Class Notes 9:

Power Series (1/3)

82A – Engineering Mathematics

Second Order Differential Equations – Series Solution

- Solution Anatomy

Differential Equation
- =0, Homogeneous
- =g(t), Non-homogeneous

Solution: $y = y_c + y_p$

- y_c ← Homogeneous ← Fundamental solutions
- y_p ← Non-homogeneous

Fundamental solutions
- → D.E. with constant coefficients → Elementary function
- → D.E. with variable coefficients → Power Series

Famous Differential Equation - Power Series

- Airy Eq. $$y'' - xy = 0$$
- Chebych Eq. $$(1 - x^2)y'' - xy' + \lambda^2 y = 0$$
- Hermit Eq. $$y'' - 2xy' + \lambda y = 0$$
- Bessel Eq. $$t^2 y'' - ty' + (t^2 - \lambda^2)y = 0$$
- Euler Eq. $$x^2 y'' - axy' + by = 0$$
- Legendre Eq. $$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

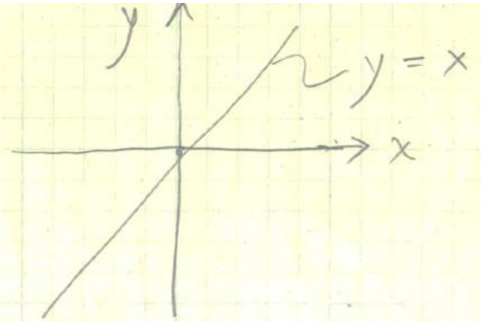
Power Series – Definition

- A power series in $(x-x_0)$ is the infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

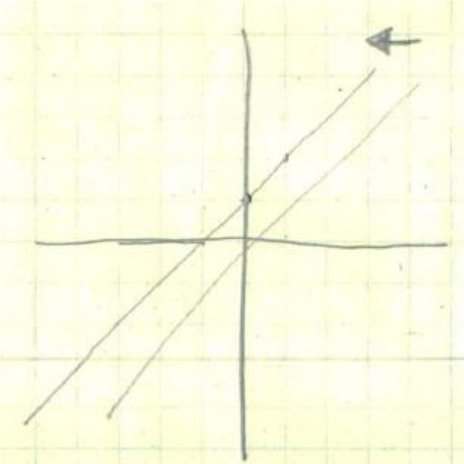
also known as a **power series centered at x_0**

Example - A Power Series Solution



x	y
-2	-2
-1	-1
0	0
1	1
2	2

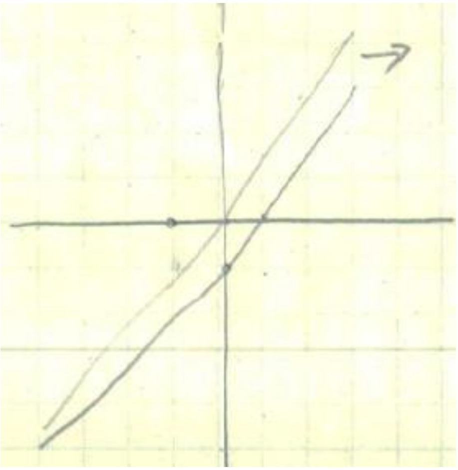
$y = x + x_0$ $x_0 = 1$



x	y
-2	-1
-1	0
0	1
1	2
2	3

shift to the left by x_0

$y = x - x_0$ $x_0 = 1$



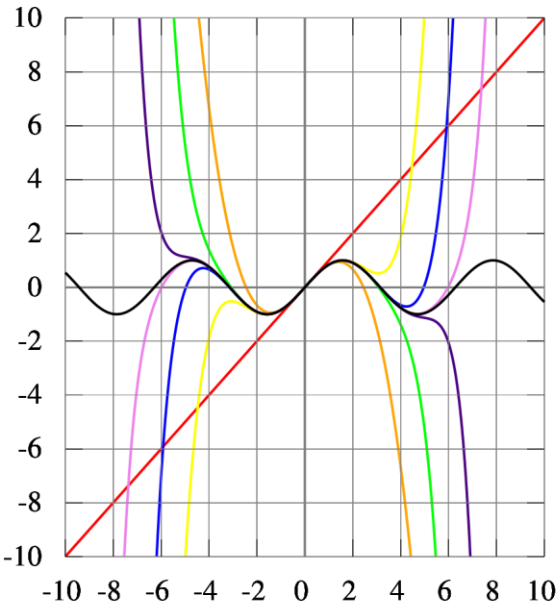
x	y
-2	-3
-1	-2
0	-1
1	0
2	1

shift to the right by x_0

Power Series – Expansion Point at x_0

- A power series of Sin centered at x_0

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

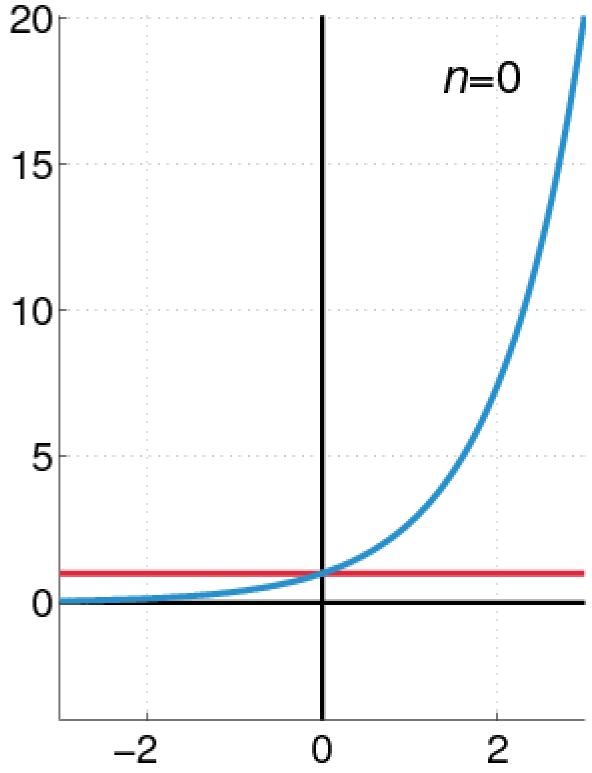


- As the degree of the Taylor polynomial rises, it approaches the correct function. This image shows sin *x* and its Taylor approximations, polynomials of degree **1**, **3**, **5**, **7**, **9**, **11** and **13**.

Power Series – Expansion Point at x_0

- A power series of exponent centered at x_0

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad \cdots \quad = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$



Power Series – Expansion Point at x_0 - Examples

- A power series developed around $(x=x_0)$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

- A power series developed around $(x=x_0=0)$

$$\sum_{n=0}^{\infty} a_n (x)^n = a_0 + a_1(x) + a_2(x)^2 + \cdots$$

- Example - Power series centered around -1

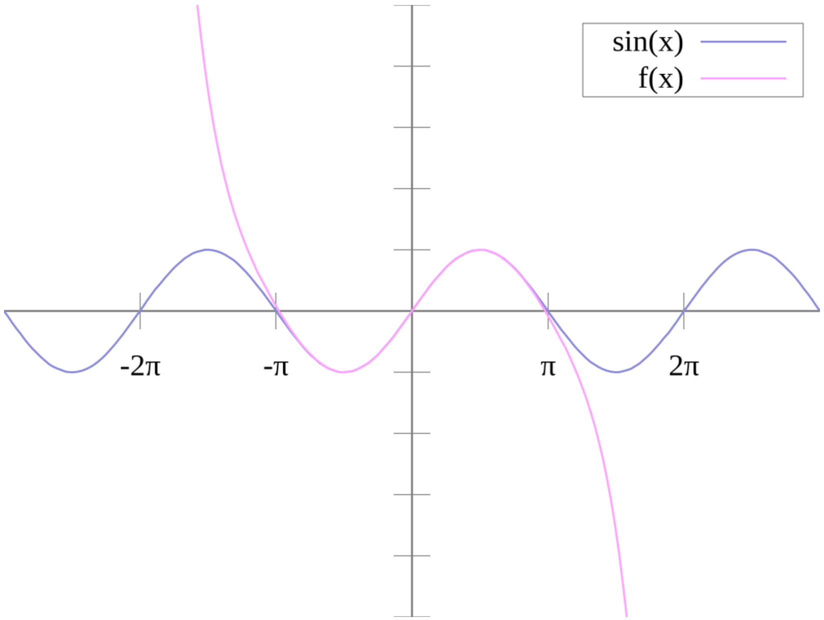
$$\sum_{n=0}^{\infty} (x+1)^n = \sum_{n=0}^{\infty} (x-(-1))^n$$

- Example - Power series centered around 0

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n (x-0)^n$$

Interval / Radius of Convergence & Error

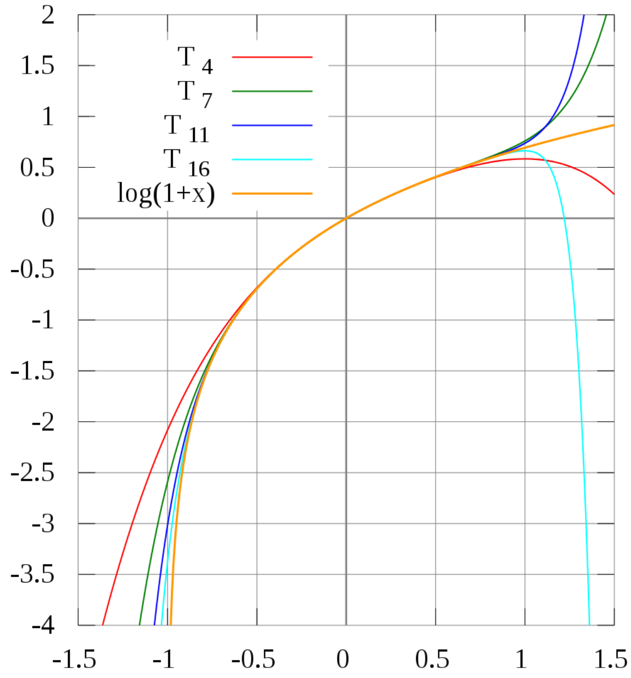
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

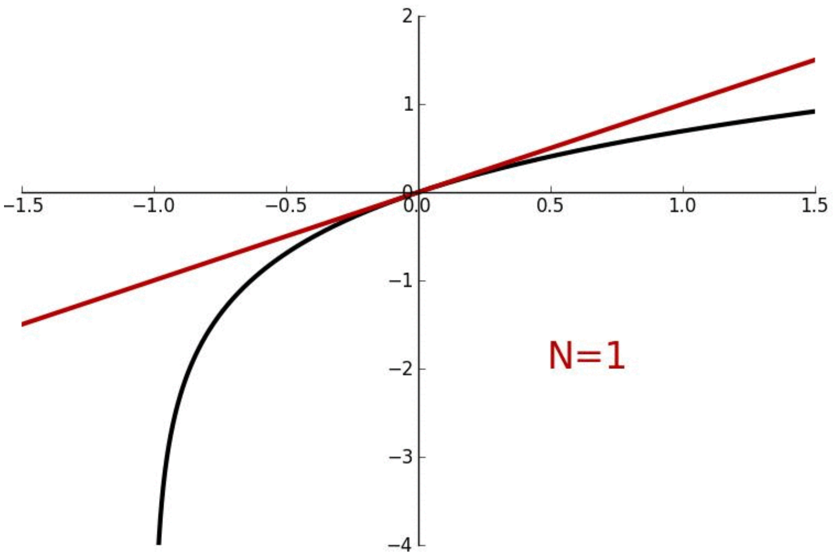


- The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full period centered at the origin.
- Error - for $-1 < x < 1$, the error is less than 0.000003
- Radius of Convergence - $-\infty < x < \infty$

Interval / Radius of Convergence & Error

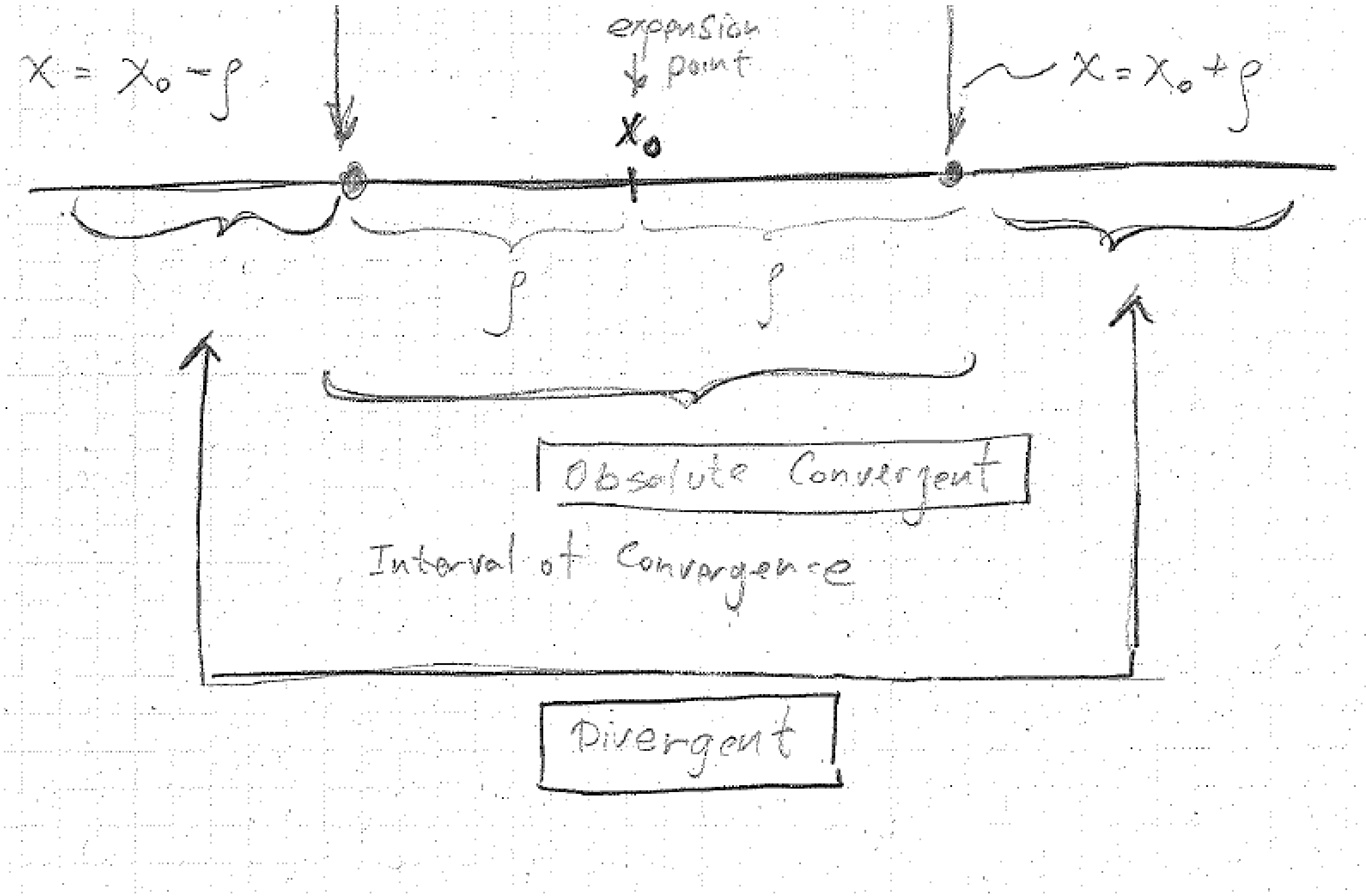
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$





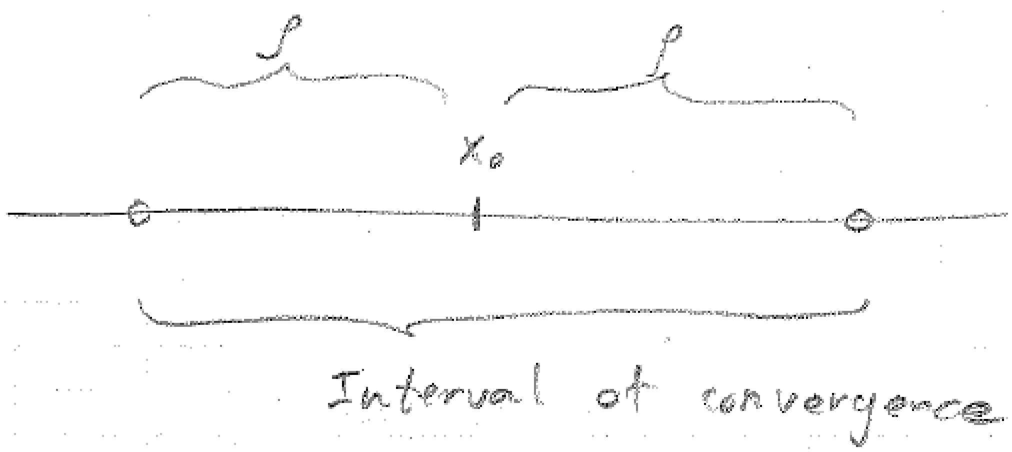
- Error - The Taylor polynomials for log(1 + x) only provide accurate approximations in the range −1 < x ≤ 1. Note that, for x > 1, the Taylor polynomials of higher degree are ***worse*** approximations.
- Radius of Convergence - $-1 < x < 1$

Radius of Convergence



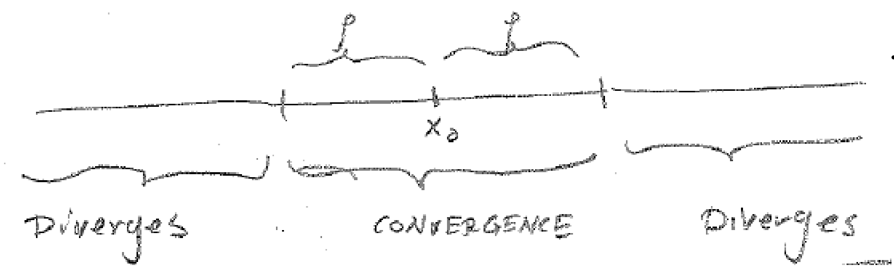
Interval of Convergence

- The interval of convergence is the set of **all** real numbers of x for which the series converges

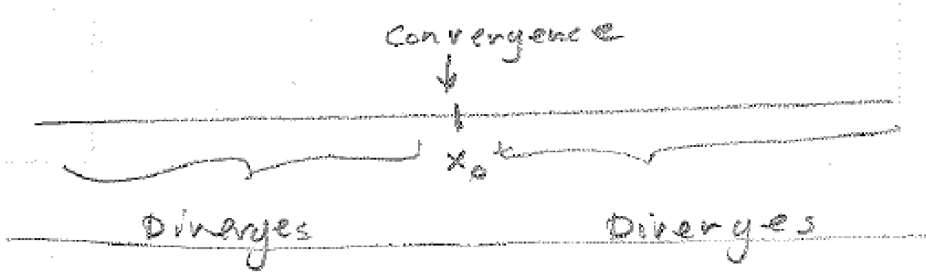


Radius of Convergence

- The radius ρ of the interval of convergence of a power series is called its **radius of convergence**
- If ρ > 0 - The power series
  - converges $|x - x_0| < \rho$ $(-\rho < x - x_0 < \rho)$
  - diverges $|x - x_0| > \rho$ $(x - x_0 > \rho \text{ or } x - x_0 < -\rho)$



- If ρ = 0 - The power series converges only at x_0



- If ρ =∞ - The power series converges for all x

Convergence

- A power series is convergent at a specified value of x if its sequence of partial sum $\{s_n(x)\}$ converges

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} a_n (x - x_0)^n \quad \begin{cases} \text{exist} \rightarrow \text{converge} \\ \text{doesn't exist} \rightarrow \text{diverge} \end{cases}$$

Absolute Convergence Of Power Series

- Absolute Convergence of Power Series
  - A power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to converge absolutely at a point x if

$$\sum_{n=0}^{\infty} \left| a_n (x - x_0)^n \right| = \sum_{n=0}^{\infty} |a_n| \left| (x - x_0)^n \right| \quad \text{converges}$$

- **Radius Of Convergence Of A Power Series (PS)**

  If a power series about $x-x_0$ converges for all values of x in $|x - x_0| < \rho$

  Then ρ is said to be radius of convergence of the PS

Determine The Radius Of Convergence (ρ) For A Given Power Series (Ratio Test)

- If $a_n \neq 0$
- If for a fixed value of x

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|\frac{1}{\rho} = L$$

$$\downarrow$$

$$\rho = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

- Then the power series at that value of x

1) Converges if $L = |x-x_0|\frac{1}{\rho} < 1 \Rightarrow |x-x_0| < \rho$

2) Diverges if $L = |x-x_0|\frac{1}{\rho} > 1 \Rightarrow |x-x_0| > \rho$

3) Inconclusive if $L = |x-x_0|\frac{1}{\rho} = 1 \Rightarrow |x-x_0| = \rho$

Determine The Radius Of Convergence (ρ) For A Given Power Series (Ratio Test) – Example

- Find which values of x does power series converges

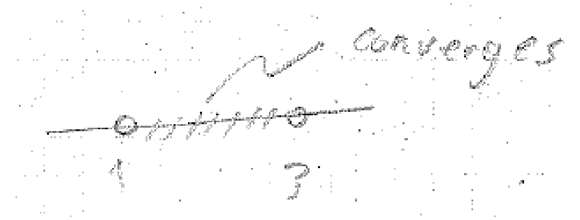
$$\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(-1)^{n+1}(n+1)(x-2)(x-2)^n}{(-1)^{n+1}n(x-2)^n}\right|$$

$$= \underbrace{|x-2|}_{\text{not a function of n}} \underbrace{\lim_{n\to\infty}\frac{n+1}{n}}_{=1} \underset{\text{converges}}{\underset{\uparrow}{<}} 1 \Rightarrow \boxed{|x-2|<1}$$

- Thus

$$\begin{cases} x-2<1 \\ -1<x-2 \end{cases} \Rightarrow \begin{cases} x<3 \\ 1<x \end{cases}$$



Determine The Radius Of Convergence (ρ) For A Given Power Series (Ratio Test) – Example (Continue)

- For x=1

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(-1)^{n} \rightarrow \text{Diverges}$$

- For x=3

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(1)^{n} \rightarrow \text{Diverges}$$

- The radius of convergence is ρ = 1

Power Series of a Given Function

- If for a given x the limit

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n \left(x - x_0\right)^n$$

  exist

- Then the series is said to be power series expansion of f(x)

$$a_0 = f(x_0)$$

The series converges for $x=x_0$

It may converge for all x

It may converge for some value of x and not for others

Power Series of a Given Function

- A power series defines a function that is

$$f(x)=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n$$

whose domain is in the interval of convergence of the series

- If the radius of convergence is
  - R>0
  - R=∞

- Then $f(x)$ on the intervals is
  - Differentiable
  - Continuous
  - Integrable

- Convergence at an end point may be
  - **lost** by differentiation
  - **gain** by integration

Analytical Functions & Power Series

- **Analytical Function – Definition**

  A function f(x) is said to be analytic at $x=x_0$ if f(x) can be differentiated at any number of times.

- For an analytic function

$$a_n = \frac{d^n}{dx^n}\left[f(x)\right]_{x=x_0}$$

exists a_n bounded for all n

Analytical Functions & Power Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} a_n n x^n = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

↑

Note that for n=0 the first term is 0. Start summing from 1

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^n = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$$

↑

Note that for n=0,n=1 the first and the second terms are 0.
Start summing from 2

Power Series (PS)
Representation of an Analytic Function

- An analytic function f(x) has a power series representation within the domain of convergence an f(x) can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\left[f(x)\right]\Big|_{x=x_0} (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

within the domain of convergence $|x - x_0| < \rho$

x_0 – The expansion point of the PS

Taylor Series

- Suppose that $\sum a_n (x - x_0)^n$ converges to f(x) for $|x - x_0| < \rho$
- Then the value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and the series is called the **Taylor Series** for f about $x=x_0$

- If

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

  - f(t) is continuous
  - Has derivative of all orders on the interval of convergence
  - The derivatives of f can be computed by differentiating the relevant series term by term

Taylor Series

- Taylor Series: for a point $x_0 \neq 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

- Maclaurin Series: for a point $x_0 = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

PS Expansions of Analytical Function (Maclaurine Series)

Intervalof convergence

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad -\infty < x < \infty$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad -\infty < x < \infty$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad \cdots \quad = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad -\infty < x < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad -1 < x < 1$$

These results can be used to obtain power series representations of other function

$e.g.\ e^{2x}$ (replace $x \to x^2$)

Interval of convergence

$$e^{2x} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \qquad -\infty < x < \infty$$

PS Expansions of Analytical Function (Maclaurine Series)

To obtain a Taylor series representation of ln x centered at $x_0 = 1$

Replace $x \to x-1$

$$\ln x = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Interval of convergence is shifted by 1 from (-1, 1] to (0, 2]

Arithmetic of Power Series

- Multiplication of Power Series

$$e^x \sin x = \overbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}^{e^x} \overbrace{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)}^{\sin x}$$

- Addition of power series

$$\overbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}^{I} + \overbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}^{II}$$

- Shifting an index of summation

$$\sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2}$$

Arithmetic of Power Series – Multiplication

$$e^x \sin x = \underbrace{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}_{e^x}\underbrace{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)}_{\sin x}$$

$$= (1)\,x + (1)\,x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

since e^x and $\sin x$ both converge on $(-\infty\ \ \infty)$ the product converges on the same interval

Arithmetic of Power Series – Addition

$$\overbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}^{I} + \overbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}^{II}$$

1. Both series should start with the same power

2. Both indices of summation should start with the same number

$$\overbrace{\sum_{\boxed{n=2}}^{\infty} n(n-1)a_n x^{\boxed{n-2}}}^{I} + \overbrace{\sum_{\boxed{n=0}}^{\infty} a_n x^{\boxed{n+1}}}^{II}$$

Arithmetic of Power Series – Addition

1. Both series should start with the same power

$$\overbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}}^{I} + \overbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}^{II}$$

$k = n - 2$

$(n = k + 2)$

$k = n + 1$

$(n = k - 1)$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^{k} + \sum_{k=1}^{\infty} a_{k-1} x^{k}$$

Arithmetic of Power Series – Addition

2. Both indices of summation should start with the same number

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^{k}+\sum_{k=1}^{\infty}a_{k-1}x^{k}$$

same

$$2\cdot 1\cdot a_2x^0+\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^{k}+\sum_{k=1}^{\infty}a_{k-1}x^{k}=2a_2+\sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2}+a_{k-1}\right]x^{k}$$

same

Arithmetic of Power Series – Shifting Index of Summation

$$\sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2}$$

start from $n = 0$ $\quad m = n - 2$

$n = m + 2$

- The index of summation in an infinite series is a dummy parameter.

Arithmetic of Power Series – Rewriting Generic Term

Generic term

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}=\sum_{m=0}^{\infty}(m+4)(m+3)a_{m+2}(x-x_0)^m$$

$$m=n-2$$

$$n=m+2$$

$n=2$ corresponds to $m=0$

Arithmetic of Power Series – Rewriting Generic Term

put x^2 into the sum

$$x^2\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} = \sum_{n=0}^{\infty}(r+n)a_n x^{n+r+1} = \sum_{m=1}^{\infty}(r+m-1)x^{r+m}$$

$$x^2 x^{r+m-1} = x^{r+m-1+2} = x^{r+m+1}$$

$$\left.\begin{aligned} m &= n+1 \\ n &= m-1 \end{aligned}\right\}$$

Series Equality

- If two power series are equal

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n = \sum_{n=1}^{\infty} b_n (x - x_0)^n$$

for each x in some open interval with center $x(0)$

Then $a_n = b_n$ for $n = 0, 1, 2, \cdots$

Determining Coefficients

Assume $$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

- What this implies about the coefficients

- Rewriting both series with the same power of x

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

$n-1=m$

$n=m+1$

replace $m \to n$

$$(n+1)a_{n+1} = a_n \;\rightarrow\; a_{n+1} = \frac{a_n}{n+1} \quad \text{for } n = 0, 1, 2, 3, \cdots$$

Determining Coefficients

$$a_1 = \frac{a_0}{2}$$

$$a_2 = \frac{a_1}{3} = \frac{a_0}{2*3} = \frac{a_0}{6}$$

$$a_3 = \frac{a_2}{4} = \frac{a_0}{4*6} = \frac{a_0}{24}$$

$$a_n = \frac{a_0}{n!}$$

$$a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

A Power Series Solution – Example

$$y' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Step 1 : calculate derivative of the assumed solution

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x^1 + 3a_3 x^2$$

Step 2 : substitute y & y’ into the diff eq.

$$y' + y = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

A Power Series Solution – Example

Step 3 : shift indices of summation

$$y' + y = \underbrace{\sum_{n=1}^{\infty} a_n n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{n=k}$$

$$= \sum_{k=0}^{\infty} a_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} \left[a_{k+1}(k+1) + a_k\right] x^k$$

Step 4 : Because $y' + y = 0$ for all x in some interval

$$\sum_{k=0}^{\infty} \underbrace{\left[a_{k+1}(k+1) + a_k\right]}_{=0} x^k = 0$$

$$a_{k+1}(k+1) + a_k = 0$$

A Power Series Solution – Example

$$a_{k+1} = -\frac{a_k}{(k+1)}$$

$$a_1 = -\frac{1}{1} a_0 = -a_0$$

$$a_2 = -\frac{1}{2} a_1 = -\frac{1}{2}(-a_0) = \frac{1}{2} a_0$$

$$a_3 = -\frac{1}{3} a_2 = -\frac{1}{3*2} a_0$$

$$a_4 = -\frac{1}{4} a_3 = -\frac{1}{432} a_0$$

Step 5 : Define the solution

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = a_0 - a_0 x + \frac{1}{2} a_0 x^2 - \frac{1}{3*2} a_0 x^3 + \frac{1}{4*3*2} a_0 x^4$$

$$= a_0 \left[1 - x + \frac{1}{2} x^2 - \frac{1}{3*2} x^3 + \frac{1}{4*3*2} x^4 \right] = a_0 \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k}_{e^{-x}}$$

$$y = a_0 e^{-x}$$

A Power Series Solution – Example

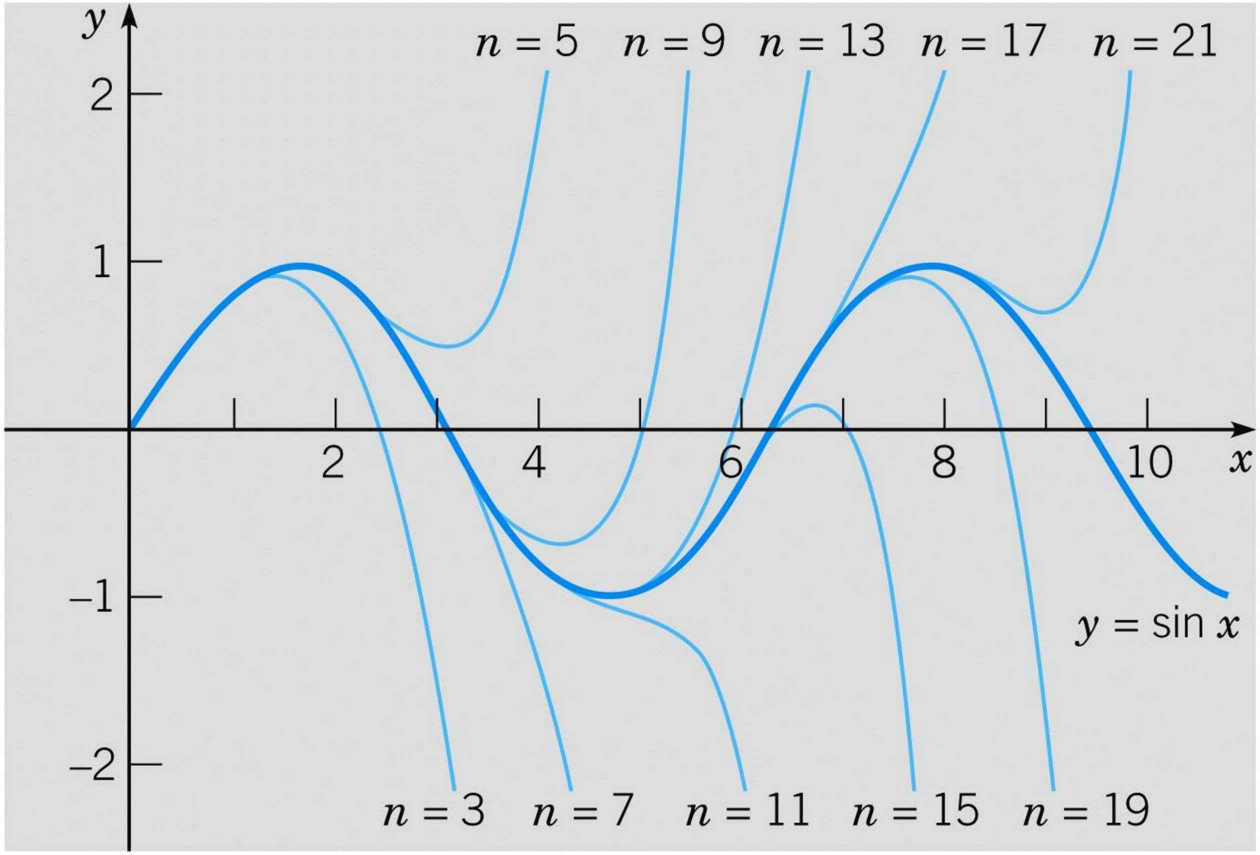


Figure 5.2.2

$$\sin(x) \cong x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$$

Famous Series Solutions

Airy Eq.	$y'' - xy = 0$
Chebychev Eq.	$(1 - x^2)y'' - xy' + \lambda^2 y = 0$
Hermite Eq.	$y'' - 2xy' + \lambda y = 0$
Bessel's Eq.	$t^2 x'' + tx' + (t^2 - \lambda^2)x = 0$
Euler's Eq.	$x^2 y'' + axy' + \beta y = 0$
Legendre's Eq.	$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$