Class Notes 7:

High Order Linear Differential Equation Homogeneous

MAE 82 – Engineering Mathematics

High Order Differential Equations – Introduction

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_1(t)\frac{dy}{dt} + P_0(t)y = \begin{cases} 0 \\ G(t) \end{cases}$$

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = \begin{cases} 0 \\ G(t) \end{cases}$$

High Order Differential Equations – General Theory

- An ***n*th order ODE** has the general form

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_1(t)\frac{dy}{dt} + P_0(t)y = G(t)$$

- We assume that $P_0, \ldots, P_n$, and G are continuous real-valued functions on some interval $I = (\alpha, \beta)$, and that P_n is nowhere zero on I.
- Dividing by P_n, the ODE becomes

$$L[y] = \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)$$

- For an *n*th order ODE, there are typically *n* initial conditions:

$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Theorem 4.1.1

- Consider the *n*th order initial value problem

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)$$

$$y(t_0) = y_0,\ \ y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

- If the functions $p_1,\ldots, p_n$, and g are continuous on an open interval *I*,
- then there exists exactly one solution $y = \phi(t)$ that satisfies the initial value problem. This solution exists throughout the interval *I*.

High Order Differential Equations – Homogeneous Equation

- As with 2nd order case, we begin with homogeneous ODE:

$$L[y] = \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0$$

- If $y_1, \ldots, y_n$ are solns to ODE, then so is linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Every solution can be expressed in this form, with coefficients determined by initial conditions, if we can solve:

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

High Order Differential Equations – Homogeneous Equation & Wronskian

- The system of equations on the previous slide has a unique solution if its determinant, or Wronskian, is nonzero at t_0:

$$W(y_1, y_2, \ldots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0$$

- Since t_0 can be any point in the interval *I*, the Wronskian determinant needs to be nonzero at every point in *I*.
- As before, it turns out that the Wronskian is either zero for every point in *I*, or it is never zero on *I*.

Theorem 4.1.3

- If $\{y_1,\ldots, y_n\}$ is a fundamental set of solutions of

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval I,

- then $\{y_1,\ldots, y_n\}$ are linearly independent on that interval.
- if $\{y_1,\ldots, y_n\}$ are linearly independent solutions to the above differential equation,
- then they form a fundamental set of solutions on the interval I

High Order Differential Equations – Nonhomogeneous Equation

- Consider the nonhomogeneous equation:

$$L[y] = \frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t)$$

- If Y_1, Y_2 are solutions to nonhomogeneous equation, then $Y_1 - Y_2$ is a solution to the homogeneous equation:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$$

- Then there exist coefficients $c_1, \ldots, c_n$ such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Thus the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$

where Y is any particular solution to nonhomogeneous ODE.

High Order Differential Equations – Homogeneous Equation with Constant Coefficients

- Consider the *n*th order linear homogeneous differential equation with constant, real coefficients:

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- As with second order linear equations with constant coefficients, $y = e^{rt}$ is a solution for values of r that make characteristic polynomial $Z(r)$ zero:

$$L[e^{rt}] = e^{rt} \underbrace{\left[a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right]}_{\text{characteristic polynomial } Z(r)} = 0$$

- By the fundamental theorem of algebra, a polynomial of degree n has n roots $r_1, r_2, \ldots, r_n$, and hence

$$Z(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n)$$

High Order Differential Equations – Homogeneous Equation with Constant Coefficients Cases

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- $a_0, a_1, \ldots, a_n$ – Real constants $a_n \neq 0$

$$y = e^{rt}$$

$$L[e^{rt}] = e^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right)$$

- Characteristic Polynomial

$$Z(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$$

- Characteristic Equation

$$Z(r) = 0$$

High Order Differential Equations – Homogeneous Equation with Constant Coefficients Cases

- Z(r) – {Polynomial of degree n}

➡ n zeros, r1, r2,…, rn

Some of which may be equal

$$Z(r) = a_n(r - r_1)(r - r_2)\cdots(r - r_n)$$

(1) Real and unequal roots

(2) Complex roots

(3) Repeated roots

Case 1: Real & Unequal Roots

- Real and no two are equal ⟶ n distinct solutions
- Complimentary Solution (Homogeneous)

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}$$

Case 1: Real & Unequal Roots – Example

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1 \end{cases}$$

- Assume

$$y = e^{rt}$$

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

Case 1: Real & Unequal Roots – Example

- Roots
$$\begin{cases} r_1 = 1 \\ r_2 = -1 \\ r_3 = 2 \\ r_4 = -3 \end{cases}$$

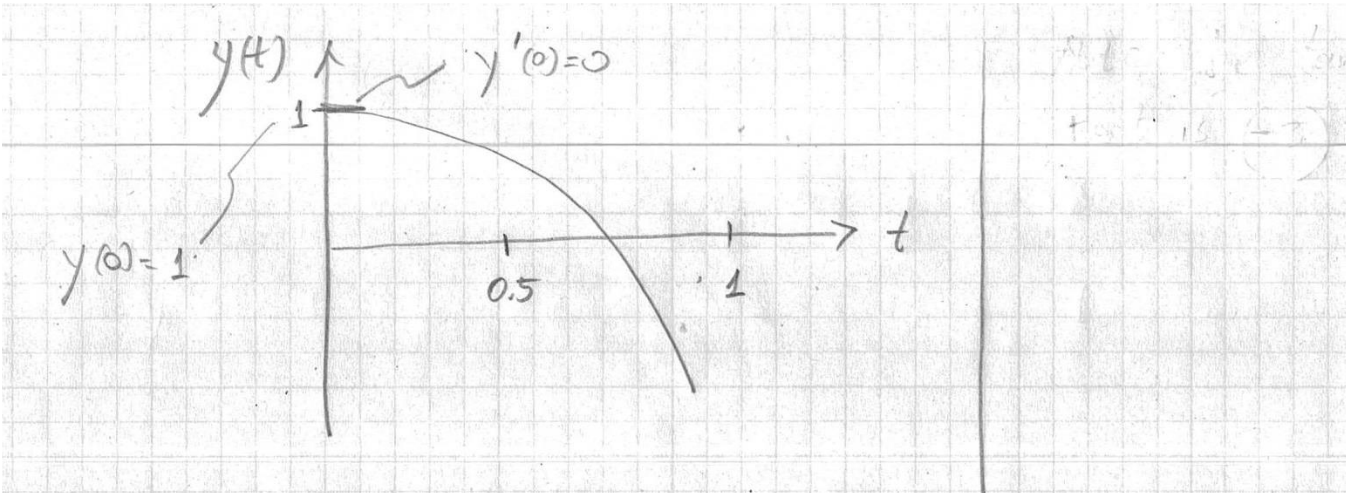
- The solution
$$\begin{cases} y = C_1 e^{t} + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-3t} \\ y' = C_1 e^{t} - C_2 e^{-t} + 2C_3 e^{2t} - 3C_4 e^{-3t} \\ y'' = C_1 e^{t} + C_2 e^{-t} + 4C_3 e^{2t} + 9C_4 e^{-3t} \\ y''' = C_1 e^{t} - C_2 e^{-t} + 8C_3 e^{2t} - 27C_4 e^{-3t} \end{cases}$$

I.C
$$\begin{cases} y(0) = 1 & \rightarrow & C_1 + C_2 + C_3 + C_4 = 1 \\ y'(0) = 0 & \rightarrow & C_1 - C_2 + 2C_3 - 3C_4 = 0 \\ y''(0) = -2 & \rightarrow & C_1 + C_2 + 4C_3 + 9C_4 = -2 \\ y'''(0) = -1 & \rightarrow & C_1 + C_2 + 8C_3 - 27C_4 = -1 \end{cases}$$

Case 1: Real & Unequal Roots – Example

$$\begin{cases} C_1 = \frac{11}{8} \\ C_2 = \frac{5}{12} \\ C_3 = -\frac{2}{3} \\ C_4 = -\frac{1}{8} \end{cases}$$

$$y = \frac{11}{8}e^{t} + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$



Case 2: Complex Roots

- Two complex conjugate roots

$$\alpha = \lambda + i\mu \rightarrow y(x) = e^{\lambda \pm i\mu} = e^{\lambda x}\left[\cos \mu x + i \sin \mu x\right]$$

- Solution

$$y_1 = e^{\lambda x} \cos \mu x$$
$$y_1 = e^{\lambda x} \sin \mu x$$

$$y(x) = C_1 e^{\lambda x} \cos \mu x + C_2 e^{\lambda x} \sin \mu x$$

Case 2: Complex Roots – Example

$$y^{(4)} - y = 0$$

$$\begin{cases} y(0) = 7/2 \\ y'(0) = -4 \\ y''(0) = 5/2 \\ y'''(0) = -2 \end{cases}$$

Case 2: Complex Roots

substituting $y = e^{rt}$

$$r^4 e^{rt} - e^{rt} = 0$$

$$e^{rt}(r^4 - 1) = 0$$

$$(r^4 - 1) = (r^2 - 1)(r^2 + 1) = 0$$

$$\left.\begin{aligned} r_1 &= \ \ 1 + 0i \\ r_2 &= -1 - 0i \end{aligned}\right\} real \quad \rightarrow \quad c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\left.\begin{aligned} r_3 &= \ 0 + 1i \\ r_4 &= \ 0 - 1i \end{aligned}\right\} complex \rightarrow \quad c_3 e^{\lambda t} \cos \mu t + c_4 e^{\lambda t} \sin \mu t$$

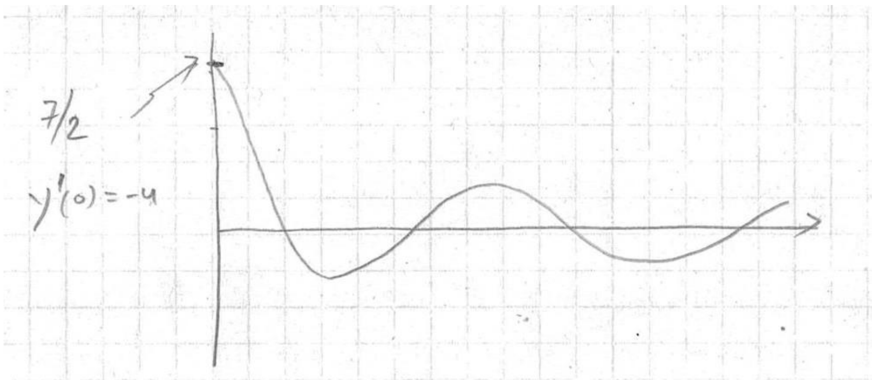
Case 2: Complex Roots

$$\begin{cases} y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \\ y' = c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t \\ y'' = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t \\ y''' = c_1 e^t - c_2 e^{-t} + c_3 \sin t - c_4 \cos t \\ y^{(4)} = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \end{cases}$$

$$\left.\begin{array}{l} y(0) = 7/2 \;\rightarrow\; c_1 + c_2 + c_3 = 7/2 \\ y'(0) = -4 \;\rightarrow\; c_1 - c_2 + c_4 = -4 \\ y''(0) = 5/2 \;\rightarrow\; c_1 + c_2 - c_3 = 5/2 \\ y'''(0) = -2 \;\rightarrow\; c_1 - c_2 - c_4 = -2 \end{array}\right\} \rightarrow \begin{cases} c_1 = 0 \\ c_2 = 3 \\ c_3 = 1/2 \\ c_4 = -1 \end{cases}$$

Case 2: Complex Roots

$$y = 3e^{-t} + \frac{1}{2}\cos t - \sin t$$

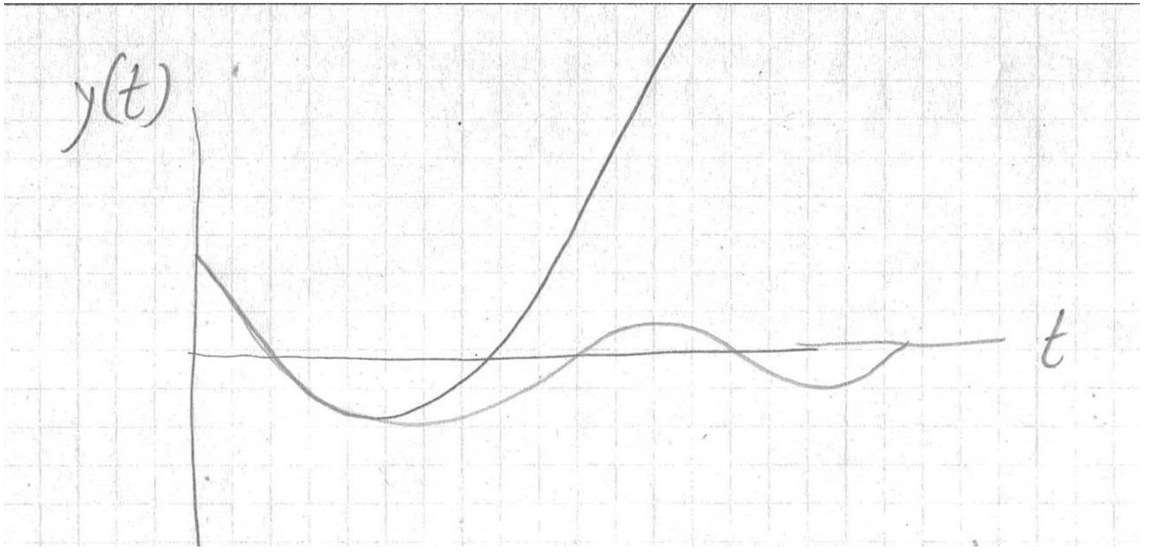


Note – Given the initial conditions $\rightarrow$ $c_1 = 0$

– For the same first IC but with $y'''(0) = -15/8$ instead of $y'''(0) = -2$ the solution is

$$y = \frac{1}{32}e^{t} + \frac{95}{32}e^{-t} + \frac{1}{2}\cos t - \frac{17}{16}\sin t$$

Case 2: Complex Roots



e^t dominate the solution even with a small coef. $1/32$, to large t

change $y'''(0) = -2 \rightarrow y'''(0) = -15/8$

Case 3: Repeated Roots

$$\alpha = \lambda + i\mu \quad y(x) = e^{(\lambda+i\mu)x} = e^{\lambda x}\left[\cos \mu x \pm i \sin \mu x\right]$$

$$\downarrow$$

$$\begin{cases} y_1 = e^{\lambda x} \cos \mu x \\ y_2 = e^{\lambda x} \sin \mu x \end{cases}$$

Complex conjugate roots $\alpha = \lambda + i\mu$ (repeated s – times)

$y_1 = e^{\lambda x} \cos \mu x$	$y_2 = e^{\lambda x} \sin \mu x$	0
$y_3 = xe^{\lambda x} \cos \mu x$	$y_4 = xe^{\lambda x} \sin \mu x$	1
$y_5 = x^2 e^{\lambda x} \cos \mu x$	$y_6 = x^2 e^{\lambda x} \sin \mu x$	2
$y_{2s+1} = x^s e^{\lambda x} \cos \mu x$	$y_{2s+2} = x^s e^{\lambda x} \sin \mu x$	s

Case 3: Repeated Roots – Example

$$y^{(4)} + 2y'' + y = 0$$

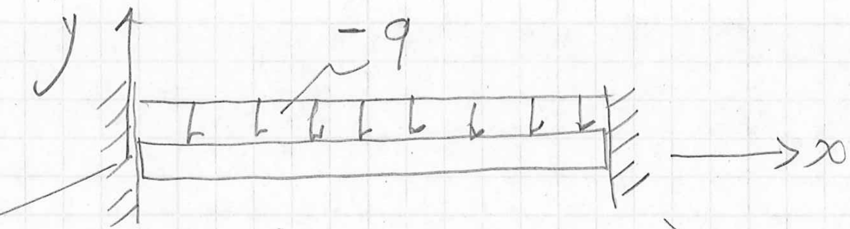
$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0$$

$$r \begin{cases} 0 + 1i \\ 0 + 1i \\ 0 - 1i \\ 0 - 1i \end{cases}$$

$$y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t) + c_3 t e^{\lambda t} \cos(\mu t) + c_4 t e^{\lambda t} \sin(\mu t)$$

$$y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t)$$

BOUNDARY VALUE PROBLEM - BEAM



$y(x=0) = 0$

$\frac{dy}{dx}(x=0) = 0$

Euler-Bernoulli Diff Eq.

$$EI\frac{d^4y}{dx^4} = -q$$

$y(x=L) = 0$

$\frac{dy}{dx}(x=L) = 0$

Repeated integrating the diff eq. we find the function $y(x)$

$$\frac{d^3y}{dx^3} = -\frac{qx}{EI} + C_1$$

$$\frac{d^2y}{dx^2} = -\frac{qx^2}{2EI} + C_1x + C_2$$

$$\frac{dy}{dx} = -\frac{qx^3}{6EI} + \frac{C_1x^2}{2} + C_2x + C_3$$

$$y(x) \quad \frac{qx^4}{24EI} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4$$

$$y(x=0)=0 \qquad y = C_4 = 0$$

$$\frac{dy}{dx}(x=0)=0 \quad \Rightarrow \quad C_3 = C_4 = 0 \qquad \frac{dy}{dx} = C_3 = 0$$

$$\left\{\begin{array}{l} \frac{dy}{dx}(x=L) = -\frac{qL^3}{6EI} + \frac{C_1 L^2}{2} + C_2 L = 0 \\ y(x=L) = -\frac{qL^4}{24EI} + \frac{C_1 L^3}{2} + \frac{C_2 L^2}{2} = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} C_1 = \frac{qL}{2EI} \\ C_2 = -\frac{qL^2}{12EI} \end{array}\right.$$

$$y(x) = -\frac{qx^4}{2EI} + \frac{qLx^3}{12EI} - \frac{qL^2x^2}{24EI} = \frac{qx^2}{24EI}\left(x^2 - 2Lx + L^2\right) = \frac{qx^2}{24EI}(x-L)^2$$

Max diflection $y = \underbrace{\frac{-q}{24EI}}_{a} \underbrace{x^2(x-L)^2}_{f(x)}$

$$y' = \left[ax^2(x-L)^2\right]' = 2ax(x-L)^2 + 2ax^2(x-L)$$

$$= 2ax(x-L)(x-L+x)$$

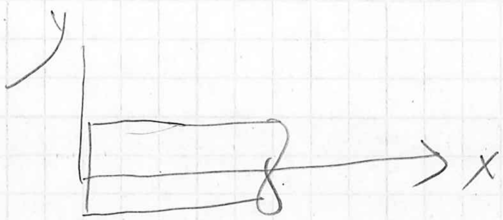
$$= 2ax(x-L)(2x-L) = 0$$

$\Rightarrow x = \frac{L}{2}$ max

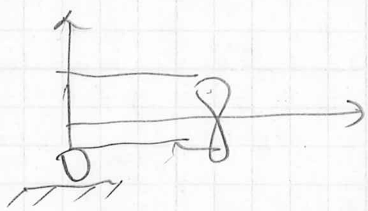
$\Rightarrow x = 0$ min

$$y_{max} = y\left(x = \frac{L}{2}\right) = \frac{q}{24EI}\left(\frac{L}{2}\right)^2\left(-\frac{L}{2}\right)^2 = \frac{qL^4}{384EI}$$

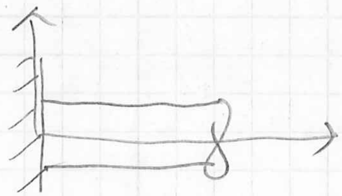
@ $x=0$



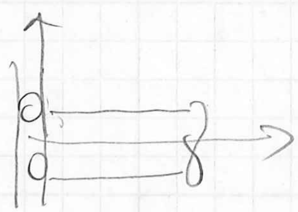
$$\frac{d^2y}{dx^2}=0 \quad ; \quad \frac{d^3y}{dx^3}=0$$



$$y=0 \quad ; \quad \frac{d^2y}{dx^2}=0$$



$$y=0 \qquad \frac{dy}{dx}=0$$



$$\frac{dy}{dx}=0 \qquad \frac{d^3y}{dx^3}=0$$

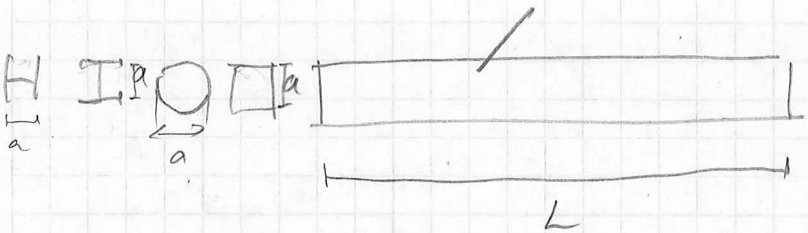
$y \rightarrow$ dsp

$y' =$ A

$y'' \longrightarrow$ M

y''' S

BEAM THEORY

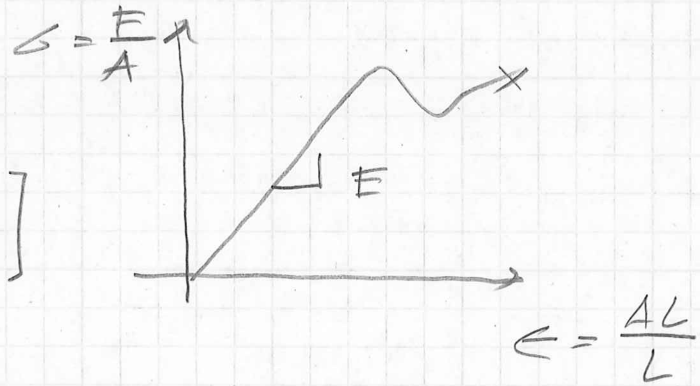


Bernoulli Euler

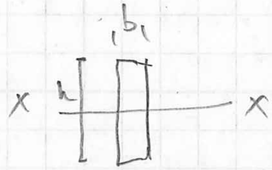
$L > 10 \cdot a$

(E) - Young Modulus

$$E = \frac{\sigma}{\varepsilon} \quad \left[\frac{N}{m^2}\right]$$



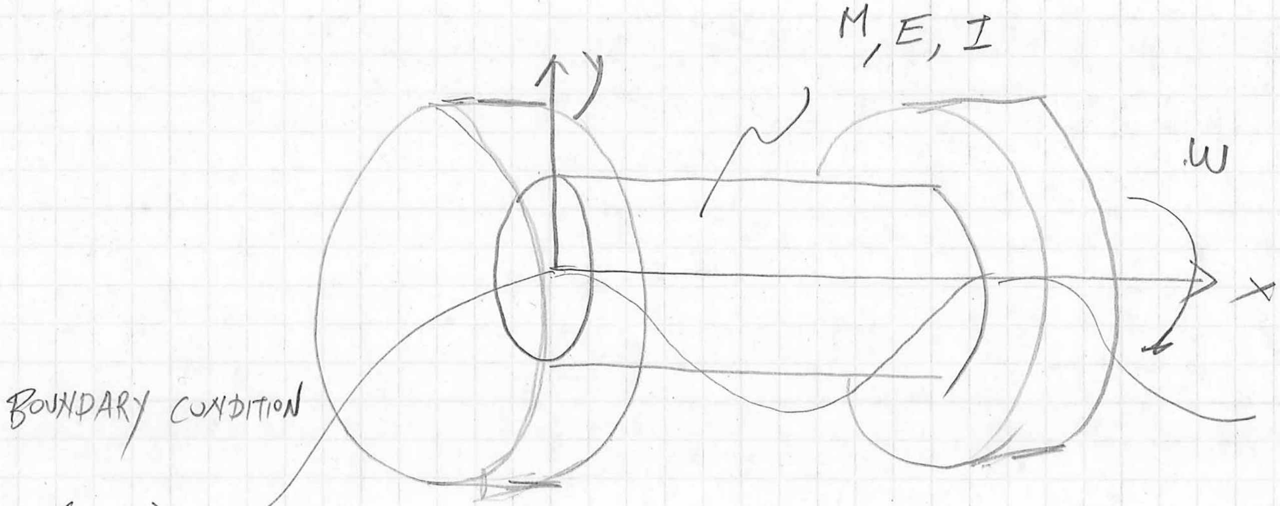
(I) - Moment of Inertia

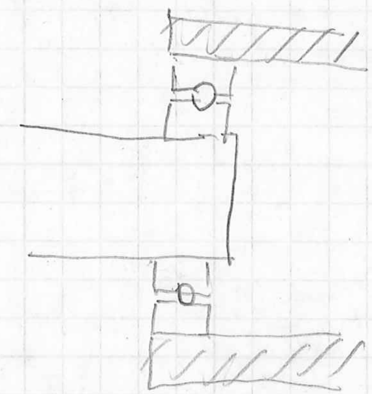


$$I_{xx} = \frac{bh^2}{12}$$

$$I = \frac{\pi}{4} r^2$$

ROTATION OF A SHAFT





$$\begin{cases} y(x=0) = 0 \\ \frac{d^2y}{dx^2}(x=0) = 0 \end{cases}$$

$$EI \frac{d^4y}{dx^4} = \omega \frac{M}{L} y$$

$$\begin{cases} y(x=L) = 0 \\ \frac{d^2y}{dx^2}(x=L) = 0 \end{cases}$$

$$\frac{d^4y}{dx^4} - \alpha^4 y = 0$$

$$\alpha^4 = \frac{\omega^2 M}{EIL}$$

Assume solution: $y = e^{st}$

The characteristic eq. $s^4 - \alpha^4 = 0$

$$(s^2 - \alpha^2)(s^2 + \alpha^2) = 0$$

$s_1 = \alpha$

$s_2 = -\alpha$

$s_3 = \alpha i$

$s_4 = -\alpha i$

$$y(x) = \overbrace{C_1 e^{\alpha x} + C_2 e^{-\alpha x}}^{\pm \alpha} + \overbrace{C_3 \cos(\alpha x) + C_4 \sin(\alpha x)}^{\pm i \alpha}$$

$$\frac{dy}{dx} = C_1 \alpha e^{\alpha x} - C_2 \alpha e^{-\alpha x} - C_3 \alpha \sin \alpha x + C_4 \alpha \cos \alpha x$$

$$\frac{d^2y}{dx^2} = C_1 \alpha^2 e^{\alpha x} + C_2 \alpha^2 e^{-\alpha x} - C_3 \alpha^2 \cos \alpha x - C_4 \alpha^2 \sin \alpha x$$

$$y(x=0) = 0 \longrightarrow \qquad C_1 + C_2 + C_3 = 0$$

$$\frac{d^2y}{dx^2}(x=0) = 0 \longrightarrow \qquad C_1 + C_2 - C_3 = 0$$

$$y(x=L) = 0 \qquad C_1 e^{\alpha L} + C_2 e^{-\alpha L} + C_3 \cos \alpha L + C_4 \sin \alpha L = 0$$

$$\frac{d^2y}{dx^2}(x=L) = 0 \longrightarrow \qquad C_1 e^{\alpha L} + C_2 e^{-\alpha L} - C_3 \cos \alpha L - C_4 \sin \alpha L = 0$$

$$\begin{cases} C_1 = 0 \\ C_2 = 0 \\ C_3 = 0 \\ C_4 \sin \alpha L = 0 \end{cases}$$

+ the case $C_4 = 0$ correspondes to the trivial solution $y = 0$. In this case the saft remains undeformed

+ A meaningful solution exists for $C_4 \neq 0$ Then the following solution is valid

$$\sin \alpha L = 0$$

$$\alpha L = \pi n \qquad n = 1, 2, 3, 4$$

$$y(x) = C_4 \sin \alpha x = C_4 \sin\left(\frac{\pi n}{L} x\right)$$

The rotating shaft starts to bend at $\alpha L = \pi n$ taking the form of a sin wave

The minimum critical frequency W_c which causes instability is given for $n = 1$

$$\alpha = \frac{\pi n}{L} \underset{\substack{\uparrow \\ n=1}}{=} \frac{\pi}{L}$$

by detention

$$\alpha^4 = \frac{W_c^2 M}{EIL} = \left(\frac{\pi}{L}\right)^4$$

$$\Rightarrow \quad W_c = \frac{\pi}{L}\sqrt{\frac{EI}{LM}}$$

For a shaft (solid Rod) $\rightarrow$

$$I = \frac{Ma^2}{4L}$$

$$W_c = \frac{\pi}{L}\sqrt{\frac{EI}{LM}} = \frac{\pi}{L}\sqrt{\frac{E}{LM}\,\frac{Ma^2}{4L}} = \frac{\pi^2 a}{2L^2}\sqrt{E}$$