Class Notes 8:

High Order Linear Differential Equation Non Homogeneous

MAE 82 – Engineering Mathematics

High Order Differential Equations – Introduction

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$y = y_c(x) + y_p(x)$$

- Solution methods for the particular solution (Nonhomogeneous)
  - Undetermined Coefficients (polynomials, Exponent, Sin/Cos)
  - Variation of Parameters (all functions – general method)

Method of Undermined Coefficients – Class A

Form of $\mathrm{y_p}$ for various form of $g(x)$

$$\boxed{\text{class}(a)} \quad g(x) = \sum_{r=0}^{m} a_r x^r \ (m_{th} \text{ degree of polynomial in } x)$$

$$y_p(x) \begin{cases} \displaystyle\sum_{r=0}^{m} A_r x^r & (a_0 \neq 0) \\ \displaystyle x\sum_{r=0}^{m} A_r x^r & (a_0 = 0, a_1 \neq 0) \\ \displaystyle x^2\sum_{r=0}^{m} A_r x^r & (a_0 = 0, a_1 = 0, a_2 \neq 0) \\ \displaystyle x^s\sum_{r=0}^{m} A_r x^r & (a_0 = a_1 = \cdots a_{s-1} = 0, a_s \neq 0) \end{cases}$$

Method of Undermined Coefficients – Class B

Form of y_p for various form of $g(x)$

class (b) $g(x) = e^{\lambda x} \sum_{r=0}^{m} a_r x^r$ (Exponential multiplied by a polynomial in x)

$$y_p(x) = x^s e^{\lambda x} \sum_{r=0}^{m} A_r x^r \ (ch(\lambda) = 0;\ \text{s} - \text{times repeated})$$

Method of Undermined Coefficients – Class C

Form of $\mathrm{y_p}$ for various form of $g(x)$

$$\boxed{\text{class}(c)} \quad g(x) = e^{\mu x}\begin{Bmatrix} \sin(\beta x) \\ \cos(\beta x) \end{Bmatrix} \sum_{r=0}^{m} a_r x^r$$

Exponential multiplied by a polynomial in x and a sin or cos in x

$$\alpha = \mu + i\beta$$

$$y_p(x) = x^s e^{\mu x}\left[\sin(\beta x)\sum_{r=0}^{m} A_r x^r + \cos(\beta x)\sum_{r=0}^{m} B_r x^r\right]$$

$$ch(\mu + i\beta) = 0 \quad repeateds - times$$

General Rule for Writing the Correct Form of the Particular Solution

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$L[y] = g(x)$$

$$L[y] = 0$$

class(a) $\quad g(x) = \sum_{r=0}^{m} a_r x^r$

$$y_p(x) = x^s \sum_{r=0}^{m} A_r x^r \quad (a_0 = a_1 = \cdots a_{s-1} = 0, a_s \neq 0)$$

$$(a_0 = a_1 = \cdots a_{s-1} = 0, a_s \neq 0)$$

class(b) $\quad g(x) = e^{\lambda x} \sum_{r=0}^{m} a_r x^r$

$$y_p(x) = x^s e^{\lambda x} \sum_{r=0}^{m} A_r x^r$$

$ch(\lambda) = 0;$ s – times repeated

class(c) $\quad g(x) = e^{\mu x} \left\{ \begin{matrix} \sin(\beta x) \\ \cos(\beta x) \end{matrix} \right\} \sum_{r=0}^{m} a_r x^r$

$$y_p(x) = x^s e^{\mu x} \left[\sin(\beta x) \sum_{r=0}^{m} A_r x^r + \cos(\beta x) \sum_{r=0}^{m} B_r x^r \right]$$

$ch(\mu + i\beta) = 0$ $\quad$ repeated s – times

General Rule for Writing the Correct Form of the Particular Solution

- Correct form of $y_p(x)$ depends on

$$\begin{cases} -\ g(x) \\ -\ L\ \ (\text{differential operator}) \end{cases}$$

- Given $g(x)$

$$y_p(x) \text{ changes with } L$$

- Given L

$$y_p(x) \text{ changes with } g(x)$$

- $y_p(x)$ should not contain a term which is a solution

  of $L[y]=0$

- Such a term should be multiplied by proper power of x as shown in forms for various classes $(a)-(c)$

Method of Undermined Coefficients – Example 1

$$2y''' - 4y'' - 2y' + 4y = g(t)$$

$$L[y] = 0$$

$$L[y] = 2y''' + 4y'' - 2y' + 4y$$

$$g(t) - \begin{cases} \sum_{r=0}^{m} a_r x^r \\ e^{\lambda x} \sum_{r=0}^{m} a_r x^r \\ e^{\mu x} \begin{Bmatrix} \sin(\beta x) \\ \cos(\beta x) \end{Bmatrix} \sum_{r=0}^{m} a_r x^r \end{cases}$$

Homo diff Eq.

$$L[y] = 0$$

$$y = e^{rt}$$

$$2r^3 - 4r^2 - 2r + 4 = 2(r^2 - 1)(r - 2) = 0$$

roots $r = 1, -1, 2$ (all simple roots)

$$y_c(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}$$

Method of Undermined Coefficients – Example 1

Differential operator (system) $L[y] = 2y''' - 4y'' - 2y' + 4y;\ y = g(t)$

Homogeneous $Eq.\ L[y]=0 \rightarrow$ Fundamental solutions $\begin{cases} y_1(t) = e^t \\ y_2(t) = e^{2t} \\ y_3(t) = e^{-t} \end{cases}$

roots $r = 1, -1, 2$ (all simple roots)

$y_c(t) = c_1e^t + c_2e^{-t} + c_3e^{2t}$

Nonhomogeneous term g(t)	Fundamental solution Yes/No	class	Particular solution $y_p(t)$
t^2	N	a	$At^2 + Bt + c$
e^t (repeat of $y_1(t)$)	Y	b	Dte^t
e^{2t} (repeat of $y_2(t)$)	Y	b	Dte^{2t}
e^{3t}	N	b	De^{3t}
$t^2 + e^t$	class 1&2	a+b	$At^2 + Bt + C + Dte^t$
$\sin t$	N	c	$A\sin t + B\cos t$
$\cos t$	N	c	$A\sin t + B\cos t$
$e^t \sin(t)$	N	c	$e^t(A\sin t + B\cos t)$
$te^t \sin(t)$	N	c	$e^t\left[(A+Bt)\sin t + (C+Dt)\cos t\right]$

Method of Undermined Coefficients – Example 2

$$y^{(4)} - 4y'' = t^2 + e^t$$

Homogeneous diff eq. $L[y] = y^{(4)} - 4y'' = 0$

$$1y^{(4)} + 0y^{(3)} - 4y'' + 0y' + 0y = 0$$

$$a_4 = 1,\ a_3 = 0,\ a_2 = -4,\ a_1 = 0,\ a_0 = 0$$

Solution $y = e^{rt}$ $\quad r^4 - 4r^2 = 0 \quad r^2(r^2 - 4) = 0$

$$r = 0,\ 0,\ +2,\ -2$$

repeated $s = 1$ (number of repetition)

General solution of $L[y] = 0$ (Homogeneous solution)

$$y_c = c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{2t} + c_4 e^{-2t}$$

$$\to y_c = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$$

Method of Undermined Coefficients – Example 2

- Particular solution (Non-homogeneous)

$$y_p = t^2\underbrace{\left(At^2 + Bt + C\right)}_{\text{Due to } t^2 \text{ (Class A)}} + \underbrace{De^t}_{\text{Due to } e^t \text{ (Class B)}}$$

$$\downarrow$$

$$a_0 = a_1 = 0$$

- Note for class A

$$g(t) = t^2 + e^t = 1t^2 + 0t + 0 + e^t$$

$$a_2 = 1,\ a_1 = 0,\ a_0 = 0$$

$$s = 2$$

$$\longrightarrow \quad y_p(x) \begin{cases} \sum_{r=0}^{m} A_r x^r & (a_0 \neq 0) \\ x\sum_{r=0}^{m} A_r x^r & (a_0 = 0, a_1 \neq 0) \\ \boxed{x^2\sum_{r=0}^{m} A_r x^r \quad (a_0 = 0, a_1 = 0, a_2 \neq 0)} \\ x^s\sum_{r=0}^{m} A_r x^r & (a_0 = a_1 = \cdots a_{s-1} = 0, a_s \neq 0) \end{cases}$$

Method of Undermined Coefficients – Example 2

- Solve for A, B, C

$$L[y_p] = t^2 + e^t \quad \Rightarrow \quad y^{(4)} - 4y'' = t^2 + e^t$$

$$\begin{aligned}
y_p &= At^4 + Bt^3 + Ct^2 + De^t \\
y_p' &= 4At^3 + 3Bt^2 + 2Ct + De^t \\
y_p'' &= 12At^2 + 6Bt + 2C + De^t \\
y_p''' &= 24At + 6B + \qquad + De^t \\
y_p^{(4)} &= 24A \qquad\qquad\qquad + De^t
\end{aligned}$$

$$24A - 4\left(12At^2 + 6Bt + 2C + De^t\right) + De^t = t^2 + e^t$$

Method of Undermined Coefficients – Example 2

t	$24B=0$	$B=0$
t^2	$-48A=1$	$A=-1/48$
t^3	————	————
e^t	$-3D=1$	$D=-1/3$

Constants: $24A-8C=0$ $\quad C=\frac{24}{8}A=3A=-\frac{3}{48}=-\frac{1}{16}$

$$y_p=-t^2\left(\frac{1}{48}t^2+\frac{1}{16}\right)-\frac{1}{3}e^t$$

- General Solution of the Differential Equation

$$y(t)=y_c+y_p=C_1+C_2t+C_3e^{2t}+C_4e^{-2t}-t^2\left(\frac{1}{48}t^2+\frac{1}{16}\right)-\frac{1}{3}e^t$$

- Use initial condition

$$\left.\begin{aligned} y'''(0)&=y_0''' \\ y''(0)&=y_0'' \\ y'(0)&=y_0' \\ y(0)&=y_0 \end{aligned}\right\}$$

To Solve for C_1, C_2, C_3, C_4

Method of Variation of Parameters – Review

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

$y_1(t)$, $y_2(t)$ are the fundamental solution of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

$$y_p = -y_1(t)\int^t \frac{y_2(s)g(s)}{W(s)}\,ds + y_2(t)\int^t \frac{y_1(s)g(s)}{W(s)}\,ds$$

$$y_p = y_1(t)\int^t \frac{\begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} g(s)}{W(s)}\,ds + y_2(t)\int^t \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} g(s)}{W(s)}\,ds$$

$$y_p = \sum_{i=1}^{2} y_i(t)\int \frac{W_i(s)g(s)}{W(s)}\,ds$$

W(s) – Wronskian $= y_1(s)y_2'(s) - y_2(s)y_1'(s)$

$W_i(s)$ - Wronskian where i-th column is replaced by zeros except the last row is 1

Method of Variation of Parameters - Example

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = g(x) \qquad x > 0$$

- Given: fundamental solutions

$$y_1(x) = x; \quad y_2 = x^2; \quad y_3 = x^3$$

- Rewrite the given differential equation in the standard form

$$L[y] = y''' - \frac{3}{x} y'' + \frac{6}{x^2} y' - \frac{6}{x^3} y = \frac{g(x)}{x^3} \qquad x > 0$$

- Check fundamental solutions

$$\begin{cases} L[y_1(x)] = 0 \\ L[y_2(x)] = 0 \\ L[y_3(x)] = 0 \end{cases}$$

Method of Variation of Parameters - Example

- Derive the Wronskians

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - 6x^3 + 2x^3 = 2x^3 \neq 0 \quad \text{as} \quad x > 0$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Method of Variation of Parameters - Example

$$y_p(x) = \sum_{i=1}^{3} y_i(x) \int^{x} \frac{W_i(s)}{W(s)} \left(\frac{g(s)}{s^3} \right) ds$$

$$y_p = x \int^{x} \frac{s^4}{2s^3} \left(\frac{g(s)}{s^3} \right) ds + x^2 \int^{x} \frac{-2s^3}{2s^3} \left(\frac{g(s)}{s^3} \right) ds + x^3 \int^{x} \frac{s^2}{2s^3} \left(\frac{g(s)}{s^3} \right) ds$$

$$y_p = x \int^{x} \frac{g(s)}{2s^2} ds - x^2 \int^{x} \frac{g(s)}{s^3} ds + x^3 \int^{x} \frac{g(s)}{2s^4} ds$$

$$y_p = \frac{1}{2} \int^{x} \left(\frac{x}{s^2} - \frac{2x^2}{s^3} + \frac{x^3}{s^4} \right) g(s) ds$$

- General Solution

$$y(x) = C_1 x + C_2 x^2 + C_3 x^3 + y_p(x)$$