Class Notes 6:

Second Order Differential Equation – Non Homogeneous – Force Vibration

MAE 82 – Engineering Mathematics

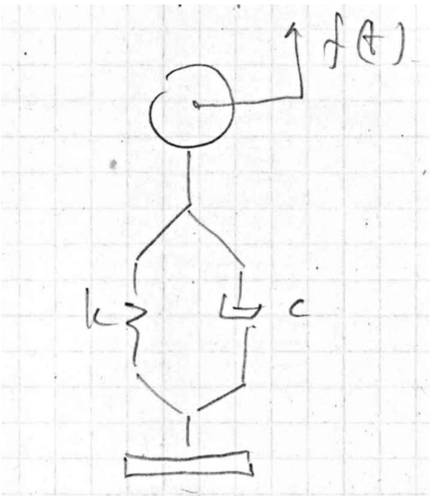
Tacoma Narrow (WA)

[Video - Galloping Gertie - The Collapse of the Tacoma Narrows Bridge]

Dynamic Balance

[Video - The vibrations resulting from the rotor destructs this Chinook helicopter]

Force Vibrations – Introduction



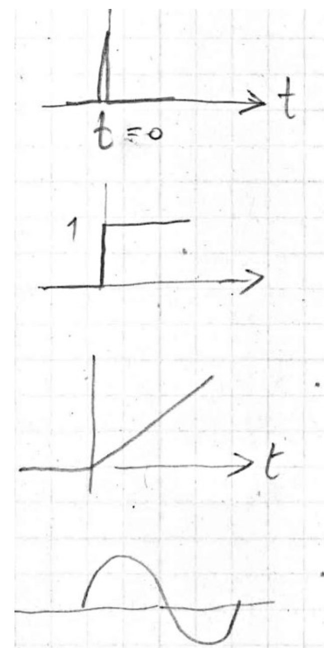
$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \text{I.C.} \begin{cases} x(t=0) = x_0 = d \\ \dot{x}(t=0) = \dot{x}_0 = v \end{cases}$$

$$w_n = \sqrt{\frac{k}{m}}; \qquad \zeta = \frac{c}{2mw_n} = \frac{2}{2\sqrt{mk}}$$

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = \frac{f(t)}{m}$$

$$f(t) = \begin{cases} f(t) = 0 & \text{Free motion} \\ f(t) \neq 0 & \text{Forced motion} \\ c = 0 & \text{Undamped motion} \\ c \neq 0 & \text{Damped motion} \end{cases}$$

$f(t) =$



impulse

step

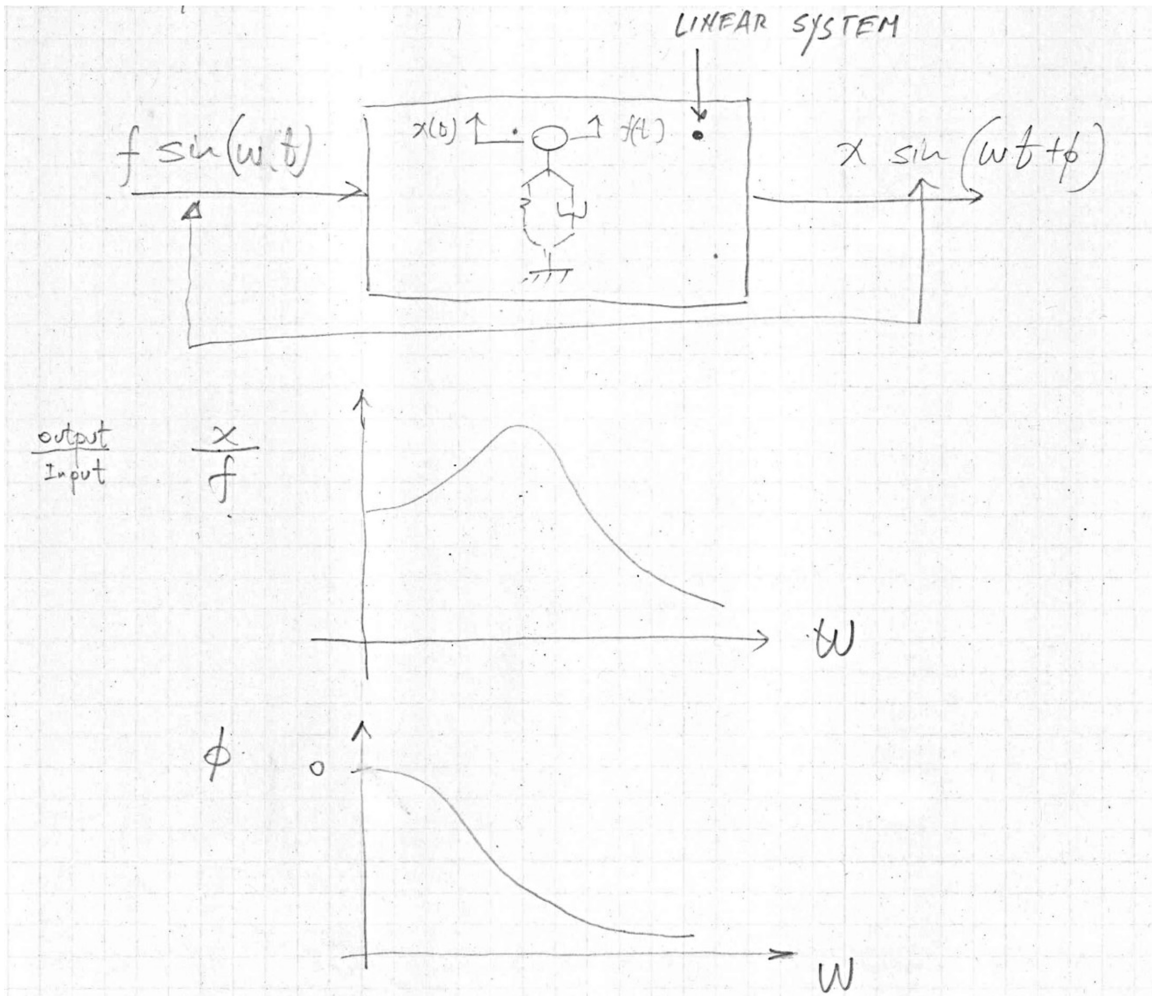
ramp

sin

Forced Vibration Tests Aircraft

[Video - CSeries Ground Vibration Test (GVT)]

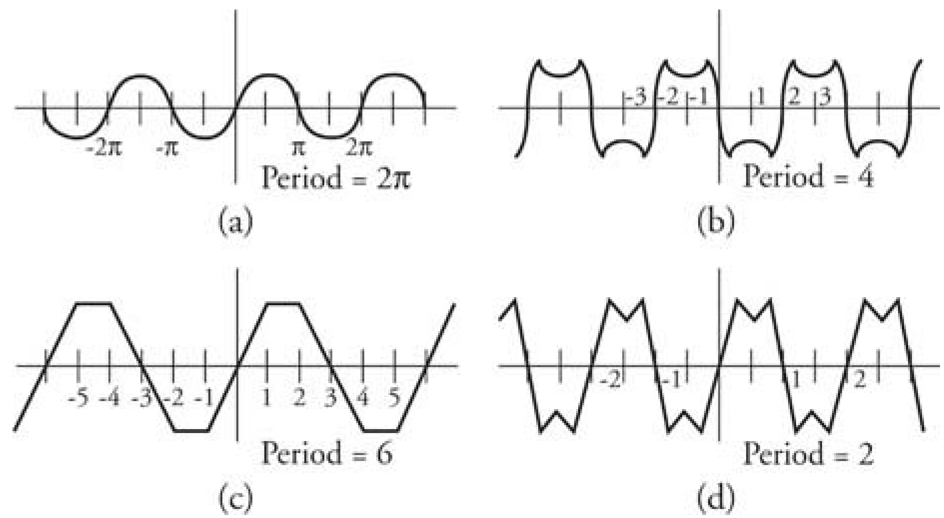
Force Vibrations – Introduction – Frequency Response – Bode Plot



Force Vibrations – Introduction
Frequency Analysis – Fourier Series

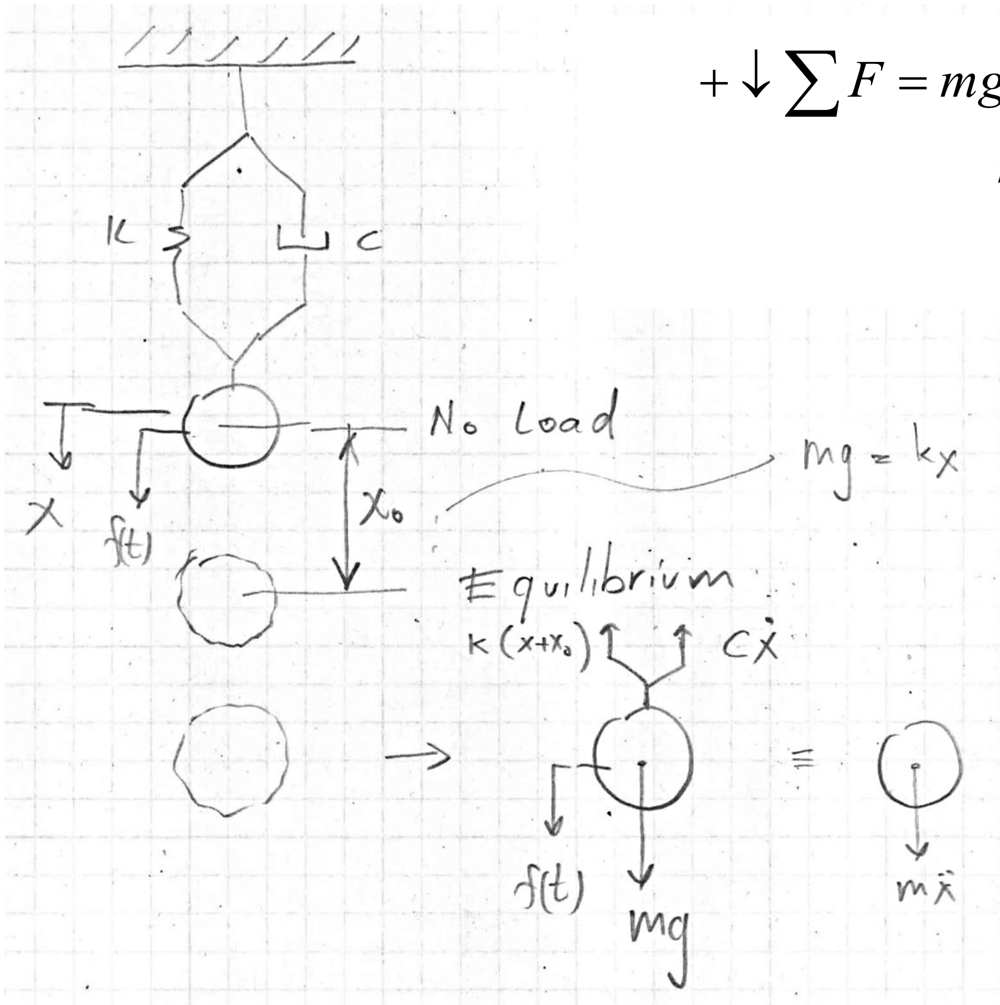
Periodic function

$$f(t+T) = f(t)$$



$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\frac{2\pi m t}{T} + \sum_{m=1}^{\infty} b_m \sin\frac{2\pi m t}{T}$$

Force Vibrations – Effect of Gravity



$$+\downarrow \sum F = mg - k(x + x_0) - c\dot{x} + f(t) = m\ddot{x}$$

$$m\ddot{x} + \underline{kx} + kx_0 + c\dot{x} \underline{- mg} = f(t)$$

$$= 0$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Force Vibrations – Classification
Damped Oscillation – Undamped Oscillation

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = \frac{F}{m}\cos wt$$

Damped oscillator

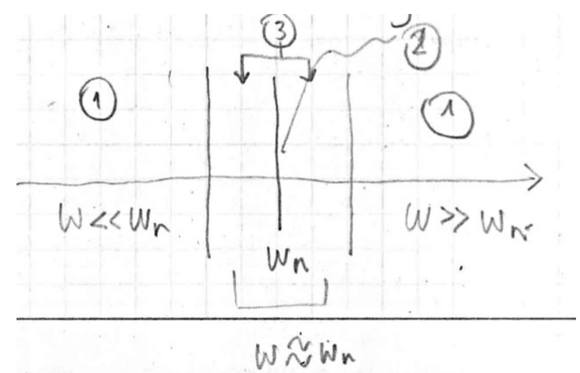
$$\zeta \neq 0$$

$$\begin{cases} 1. \quad 0 < \zeta < 1 \\ 2. \quad \zeta = 1 \\ 3. \quad \zeta > 1 \end{cases}$$

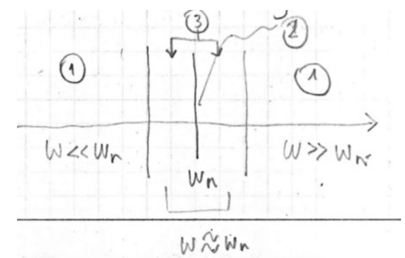
Undamped oscillator

$$\zeta = 0$$

$$\left. \begin{array}{ll} 1. & w \neq w_n \\ 2. & w = w_n \\ 3. & w \approx w_n \end{array} \right\}$$



Force Vibrations
Undamped Oscillation – Case 1



Undamped oscillator $\zeta = 0 \quad w \neq w_n$ (Case 1)

$$m\ddot{x} + kx = F\cos wt \;\rightarrow\; \ddot{x} + w_n{}^2 x = \frac{F}{m}\cos wt$$

$$x = x_c + x_p$$

$$x = \underbrace{c_1 \cos\overbrace{(w_n t) + c_2 \sin(w_n}^{w_n\,:\,system} t)}_{x_c} + \underbrace{A\cos\overbrace{(wt) + B\sin(}^{w\,:\,input} wt)}_{x_p}$$

$$x_h$$

Differentiating x_p

$$\underbrace{-Aw^2\cos(wt) - Bw^2\sin(wt)}_{\ddot{x}} + w_n{}^2(\underbrace{A\cos(wt) + B\sin(wt)}_{x}) = \frac{F}{m}\cos wt$$

B must be $B = 0$ since it is a function of cos only

Force Vibrations
Undamped Oscillation – Case 1

- Using the undermined coefficient method

$$A({w_n}^2 - w^2)\cos wt = \frac{F}{m}\cos wt$$

- Solve for A

$$A = \frac{F}{m({w_n}^2 - w^2)}$$

- Resulted in

$$x(t) = c_1\cos(w_n t) + c_2\sin(w_n t) + \frac{F}{m({w_n}^2 - w^2)}\cos wt$$

Force Vibrations
Undamped Oscillation – Case 1

- Given the solution

$$x(t) = c_1 \cos(w_n t) + c_2 \sin(w_n t) + \frac{F}{m({w_n}^2 - w^2)} \cos wt$$

- Using the Initial Conditions

$$I.C. \begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases}$$

- Resulted in

$$x(t=0) = c_1 \overset{1}{\cos(w_n 0)} + c_2 \overset{0}{\sin(w_n 0)} + \frac{F}{m({w_n}^2 - w^2)} \overset{1}{\cos(w0)} = x_0$$

$$c_1 = x_0 - \frac{F}{m({w_n}^2 - w^2)}$$

$$\dot{x}(t=0) = -c_1 w_n \overset{0}{\sin(w_n 0)} + c_2 w_n \overset{1}{\cos(w_n 0)} - \frac{F}{m({w_n}^2 - w^2)} w \overset{0}{\sin(w0)} = \dot{x}_0$$

$$c_2 = \frac{\dot{x}_0}{w_n}$$

Force Vibrations
Undamped Oscillation – Case 1

$$x(t) = \left(x_0 - \frac{F}{m(w_n^{\ 2} - w^2)} \right) \cos w_n t + \frac{\dot{x}_0}{w_n} \sin w_n t + \frac{F}{m(w_n^{\ 2} - w^2)} \cos wt$$

static difflection $\delta = \frac{F}{k}$

frequency ratio $r = \frac{w}{w_n}$

$$x(t) = x_0 - \left(\frac{F}{m w_n^{\ 2}(1 - r^2)} \right) \cos w_n t + \frac{\dot{x}_0}{w_n} \sin w_n t + \frac{F}{m w_n^{\ 2}(1 - r^2)} \cos wt$$

$$x(t) = \left(x_0 - \frac{\delta}{1 - r^2} \right) \cos w_n t + \frac{\dot{x}_0}{w_n} \sin w_n t + \frac{\delta}{1 - r^2} \cos wt$$

$$x(t) = \underbrace{x_0 \cos w_n t + \frac{\dot{x}_0}{w_n} \sin w_n t}_{\text{Free response(Depends on I.C.)}} + \underbrace{\frac{\delta}{1 - r^2} \left(\cos wt - \cos w_n t \right)}_{\text{Force response(Depends on the input)}}$$

Free response(Depends on I.C.) Force response(Depends on the input)

Force Vibrations
Undamped Oscillation – Case 1

- Examining the factor of the force Response

$$x(t) = x_0 \cos w_n t + \frac{\dot{x}_0}{w_n} \sin w_n t + \boxed{\frac{\delta}{1-r^2}} \left(\cos wt - \cos w_n t\right)$$

Free response(Depends on I.C.) Force response(Depends on the input)

- Static Deflection Factor

$$\delta = \frac{F}{k} \qquad \delta \propto F$$

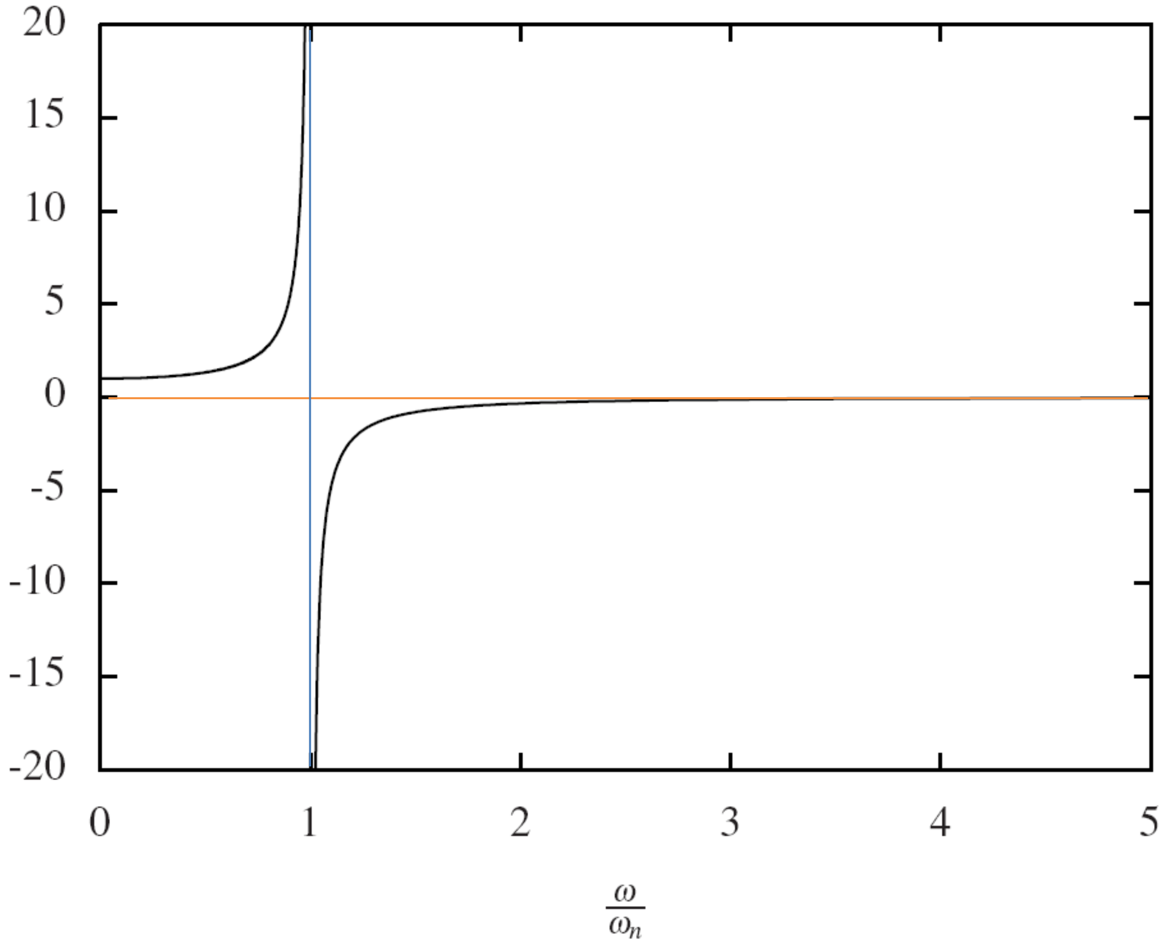
- Magnitude Factor

$$M = \frac{1}{1-r^2} = \frac{1}{1\text{-}\left(\dfrac{w}{w_n}\right)^2}$$

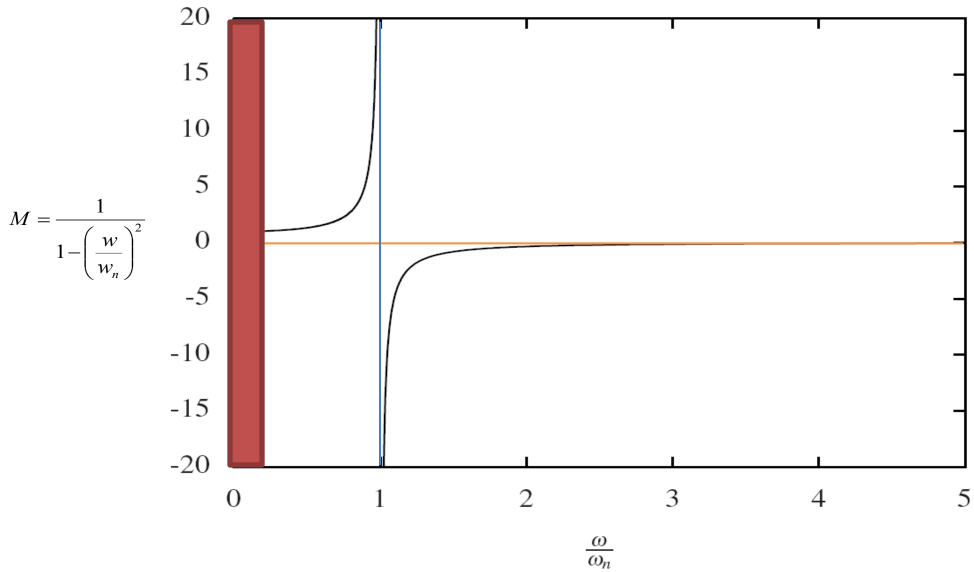
M – The amount by which the static deflection is either amplified or attenuated. It is a function of the ratio between the forcing frequency (w) and the natural frequency (w_n)

Force Vibrations
Undamped Oscillation – Case 1

$$M = \frac{1}{1-\left(\frac{w}{w_n}\right)^2}$$



Force Vibrations
Undamped Oscillation – Case 1

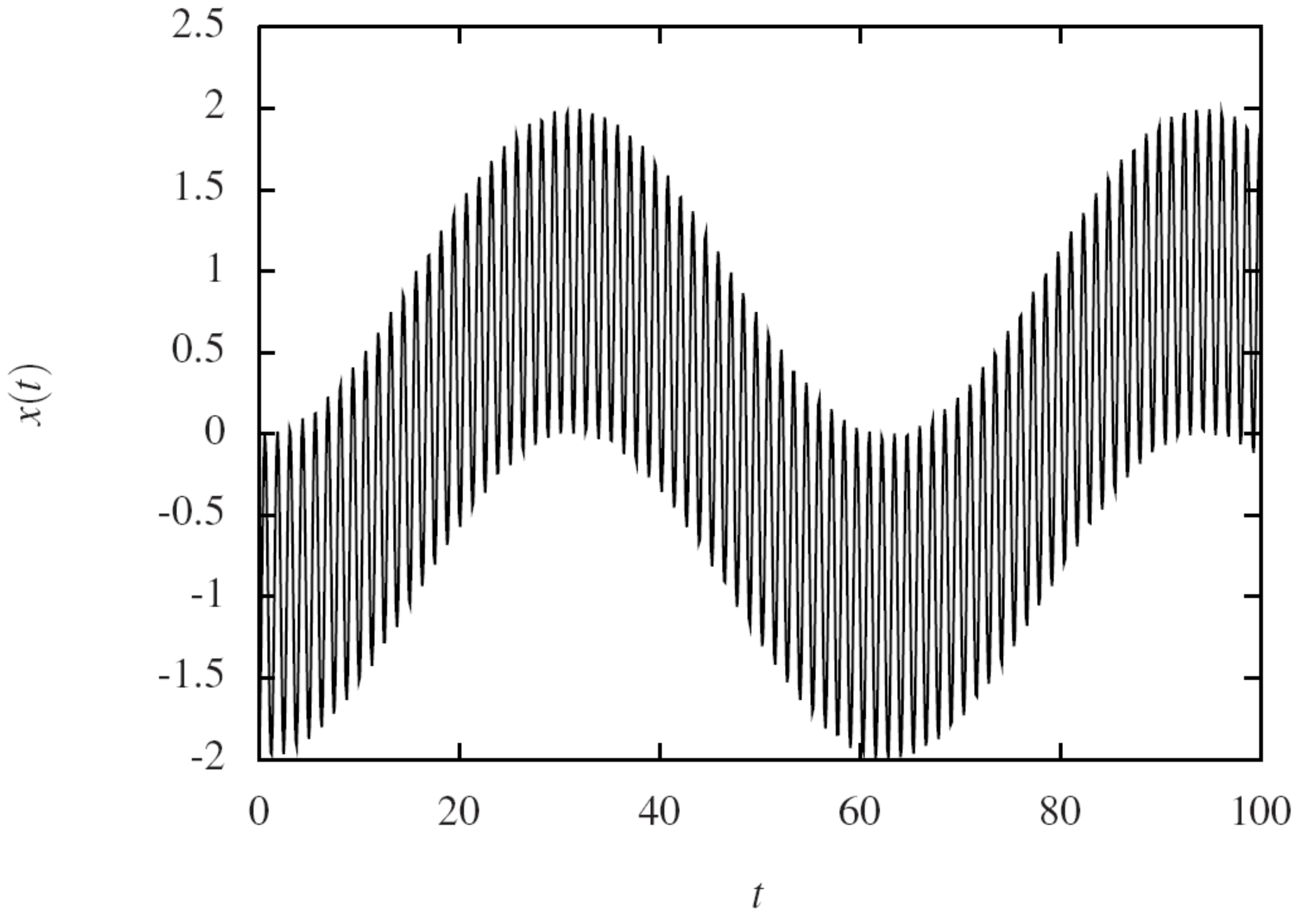


$(*)\ For\ x(0)=0,\ \dot{x}(0)=0$ and $w << w_n$ $\left(\frac{w}{w_n}\right) \to 0$

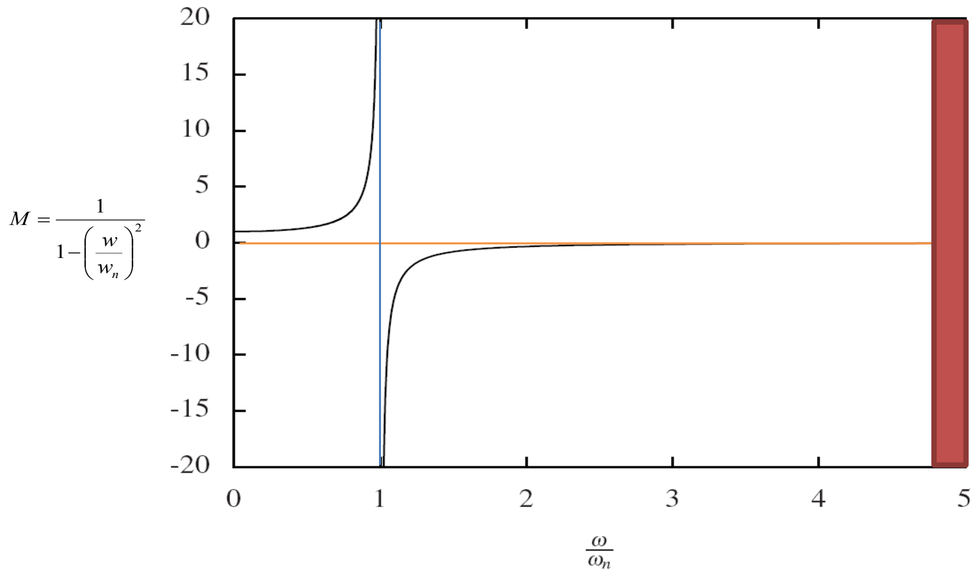
$$x(t) = \frac{\delta}{1-r^2}(\cos wt - \cos w_n t)$$

$$x(t) \approx \delta(\cos wt - \cos w_n t)$$

Force Vibrations
Undamped Oscillation – Case 1



Force Vibrations
Undamped Oscillation – Case 1

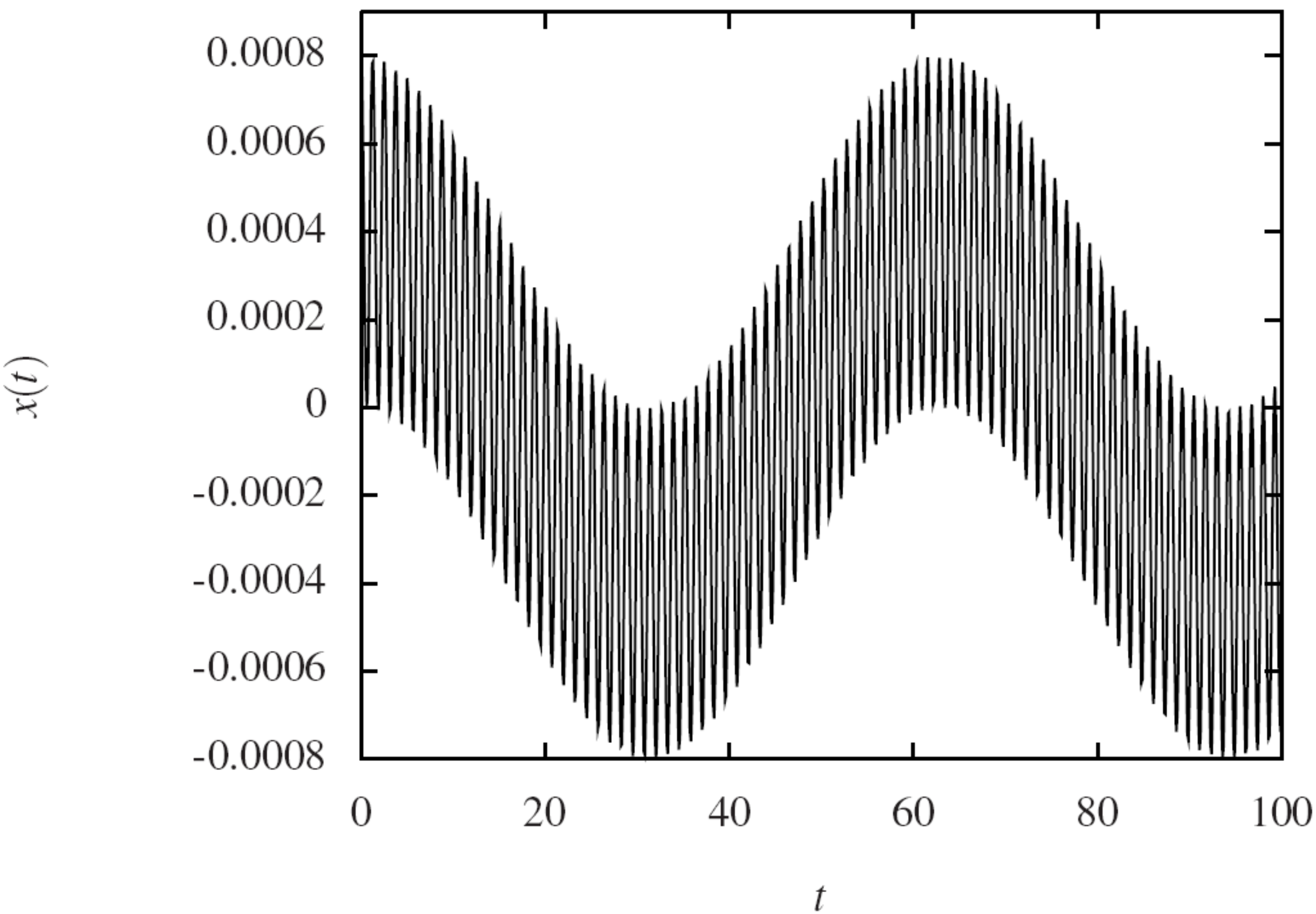


$$(**)\ For\ x(0)=0,\ \dot{x}(0)=0 \text{ and } w >> w_n \quad \left(\frac{w}{w_n}\right) \to \infty$$

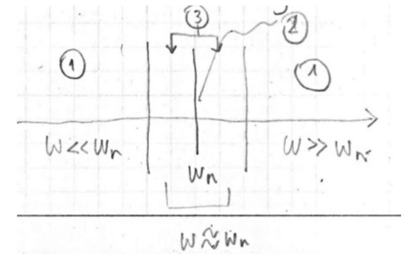
$$M = \frac{1}{1-r^2} \to 0$$

$$x(t) = \delta(\cos wt - \cos w_n t)$$

Force Vibrations
Undamped Oscillation – Case 1



Force Vibrations
Undamped Oscillation – Case 3



- Undamped Oscillator $\zeta = 0 \quad \omega \approx \omega_n$ (case 3)

$$\ddot{x} + \omega_n x = \frac{F}{m}\cos\omega t$$

$$\text{IC} \quad \begin{cases} x(0) = 0 \\ \dot{x}(0) = \dot{x}, \end{cases} \quad \omega \approx \omega_{\mathrm{n}}$$

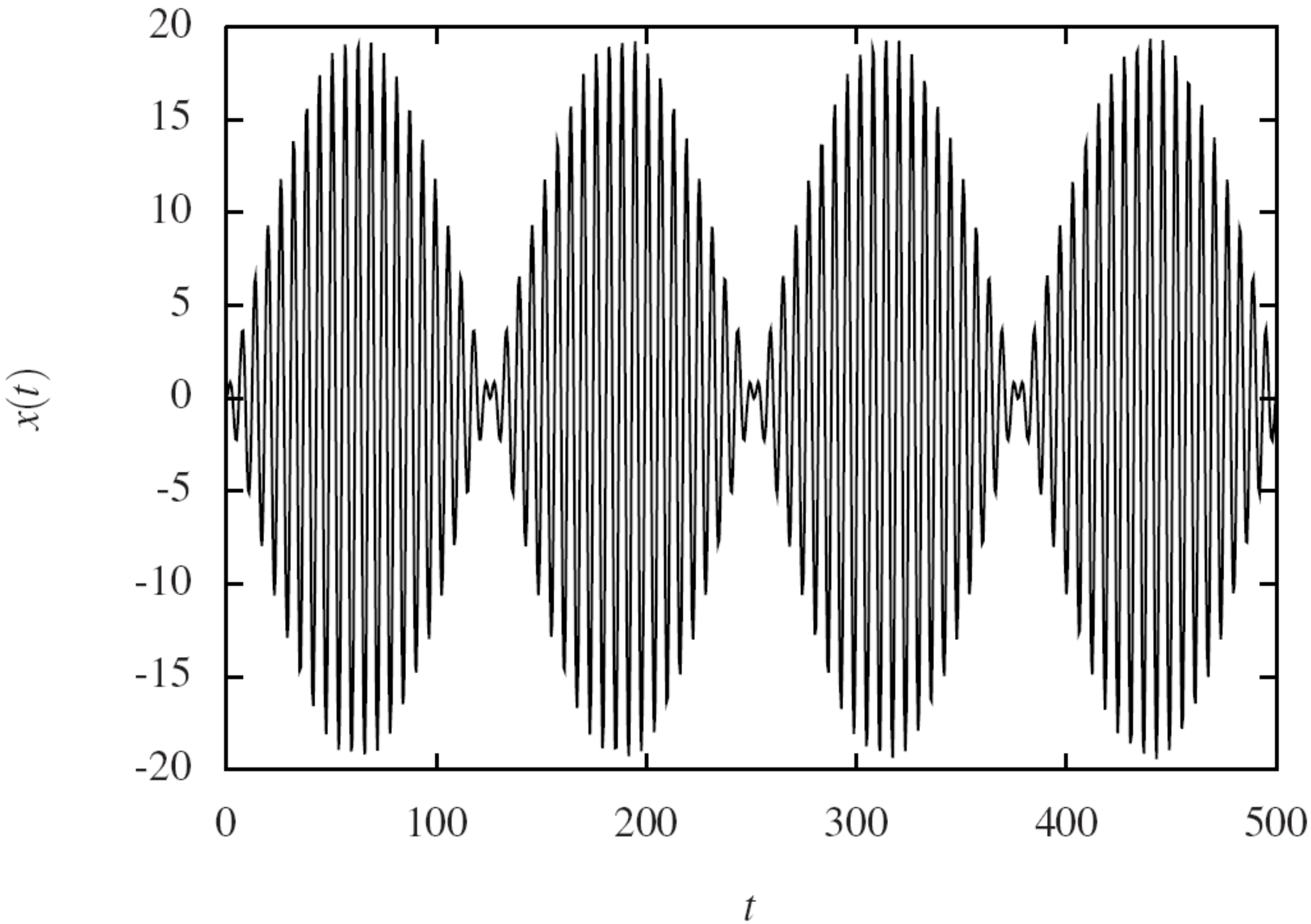
- The solution is similar to the case when $\omega \neq \omega_n$

$$x(t) = \left(x_0 - \frac{F}{m\left(\omega_n^2 - \omega^2\right)} \right)\cos\omega_n t + \frac{x_0}{\omega_n}\sin\omega_n t + \frac{F}{m\left(\omega_n^2 - \omega^2\right)}\cos\omega t$$

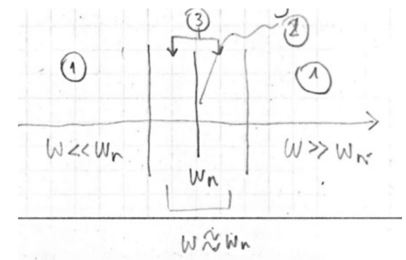
$$x(t) = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F}{m\left(\omega_n^2 - \omega^2\right)}\left(\cos\omega t - \cos\omega_n t\right)$$

- Same as $\omega \neq \omega_n$

Force Vibrations
Undamped Oscillation – Case 3



Force Vibrations
Undamped Oscillation – Case 2 (Resonance)



- Undamped Oscillator $\zeta = 0 \quad \omega = \omega_n$

$$\ddot{x} + \omega_n^2 x = \frac{F}{m}\cos\omega_n t$$

$$\lambda^2 + \omega_n^2 = 0$$

$$\begin{cases} \lambda_1 \\ \lambda_2 \end{cases} = \pm\omega_n$$

$$x = x_c + x_p$$

$$x = \underbrace{C_1\cos\overbrace{\omega_n t + C_2\sin\omega_n}^{\omega_n} t}_{x_c} + \boxed{t^1}\left(A\cos\overbrace{\omega_n t + B\sin\omega_n}^{\omega_n} t\right)$$

Force Vibrations
Undamped Oscillation – Case 2 (Resonance)

$$x = \underbrace{C_1 \cos \overbrace{\omega_n t + C_2 \sin \omega_n}^{\omega_n} t}_{x_c} + t\left(A \cos \overbrace{\omega_n t + B \sin \omega_n}^{\omega_n} t\right)$$

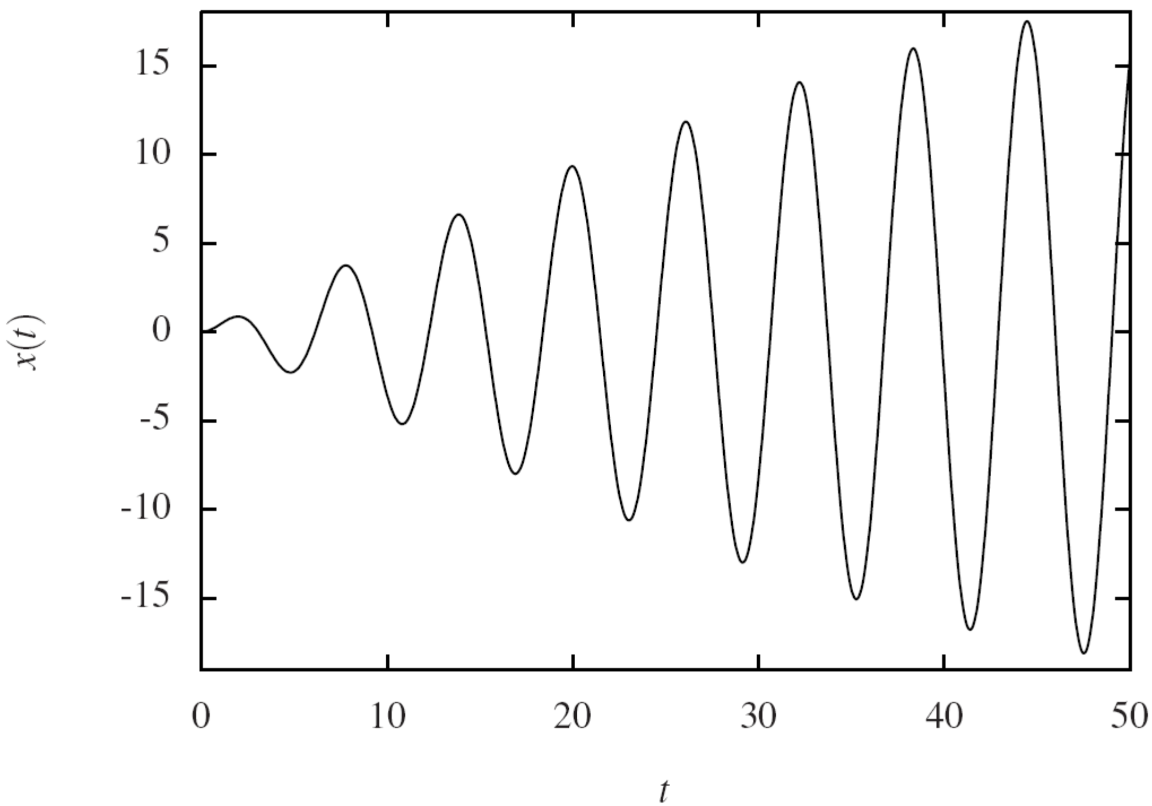
IC $\begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases}$

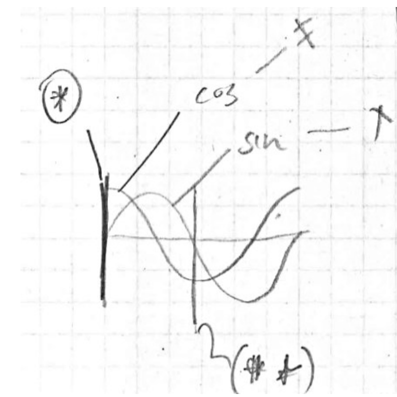
$$x(t) = \underbrace{x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}_{x_n} + \underbrace{\frac{\delta \omega_n t}{2} \sin \omega_n t}_{x_p}$$

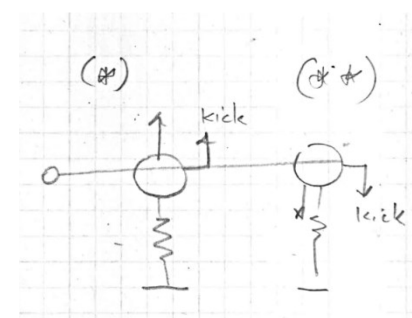
IC Forced

Force Vibrations
Undamped Oscillation – Case 2 (Resonance)

- For IC $\begin{cases} x(0)=0 \\ \dot{x}(0)=0 \end{cases}$ $\quad x(t)=\left(\dfrac{\delta\omega_n}{2}\right)t\sin(\omega_n t)$







Glass

[Video - Wine glass resonance in slow motion]

Force Vibrations – Damped Oscillations

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \underbrace{a}_{\frac{F}{m}}\cos\omega t$$

$$x_p = a_1\cos\omega t + a_2\sin\omega t$$

$$x = x_c + x_p$$

$$x = \underbrace{\begin{Bmatrix} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} & \lambda_1 \neq \lambda_2 \ \text{real} \\ (C_1 + C_2 t)e^{\lambda t} & \lambda_1 = \lambda_2 = \lambda \\ Ae^{-\zeta\omega_n t}\sin(\omega_\mathrm{d} t + \phi) & \lambda = -\zeta\omega_n \pm j\omega_d \end{Bmatrix}}_{\lambda_c:\mathrm{C}_1,C_2,A,\phi\ \text{Based on I.C.}} + \underbrace{a_1\cos\omega t + a_2\sin\omega t}_{a_1,\,a_2}$$

Force Vibrations – Damped Oscillations

- Substitute x_p into the differential equation

$$\overbrace{-\omega^2 a_1 \cos \omega t - \omega^2 a_2 \sin \omega t}^{\ddot{x}} + 2\zeta\omega_n \overbrace{\left[-\omega a_1 \sin \omega t + \omega a_2 \cos \omega t\right]}^{\dot{x}}$$

$$+\,\omega_n^2 \underbrace{\left[a_1 \cos \omega t + a_2 \sin \omega t\right]}_{x} = a \cos \omega t$$

$$\begin{cases} -\omega^2 a_1 + 2\zeta\omega_n \omega a_2 + \omega_n^2 a_1 = a \\ -\omega^2 a_2 + 2\zeta\omega_n \omega a_1 + \omega_n^2 a_2 = 0 \end{cases}$$

Coefficient of $\cos \omega t$ (first equation)

Coefficient of $\sin \omega t$ (second equation)

$$\begin{bmatrix} \left(\omega_n^2 - \omega^2\right) & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & \left(\omega_n^2 - \omega^2\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\Delta = \left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2$$

Force Vibrations – Damped Oscillations

$$a_1 = \frac{a\left(\omega_n^2 - \omega^2\right)}{\Delta}; \quad a_2 = \frac{2\zeta\omega_n\omega a}{\Delta}$$

$$x_p = \frac{\overbrace{a}^{\frac{F}{m}}}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}\left[\left(\omega_n^2 - \omega^2\right)\cos\omega t + \left(2\zeta\omega_n\omega\right)\sin\omega t\right]$$

- Convert the solution to the form of a single trigonometric function with a phase shift

$$\begin{cases} x_p = C\cos\left(\omega t + \phi\right) = C\left(\cos\phi\cos\omega t - \sin\phi\sin\omega t\right) \\ x_p = A\cos\omega t + B\sin\omega t \end{cases}$$

- where $\begin{cases} A = C\cos\phi \\ B = -C\sin\phi \end{cases}, \quad C = \sqrt{A^2 + B^2}, \quad \phi = \tan^{-1}\left(-\frac{B}{A}\right)$

Force Vibrations – Damped Oscillations

$$A = \frac{F}{m} \frac{\omega_n^2 - \omega^2}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2} \qquad B = \frac{F}{m} \frac{2\zeta\omega_n\omega}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}$$

$$C = \sqrt{A^2 + B^2}$$

$$= \frac{F}{m}\sqrt{\frac{\left(\omega_n^2 - \omega^2\right)^2}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^2} + \frac{\left(2\zeta\omega_n\omega\right)^2}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^2}}$$

$$= \frac{F}{m}\sqrt{\frac{\cancel{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^{\cancel{2}}}}$$

$$= \frac{F}{m}\frac{1}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

$$\phi = \tan^{-1}\left(-\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

Force Vibrations – Damped Oscillations

$$x_p = \frac{F}{m}\frac{1}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}\cos(\omega t + \phi)$$

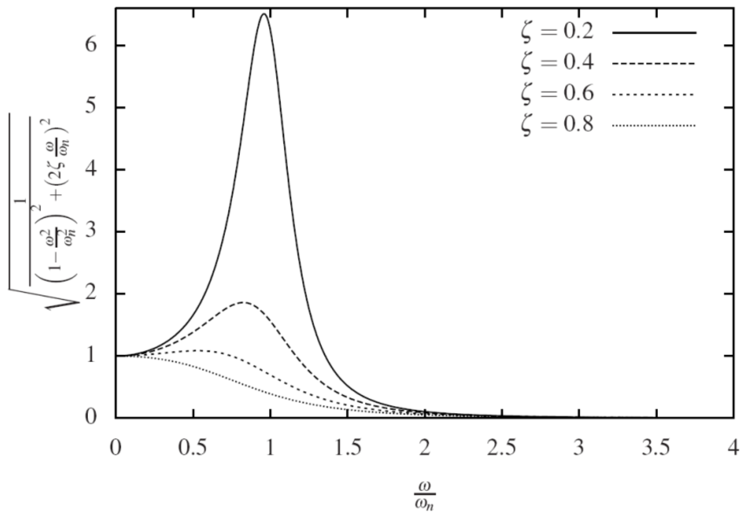
- Factor ω_n^2 out of the denominator

$$x_p(t) = \frac{F}{\omega_n^2 m}\sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}\cos(\omega t + \phi)$$

$$\frac{F}{\omega_n^2 m} = \frac{F}{\frac{k}{m}m} = \frac{F}{k} = \delta$$

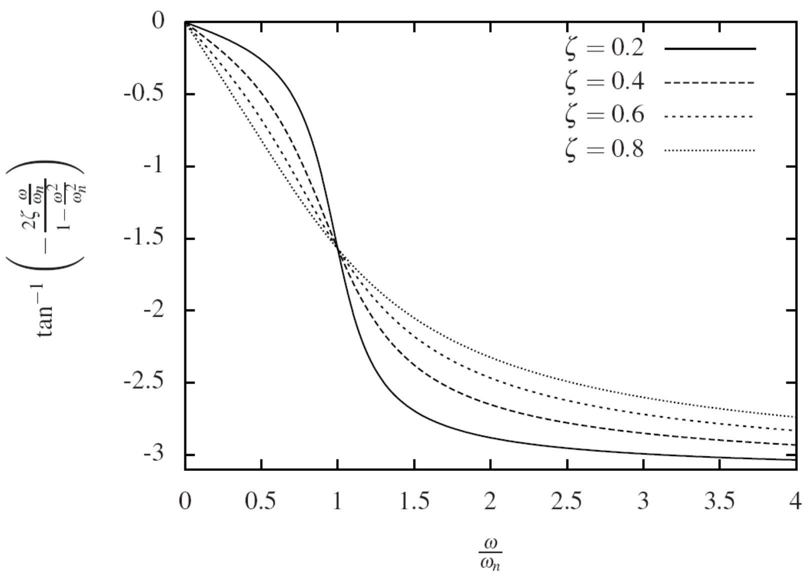
$$x_p(t) = \delta\underbrace{\sqrt{\frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}_{M}\cos(\omega t + \phi)$$

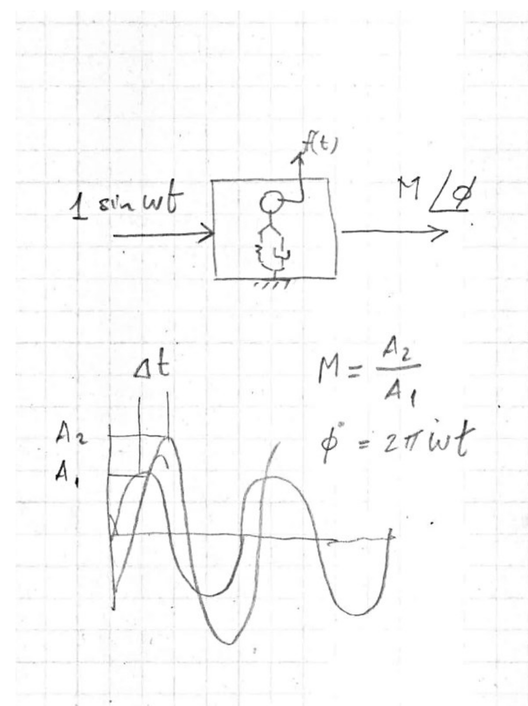
Force Vibrations – Damped Oscillations



$$M = \sqrt{\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\phi = \tan^{-1}\left[-\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right]$$





Single DOF - Resonance

[Video - SDOF Resonance Vibration Test]()