order to solve our equation near an ordinary point x_0 ,

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0$$

we will assume a series representation of the unknown solution function y :

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- As long as we are within the interval of convergence, this representation of y is continuous and has derivatives of all orders.

Theorem 5.3.1

Existence of Power Series Solution

- If x_0 is an ordinary point of the differential equation

$$P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y = 0$$

- **Then** the general solution for this equation is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary, and y_1, y_2 are linearly independent series solutions that are analytic at x_0 .

- The solutions y_1, y_2 form a fundamental set of solutions
- The power series solution converges at least on some interval defined by

$$|x - x_0| < \rho$$

- **The radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q .**

Minimum of the radii of convergence – Example

Find ρ_{\min}

For $(x^2 - 2x + 5)y'' + xy' - y = 0$

$$y'' + \frac{x}{(x^2 - 2x + 5)} y' - \frac{1}{(x^2 - 2x + 5)} y = 0$$

about an ordinary points $\left\{ \begin{array}{l} x_0 = 0 \\ x_0 = -1 \end{array} \right. \quad x^2 - 2x + 5 = 0 \rightarrow \left. \begin{array}{l} x_1 \\ x_2 \end{array} \right\} = 1 \pm 2i$

- Because $\left\{ \begin{array}{l} x_0 = 0 \\ x_0 = -1 \end{array} \right.$ are two ordinary points of the diff. eq. the theorem

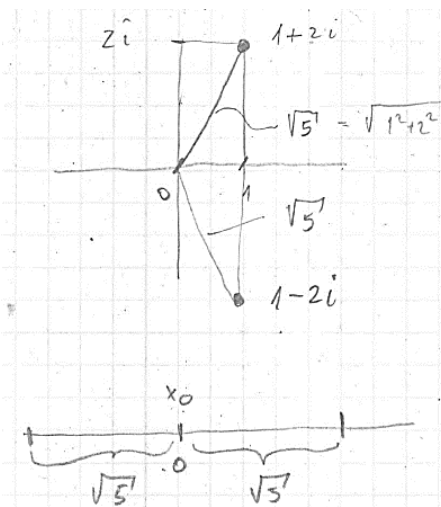
guarantees that we can find two power series centered around the two x_0

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Minimum of the radii of convergence – Example

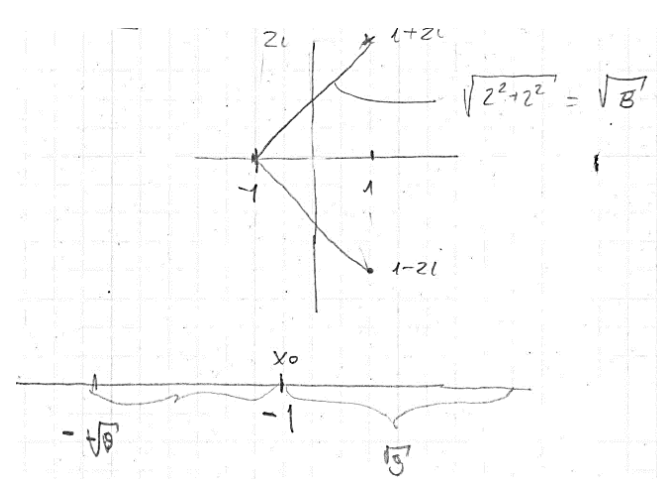
The radius of convergence

$$x_0 = 0$$



$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$x_0 = -1$$



$$y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$$

- Each power series for each will converge around x_0 at least with a minimal radius of ρ

Power Solution Around $x_0 = 0$ and $x_0 \neq 0$

- For the sake of simplicity, find only power series solution about the ordinary point $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- If $x_0 \neq 0$ is also an ordinary point substitute the solution for $x \rightarrow x - x_0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Series Solution – Ordinary Point – Example 1

$$y'' + y = 0 \quad -\infty < x < \infty$$

$$P(x) = 1; Q(x) = 0 \quad R(x) = 1$$

→ Every point is an ordinary point

Look for a solution in the form of a power series about $x_0 = 0$

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$y'' = 2a_2 + n(n-1)a_nx^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

substituting in the diff. eq.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{y''} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{y} = 0$$

Series Solution – Ordinary Point – Example 1

$$\begin{array}{ccc}
 & n^* = n - 2 & \\
 & n = n^* + 2 & \\
 \underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}_{y''} & \xrightarrow{\quad \downarrow \quad} & \underbrace{\sum_{n^*=0}^{\infty} (n^*+2)(n^*+2-1)a_{n^*+2} x^{n^*+2-2}}_{y''} \\
 \underbrace{\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n}_{y''} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_y = 0 \\
 \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n] x^n = 0
 \end{array}$$

for this equation to be satisfactory for all x the coefficient of each power of x must be zero

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad n = 0, 1, 2, 3, \dots$$

Series Solution – Ordinary Point – Example 1

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad n = 0, 1, 2, 3 \dots$$

$$n = 0 \quad 2 \cdot 1 \cdot a_2 + a_0 = 0 \quad \rightarrow a_2 = -a_0 / (2 \cdot 1)$$

$$n = 1 \quad 3 \cdot 2 \cdot a_3 + a_1 = 0 \quad \rightarrow a_3 = -a_1 / (3 \cdot 2)$$

$$n = 2 \quad 4 \cdot 3 \cdot a_4 + a_2 = 0 \quad \rightarrow a_4 = -a_2 / (4 \cdot 3)$$

$$n = 3 \quad 5 \cdot 4 \cdot a_5 + a_3 = 0 \quad \rightarrow a_5 = -a_3 / (5 \cdot 4)$$

$$n = 4 \quad 6 \cdot 5 \cdot a_6 + a_4 = 0 \quad \rightarrow a_6 = -a_4 / (6 \cdot 5)$$

$$n = 5 \quad 7 \cdot 6 \cdot a_7 + a_5 = 0 \quad \rightarrow a_7 = -a_5 / (7 \cdot 6)$$

Series Solution – Ordinary Point – Example 1

$$\text{Even} \left\{ \begin{array}{l} a_2 = -\frac{a_0}{2 \cdot 1} \\ a_4 = -\frac{a_2}{4 \cdot 3} = +\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} \\ a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \end{array} \right.$$

$$\boxed{n = 2k}$$

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!} a_0$$

$$k = 1, 2, 3, 4, \dots$$

$$\text{Odd} \left\{ \begin{array}{l} a_3 = -\frac{a_1}{3 \cdot 2} \\ a_5 = -\frac{a_3}{5 \cdot 4} = +\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \end{array} \right.$$

$$\boxed{n = 2k + 1}$$

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1$$

$$k = 1, 2, 3, 4, \dots$$

Series Solution – Ordinary Point – Example 1

$$\begin{aligned}
 y &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 \\
 &\quad + \cdots + \underbrace{\frac{(-1)^n a_0}{(2n)!}}_{\text{even}} x^{2n} + \underbrace{\frac{(-1)^n a_1}{(2n+1)!}}_{\text{odd}} x^{2n+1} \\
 &= a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right] \\
 &\quad + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right] \\
 &= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos x} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin x}
 \end{aligned}$$

Series Solution – Ordinary Point – Example 1

$$y'' + y = 0$$

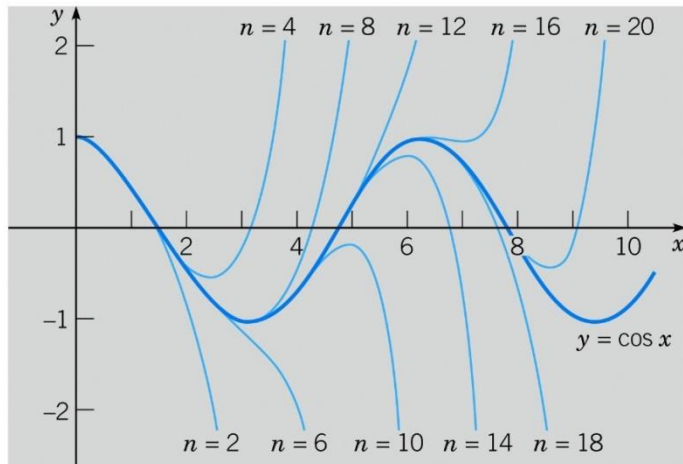
$$y = a_0 \cos(x) + a_1 \sin(x)$$

$$y' = -a_0 \sin(x) + a_1 \cos(x)$$

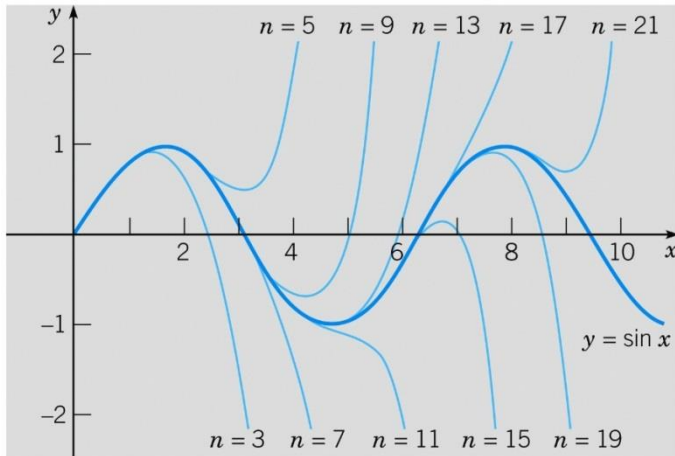
$$\text{For } \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \begin{cases} y(0) = a_0 \cdot 1 + a_1 \cdot 0 = 0 \\ y'(0) = -a_0 \cdot 0 + a_1 \cdot 1 = 1 \end{cases} \quad y = \sin x$$

$$\text{For } \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases} \quad \begin{cases} y(0) = a_0 \cdot 1 + a_1 \cdot 0 = 1 \\ y'(0) = -a_0 \cdot 0 + a_1 \cdot 1 = 0 \end{cases} \quad y = \cos x$$

Series Solution – Ordinary Point – Example 1



$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Series Solution – Ordinary Point – Example 2 – Airy's Equation

$$y'' - xy = 0$$

$$P(x) = 1; Q(x) = 0 \quad R(x) = -x$$

Assume a solution

$$\left\{ \begin{array}{l} y = \sum_{n=0}^{\infty} a_n x^n \\ y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \text{ (see previous example)} \end{array} \right.$$

substituting the series in the diff eq. $y'' = xy$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{\boxed{n}} = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{\boxed{n+1}}$$

$$\tilde{n} = n + 1$$

$$n = \tilde{n} - 1$$

Series Solution – Ordinary Point – Example 2 – Airy's Equation

$$\sum_{\boxed{n=0}}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{\boxed{\tilde{n}=1}}^{\infty} a_{\tilde{n}-1}x^{\tilde{n}}$$

start the sum from $n=1$

$$\underbrace{2 \cdot 1 \cdot a_2}_{n=0} + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n$$

For the equation to be satisfied for all x is some interval, the coefficients of the same powers of x must be equal

$$x^0 : \quad 2 \cdot 1 \cdot a_2 x^0 = 0x^0 \rightarrow a_2 = 0$$

$$n \geq 1 \quad x^n : \quad (n+2)(n+1)a_{n+2} = a_{n-1}$$

a_{n+2} is given in terms of a_{n-1}

Series Solution – Ordinary Point – Example 2 – Airy's Equation

$$x^0 : \quad 2 \cdot 1 \cdot a_2 x^0 = 0x^0 \rightarrow a_2 = 0$$

$$n \geq 1 \quad x^n : \quad (n+2)(n+1)a_{n+2} = a_{n-1}$$

a_{n+2} is given in terms of a_{n-1}

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

→ steps of 3

$$n = 1, 4, 7, 10$$

$$a_0 \rightarrow a_3 \rightarrow a_6$$

$$n = 2, 5, 8, 11$$

$$a_1 \rightarrow a_4 \rightarrow a_7$$

$$n = 3, 6, 9, 12$$

$$a_2 \rightarrow a_5 \rightarrow a_8 \rightarrow a_2 = a_5 = a_8 = a_{11} = \dots = 0$$

↑

0

Series Solution – Ordinary Point – Example 2 – Airy's Equation

For a_0, a_3, a_6, a_9 $n = 1, 4, 7, 10$

$$a_3 = \frac{a_0}{2 \cdot 3}; \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}; \quad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 8 \cdot 9}$$

The general formula

$$n \geq 4 \quad a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n-1) \cdot (3n)}$$

Series Solution – Ordinary Point – Example 2 – Airy's Equation

For a_1, a_4, a_7, a_{10} $n = 2, 5, 8, 11$

$$a_4 = \frac{a_1}{3 \cdot 4}; \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}; \quad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$$

The general formula

$$n \geq 4 \quad a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \cdot 3n \cdot (3n+1)}$$

Series Solution – Ordinary Point – Example 2 – Airy's Equation

The General solution

$$y = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot (3n-1) \cdot (3n)} \right]$$
$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \cdot (3n) \cdot (3n+1)} \right]$$

Notes

- Both series converge for all x
- The general solution is $y = a_0 y_1(x) + a_1 y_2(x) \quad -\infty < x < \infty$

Series Solution – Ordinary Point – Example 2 – Airy's Equation

$$y = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot (3n-1) \cdot (3n)} \right]$$

$$+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \cdot (3n) \cdot (3n+1)} \right]$$

$$y = a_0 y_1(x) + a_1 y_2(x) \quad -\infty < x < \infty$$

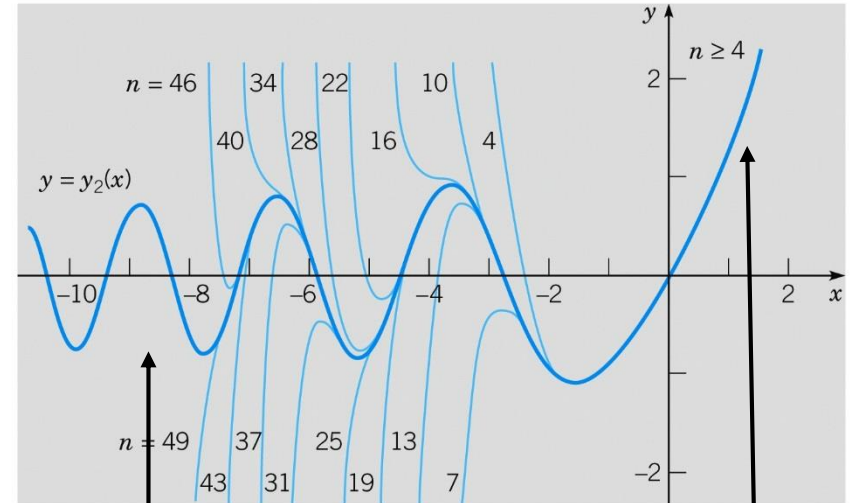
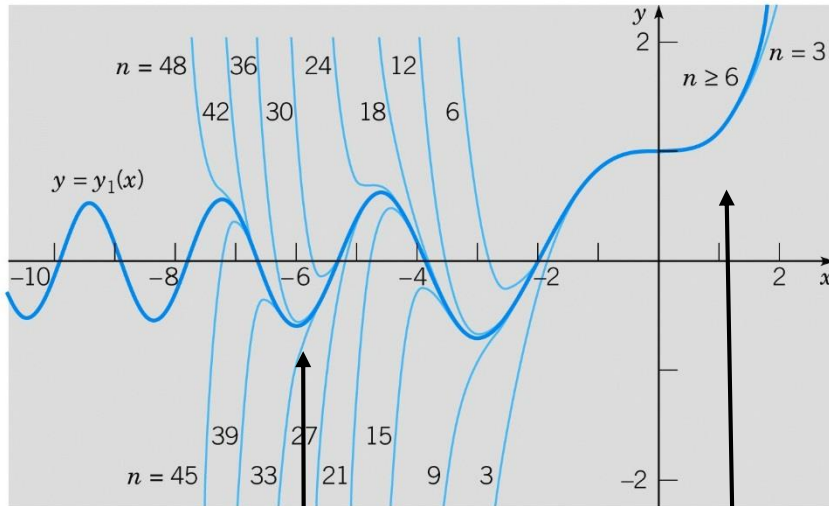


Figure 5.2.3
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Figure 5.2.4
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oscillatory for $x < 0$

Decay in Amp; Increase in Freq

monotone for $x > 0$

Example – Airy's Equation

$$\text{For a given I.C. } \begin{cases} y(x=0) = y_0 \\ y'(x=0) = y'_0 \end{cases}$$

$$y = a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \dots + \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot (3n-1) \cdot (3n)} \right] \\ + a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \cdot (3n) \cdot (3n+1)} \right]$$

$$\begin{cases} y(x=0) = a_0 = y_0 \\ y'(x=0) = a_1 = y'_0 \end{cases}$$

see also example 3, p 202 – Airy's Equation for $X_0=1$
(ordinary point)

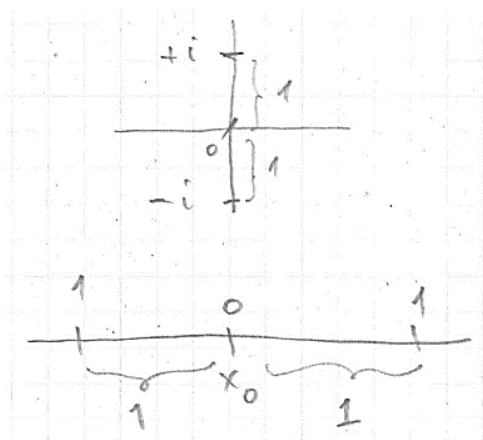
Series Solution – Ordinary Point – Example 3

$$(x^2 + 1)y'' + xy' - y = 0$$

$$y'' + \frac{x}{x^2 + 1}y' - \frac{1}{x^2 + 1}y = 0$$

$$x^2 + 1 = 0 \rightarrow x = \pm i$$

For $x_0 = 0$



$$\rho = 1$$

converge at least $|x| < 1$

$$\left\{ \begin{array}{l} y = \sum_{n=0}^{\infty} c_n x^n \\ y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \end{array} \right.$$

Series Solution – Ordinary Point – Example 3

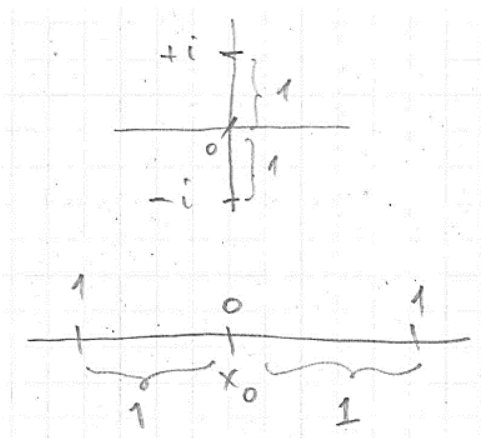
$$(x^2 + 1)y'' + xy' - y = 0 \quad \text{For } x_0 = 0$$

$$y'' + \frac{x}{x^2 + 1} y' - \frac{1}{x^2 + 1} y = 0$$

$$x^2 + 1 = 0 \rightarrow x = \pm i$$

For $x_0 = 0$

$$\rho = 1$$



converge at least $|x| < 1$

Series Solution – Ordinary Point – Example 3

$$(x^2 + 1)y'' + xy' - y = 0 \quad \text{For } x_0 = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad ; \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad ; \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\underbrace{(x^2 + 1) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{y''} + x \underbrace{\sum_{n=1}^{\infty} n c_n x^{n-1}}_{y'} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_y$$

Series Solution – Ordinary Point – Example 3

$$= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \underbrace{(2)(1)c_2 x^0}_{k=0} + \underbrace{(3)(2)c_3 x^1}_{k=1} + \sum_{k=2}^{\infty} (k+2)(k+1)c_k x^k + \underbrace{1 \cdot c_1 x^1}_{k=1} + \sum_{k=2}^{\infty} k c_k x^k$$

$$- c_0 x^0 - c_1 x^1 - \sum_{k=2}^{\infty} c_k x^k = 0$$

$$(2c_2 - c_0) + (6c_3)x + \sum_{k=2}^{\infty} [k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k] x^k = 0$$

$$(2c_2 - c_0) + (6c_3)x + \sum_{k=2}^{\infty} [(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}] x^k = 0$$

Series Solution – Ordinary Point – Example 3

$$\underbrace{(2c_2 - c_0)}_{=0} + \underbrace{(6c_3)}_{=0}x + \sum_{k=2}^{\infty} \left[\underbrace{(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}}_{=0} \right] x^k = 0$$

For the equation to be satisfied for all x in some interval, the coefficients of the same powers of x must be equal

$$\begin{aligned} x^0 &\rightarrow \begin{cases} 2c_2 - c_0 = 0 \\ 6c_3 = 0 \\ (k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0 \end{cases} \\ x^1 &\rightarrow \\ x^k &\rightarrow \end{aligned}$$

$$c_2 = \frac{1}{2}c_0$$

$$c_3 = 0$$

$$c_{k+2} = \frac{1-k}{k+2}c_k, k = 2, 3, 4, \dots$$

Series Solution – Ordinary Point – Example 3

$$c_{k+2} = \frac{1-k}{k+2} c_k, k = 2, 3, 4, \dots$$

$$k = 2 \quad c_4 = -\frac{1}{4} c_2 = -\frac{1}{2 \cdot 4} c_0 = -\frac{1}{2^2 \cdot 2!} c_0$$

$$k = 3 \quad c_5 = -\frac{2}{5} c_3 = 0$$

$$k = 4 \quad c_6 = -\frac{3}{6} c_4 = -\frac{3}{2 \cdot 4 \cdot 6} c_0 = -\frac{1 \cdot 3}{2^3 \cdot 3!} c_0$$

$$k = 5 \quad c_7 = -\frac{4}{7} c_5 = 0$$

$$k = 6 \quad c_8 = -\frac{5}{8} c_6 = -\frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} c_0 = -\frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!} c_0$$

$$k = 7 \quad c_9 = -\frac{6}{9} c_7 = 0$$

$$k = 8 \quad c_{10} = -\frac{7}{10} c_8 = \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} c_0 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 5!} c_0$$

Series Solution – Ordinary Point – Example 3

$$\begin{aligned} y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots \\ &= c_0 \underbrace{\left[1 + \frac{1}{2}x^2 - \frac{1}{2^2 2!}x^4 + \frac{1 \cdot 3}{2^3 3!}x^6 - \frac{1 \cdot 3 \cdot 5}{2^4 4!}x^8 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 5!}x^{10} - \dots \right]}_{y_1} + c_1 \underbrace{x}_{y_2} \\ &= c_0 y_1 + c_1 y_2 \end{aligned}$$

$$\begin{cases} y_1 = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{2n} \\ y_2 = x \end{cases} \quad \left| \quad \begin{array}{l} -1 < x < 1 \\ |x| < 1 \end{array} \right.$$

Series Solution – Ordinary Point – Example 4 (Three Recurrence Relation)

$$y'' - (1+x)y = 0 \quad \text{For } x_0 = 0$$

$$y'' - xy - y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{y''} - x \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{xy} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Series Solution – Ordinary Point – Example 4 (Three Recurrence Relation)

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$k = n - 2 \quad k = n + 1 \quad k = n$$

$$n = k + 2 \quad n = k - 1$$

$$n : 2 \rightarrow \infty \quad n : 0 \rightarrow \infty$$

$$k : 0 \rightarrow \infty \quad k : 1 \rightarrow \infty$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Start at $k = 2$

$$\underbrace{(2)(1)c_2 x^0}_{k=0} + \underbrace{(3)(2)c_3 x^1}_{k=1} + \sum_{k=2}^{\infty} (k+2)(k+1)c_{k+2}x^k$$

$$\underbrace{-c_0 x^1}_{k=1} - \sum_{k=2}^{\infty} c_{k-1}x^k$$

$$\underbrace{c_0 x^0}_{k=0} + \underbrace{c_1 x^1}_{k=1} - \sum_{k=2}^{\infty} c_k x^k = 0$$

Series Solution – Ordinary Point – Example 4 (Three Recurrence Relation)

$$\underbrace{(-c_0 + 2c_2)}_{=0} + \underbrace{(6c_3 - c_1 - c_0)}_{=0}x + \sum_{k=2}^{\infty} \underbrace{[(k+2)(k+1)c_{k+2}x^k - c_{k-1} - c_k]}_{=0}x^k = 0$$

$$\left\{ \begin{array}{l} -c_0 + 2c_2 = 0 \quad c_2 = \frac{1}{2}c_0 \\ 6c_3 - c_1 - c_0 = 0 \quad c_3 = \frac{1}{6}(c_0 + c_1) \\ (k+2)(k+1)c_{k+2} - c_{k-1} - c_k = 0 \end{array} \right.$$

$$k = 2, 3, \dots \quad c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}$$

Series Solution – Ordinary Point – Example 4 (Three Recurrence Relation)

$$c_2 = \frac{1}{2}c_0$$

$$c_3 = \frac{1}{6}(c_0 + c_1)$$

$$k = 2 \quad c_4 = \frac{c_2 + c_1}{3 \cdot 4}$$

$$k = 3 \quad c_5 = \frac{c_3 + c_2}{4 \cdot 5}$$

$$k = 4 \quad c_6 = \frac{c_4 + c_3}{5 \cdot 6}$$

Series Solution – Ordinary Point – Example 4 (Three Recurrence Relation)

$c_0 \neq 0; c_1 = 0$	$c_0 = 0; c_1 \neq 0$
$c_2 = \frac{1}{2}c_0$	$c_2 = \frac{1}{2}c_0 = 0$
$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{6}$	$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{6}$
$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{c_0}{24}$	$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{3 \cdot 4} = \frac{c_1}{12}$
$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{4 \cdot 5} \left[\frac{1}{6} + \frac{1}{2} \right] = \frac{c_0}{30}$	$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{4 \cdot 5 \cdot 6} = \frac{c_1}{120}$

$$y = c_0 y_1 + c_1 y_2 \quad | \quad y = \sum_{n=0}^{\infty} c_n x^n$$

$$c_0 y_1 = c_0 \left[1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \dots \right]$$

$$c_1 y_2 = c_1 \left[x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \dots \right]$$

Series Solution – Ordinary Point – Example 5 (Non-polynomial Coefficients)

$$y'' + (\cos x)y = 0$$

$$x_0 = 0 \Rightarrow \text{Ordinary Point}$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y'' + (\cos x)y = \overbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}^{y''} + \overbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}^{\cos x} \overbrace{\sum_{n=0}^{\infty} c_n x^n}^y$$

$$= \left(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) (c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)$$

$$= (c_0 + 2c_2) + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \dots = 0$$

$$\begin{cases} c_0 + 2c_2 = 0 \\ 6c_3 + c_1 = 0 \\ 12c_4 + c_2 - \frac{1}{2}c_0 = 0 \\ 20c_5 + c_3 - \frac{1}{2}c_1 = 0 \end{cases}$$

Series Solution – Ordinary Point – Example 5 (Non-polynomial Coefficients)

$$\left[\begin{array}{l} c_2 = -\frac{1}{2}c_0 \\ c_3 = -\frac{1}{6}c_1 \\ c_4 = \frac{1}{12}c_0 \\ c_5 = \frac{1}{30}c_1 \end{array} \right]$$

$$n = 0, 2, 4, 6, 8 \quad c_0 y_1 = c_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \dots \right)$$

$$n = 1, 3, 5, 7 \quad c_1 y_2 = c_1 \left(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \dots \right)$$

$$y = c_0 y_1 + c_1 y_2$$

Because the differential equation has no finite singular points, both power series converge for $|x| < \infty$ $-\infty < x < \infty$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

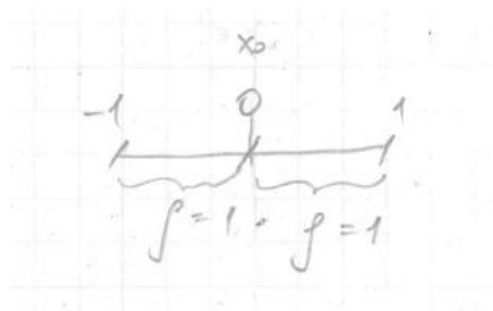
Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$(1-x^2)y'' - 2xy' + \tilde{n}(\tilde{n}+1)y = 0$$

$$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \tilde{n}(\tilde{n}+1)y = 0$$

$$1-x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$x_0 = 0$ Regular point



$$\Rightarrow \rho_{\min} = 1$$

$$y = \sum_{n=0}^{\infty} c_n x^n \quad ; \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad ; \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$\begin{aligned}
 & y'' - x^2 y'' - 2xy' + \tilde{n}(\tilde{n} + 1)y \\
 &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{y''} - \underbrace{x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{x^2 y''} - \underbrace{2x \sum_{n=1}^{\infty} n c_n x^{n-1}}_{2xy'} + \underbrace{\tilde{n}(\tilde{n} + 1) \sum_{n=0}^{\infty} c_n x^n}_{n(n+1)y} \\
 &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} - \underbrace{2 \sum_{n=1}^{\infty} n c_n x^n}_{k=n} + \underbrace{\tilde{n}(\tilde{n} + 1) \sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
 &= \underbrace{\sum_{\substack{k=0 \\ \text{Start with 2}}}^{\infty} (k+2)(k+1)c_{k+2} x^k}_{\text{Start with 2}} - \underbrace{\sum_{k=2}^{\infty} k(k-1)c_k x^k}_{\text{Start with 2}} - 2 \underbrace{\sum_{\substack{k=1 \\ \text{Start with 2}}}^{\infty} k c_k x^k}_{\text{Start with 2}} + \tilde{n}(\tilde{n} + 1) \underbrace{\sum_{\substack{k=0 \\ \text{Start with 2}}}^{\infty} c_k x^k}_{\text{Start with 2}} \\
 &= \underbrace{2c_2}_{k=0} + \underbrace{3 \cdot 2c_3 x}_{k=1} - \underbrace{2c_1 x}_{k=1} + \tilde{n}(\tilde{n} + 1) \left[\underbrace{c_0}_{k=0} + \underbrace{c_1 x}_{k=1} \right] \\
 &+ \sum_{k=2}^{\infty} \left[(k+2)(k+1)c_{k+2} + \underbrace{(-k(k-1) - 2k + \tilde{n}(\tilde{n} + 1))}_{\substack{-k^2 - k - 2k + \tilde{n}(\tilde{n} + 1) \\ = \tilde{n}^2 + \tilde{n} - k^2 - k \\ = \tilde{n}^2 + \tilde{n}k + \tilde{n} - \tilde{n}k - k^2 - k \\ = (\tilde{n} - k)(\tilde{n} + k + 1)}} c_k x^k \right] x^k
 \end{aligned}$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$= [2c_2 + \tilde{n}(\tilde{n} + 1)c_0] + \left[\underbrace{6c_3 - 2c_1 + \tilde{n}(\tilde{n} + 1)c_1}_{\substack{(\tilde{n}^2 + \tilde{n} - 2)c_1 \\ = (\tilde{n} - 1)(\tilde{n} + 2)c_1}} \right] x$$

$$+ \sum_{k=2}^{\infty} [(k + 2)(k + 1)c_{k+2} + (\tilde{n} - k)(\tilde{n} + k + 1)c_k] x^k = 0$$

For the equation to be satisfied for all x in some interval, the coefficients of the same powers of x must be equal

$$\begin{cases} 2c_2 + \tilde{n}(\tilde{n} + 1)c_0 = 0 \\ 6c_3 + (\tilde{n} - 1)(\tilde{n} + 2)c_1 = 0 \\ (k + 2)(k + 1)c_{k+2} + (\tilde{n} - k)(\tilde{n} + k + 1)c_k = 0 \end{cases}$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$\begin{cases} 2c_2 + \tilde{n}(\tilde{n}+1)c_0 = 0 \\ 6c_3 + (\tilde{n}-1)(\tilde{n}+2)c_1 = 0 \\ (k+2)(k+1)c_{k+2} + (\tilde{n}-k)(\tilde{n}+k+1)c_k = 0 \end{cases}$$

$$c_2 = -\frac{\tilde{n}(\tilde{n}+1)}{\underset{=2!}{2}} c_0$$

$$c_3 = -\frac{(\tilde{n}-1)(\tilde{n}+2)}{\underset{=3!}{6}} c_1$$

$$c_{k+2} = -\frac{(\tilde{n}-k)(\tilde{n}+k+1)}{(k+2)(k+1)} c_k \quad k = 2, 3, 4, 5, \dots$$

$$k = 2 \rightarrow c_4 = -\frac{(\tilde{n}-2)(\tilde{n}+3)}{4 \cdot 3} c_2 = -\frac{(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)}{4!} c_0$$

$$k = 3 \rightarrow c_5 = -\frac{(\tilde{n}-3)(\tilde{n}+4)}{5 \cdot 4} c_3 = -\frac{(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)}{5!} c_1$$

$$k = 4 \rightarrow c_6 = -\frac{(\tilde{n}-4)(\tilde{n}+5)}{6 \cdot 5} c_4 = -\frac{(\tilde{n}-4)(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)(\tilde{n}+5)}{6!} c_0$$

$$k = 5 \rightarrow c_7 = -\frac{(\tilde{n}-5)(\tilde{n}+6)}{7 \cdot 6} c_5 = -\frac{(\tilde{n}-5)(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)(\tilde{n}+6)}{7!} c_1$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$y_1(x) = c_0 \left[1 - \frac{\tilde{n}(\tilde{n}+1)}{2!} x^2 + \frac{(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)}{4!} x^4 - \frac{(\tilde{n}-4)(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)(\tilde{n}+5)}{6!} x^6 + \dots \right]$$

$$y_2(x) = c_1 \left[x - \frac{(\tilde{n}-1)(\tilde{n}+2)}{3!} x^3 + \frac{(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)}{5!} x^5 - \frac{(\tilde{n}-5)(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)(\tilde{n}+6)}{7!} x^7 + \dots \right]$$

- Note for \tilde{n} as even integer,
 - y_1 terminates with x^n ,
 - y_2 becomes infinite series
- Note for \tilde{n} as odd integer (vice versa)
 - y_1 becomes infinite series
 - Y_2 terminates with x^n ,
- For example $\tilde{n} = 4$

$$y_1(x) = c_0 \left[1 - \frac{4 \cdot 5}{2!} x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!} x^4 \right] = c_0 \left[1 - 10x^2 + \frac{35}{3} x^4 \right]$$

$$y_2(x) = c_1 \left[x - \frac{3 \cdot 6}{3!} x^3 + \frac{1 \cdot 3 \cdot 6 \cdot 8}{5!} x^5 - \frac{-1 \cdot 1 \cdot 3 \cdot 6 \cdot 8 \cdot 10}{7!} x^7 + \dots \right]$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

$$y_1(x) = c_0 \left[1 - \frac{\tilde{n}(\tilde{n}+1)}{2!} x^2 + \frac{(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)}{4!} x^4 - \frac{(\tilde{n}-4)(\tilde{n}-2)\tilde{n}(\tilde{n}+1)(\tilde{n}+3)(\tilde{n}+5)}{6!} x^6 + \dots \right]$$
$$y_2(x) = c_1 \left[x - \frac{(\tilde{n}-1)(\tilde{n}+2)}{3!} x^3 + \frac{(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)}{5!} x^5 - \frac{(\tilde{n}-5)(\tilde{n}-3)(\tilde{n}-1)(\tilde{n}+2)(\tilde{n}+4)(\tilde{n}+6)}{7!} x^7 + \dots \right]$$

- Note For $\tilde{n} > 0$ (positive), one of the two solutions will be n-th degree polynomial solution to the Legendre's equation

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

- Because a constant multiple of a solution of the Legendre's equation is also a solution
- Then it is traditional to choose specific values for c_0, c_1 depending on whether n is an even or odd positive integer respectively

- For $\tilde{n} = 0 \rightarrow c_0 = 1$

$$\tilde{n} = 2, 4, 6 \rightarrow c_0 = (-1)^{\frac{\tilde{n}}{2}} \frac{1 \cdot 3 \cdots (\tilde{n} - 1)}{2 \cdot 4 \cdots \tilde{n}}$$

- For $\tilde{n} = 1 \rightarrow c_1 = 0$

$$\tilde{n} = 3, 5, 7 \rightarrow c_1 = (-1)^{\frac{(\tilde{n}-1)}{2}} \frac{1 \cdot 3 \cdots \tilde{n}}{2 \cdot 4 \cdots (\tilde{n} - 1)}$$

- For example $\tilde{n} = 4$

$$y_1(x) = (-1)^{\frac{4}{2}} \frac{1 \cdot 3}{2 \cdot 4} \left[1 - 10x^2 + \frac{35}{3}x^4 \right] = \frac{1}{8} [35x^4 - 30x^2 + 3]$$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

- These specific n-th degree polynomial solutions are called Legendre polynomials and are denoted by $P_n(x)$
- Note that for

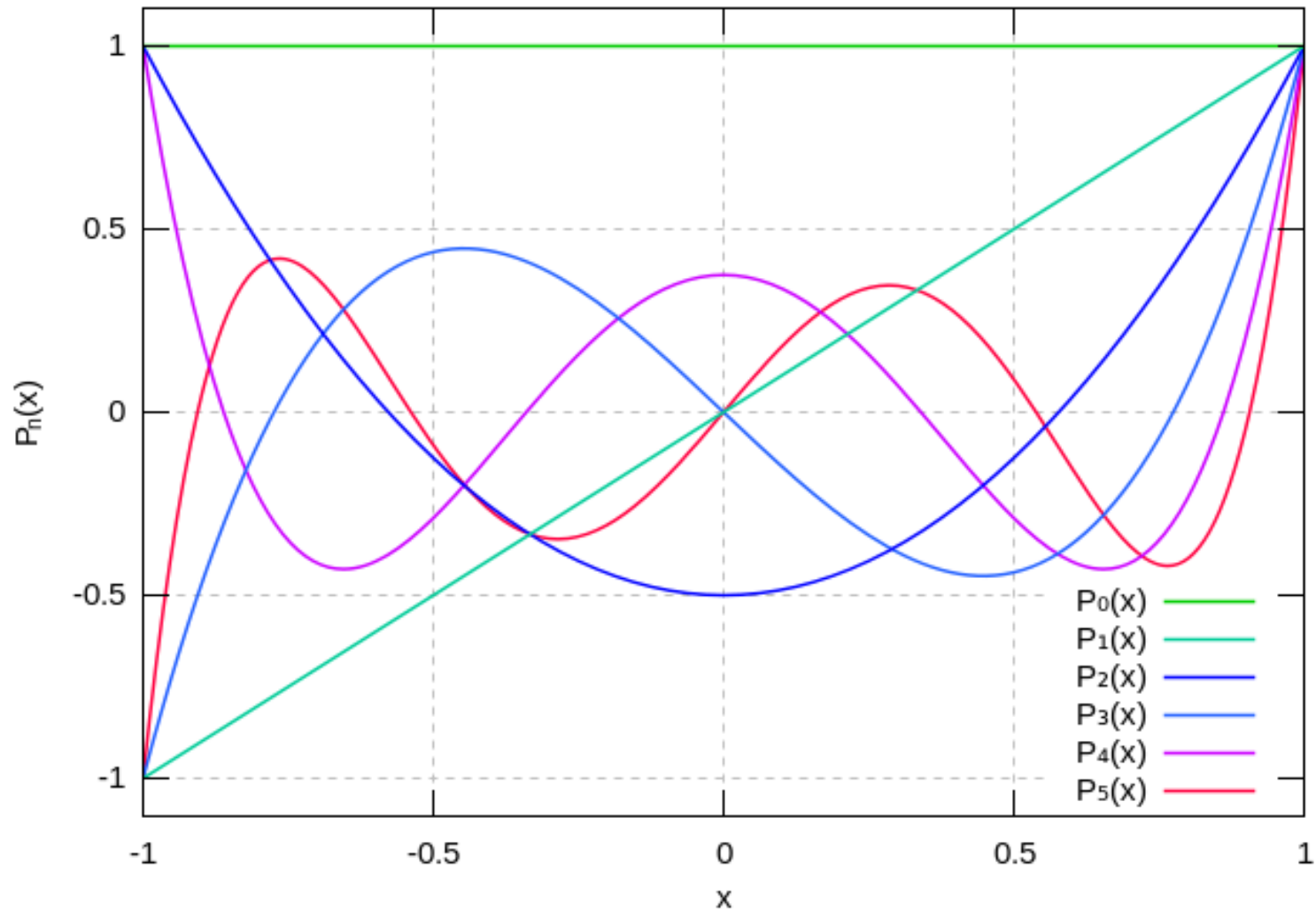
n	Legendre's Equation	Legendre Polynomial (Particular Solution)
0	$(1-x^2)y'' - 2xy' = 0$	$P_0(x) = 1$
1	$(1-x^2)y'' - 2xy' + 2y = 0$	$P_1(x) = x$
2	$(1-x^2)y'' - 2xy' + 6y = 0$	$P_2(x) = \frac{1}{2}(3x^2 - 1)$
3	$(1-x^2)y'' - 2xy' + 12y = 0$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Series Solution – Ordinary Point – Example 6 (Legendre’s Equation)

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
8	$\frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
9	$\frac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$
10	$\frac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

Series Solution – Ordinary Point – Example 6 (Legendre's Equation)

legendre polynomials



Legendre Polynomials

