Class Notes 5:

Second Order Differential Equation – Non Homogeneous

82A – Engineering Mathematics

Second Order Linear Differential Equations –

Homogeneous & Non Homogenous v

$$y'' + p(t)y' + q(t)y = \begin{cases} 0 & \text{Homogeneous} \\ g(t) & \text{Non-homogeneous} \end{cases}$$

- p, q, g are given, continuous functions on the open interval I

Second Order Linear Differential Equations –

Homogeneous & Non Homogenous – Structure of the General Solution

$$y'' + p(x)y' + q(x)y = \begin{cases} g(x), & \text{Non-homogeneous} \\ 0, & \text{Homogeneous} \end{cases}$$

I.C. $\begin{cases} y(t=0) = y_0 \\ y'(t=0) = y'_0 \end{cases}$

- Solution:

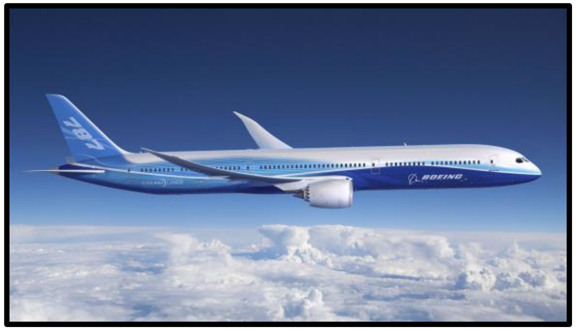
$$y = y_c(x) + y_p(x)$$

where

$y_c(x)$: solution of the homogeneous equation (complementary solution)

$y_p(x)$: any solution of the non-homogeneous equation (particular solution)

Second Order Linear Differential Equations – Non Homogenous



$$y'' + p(t)y' + q(t) = f(t)$$

I.C. $$\begin{cases} y(t=0) = y_0 \\ y'(t=0) = y'_0 \end{cases}$$

Theorem (3.5.1)

- If Y_1 and Y_2 are solutions of the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Then Y_1 - Y_2 is a solution of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

- If, in addition, $\{y_1, y_2\}$ forms a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem (3.5.2) – General Solution

- The general solution of the **nonhomogeneous** equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1 and y_2 form a fundamental solution set for the homogeneous equation, c_1 and c_2 are arbitrary constants, and $Y(t)$ is a specific solution to the nonhomogeneous equation.

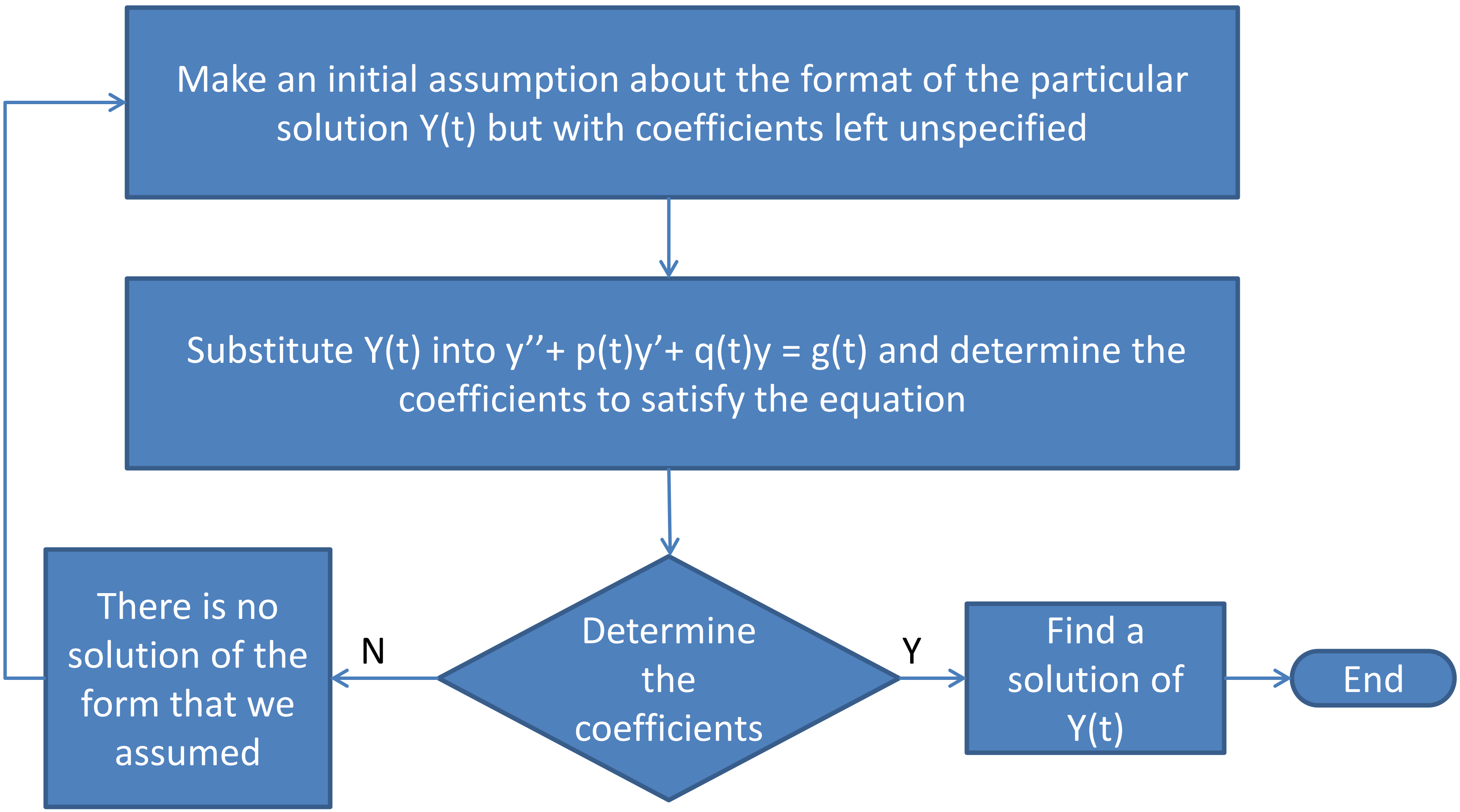
Second Order Linear Non Homogenous Differential Equations –

Methods for Finding the Particular Solution

- The methods of **undetermined coefficients**
- The methods of **variation of parameters**

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Block Diagram



Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Block Diagram

- **Advantages**
  - **Straight Forward Approach -** It is a straight forward to execute once the assumption is made regarding the form of the particular solution $Y(t)$

- **Disadvantages**
  - **Constant Coefficients** - Homogeneous equations with constant coefficients
  - **Specific Nonhomogeneous Terms** - Useful primarily for equations for which we can easily write down the correct form of the particular solution $Y(t)$ in advanced for which the Nonhomogenous term is restricted to
    - Polynomic
    - Exponential
    - Trigonematirc (sin / cos)

Second Order Linear Non Homogenous Differential Equations –

Particular Solution For Non Homogeneous Equation Class A

- The particular solution y_p for the nonhomogeneous equation

$$ay'' + by' + cy = g(x)$$

- Class A

$$g(x) = \begin{cases} P_n(x) \rightarrow \text{Polynomial in x} \\ a_0 x^n + a_1 x^{n-1} + \ldots + a_n \end{cases}$$

$$y_p = \begin{cases} A_0 x^n + A_1 x^{n-1} + \ldots + A_n & c \neq 0 \\ x\left(A_0 x^n + A_1 x^{n-1} + \ldots + A_n\right) & c = 0, b \neq 0 \\ x^2\left(A_0 x^2 + A_1 x^{n-1} + \ldots + A_n\right) & c = b = 0 \end{cases}$$

Second Order Linear Non Homogenous Differential Equations –

Particular Solution For Non Homogeneous Equation Class B

- The particular solution y_p for the nonhomogeneous equation

$$ay'' + by' + cy = g(x)$$

- Class B

$$g(x) = \begin{cases} e^{\alpha x} P_n(x) \\ e^{\alpha x}(a_0 x^n + a_1 x^{n-1} + \dots + a_n) \end{cases}$$

$$g(x) = \begin{cases} e^{\alpha x}(A_0 x^n + A_1 x^{n-1} + \dots + A_n) \\ \rightarrow \alpha \text{ is not a root of the characteristic equation } ch(\alpha) \neq 0 \end{cases}$$

$$g(x) = \begin{cases} x e^{\alpha x}(A_0 x^n + A_1 x^{n-1} + \dots + A_n) \\ \rightarrow \alpha \text{ is a simple root of the characteristic equation } ch(x) = 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 e^{\alpha x}(A_0 x^n + A_1 x^{n-1} + \dots + A_n) \\ \rightarrow \alpha \text{ is a double root of the characteristic equation } ch(x) = 0 \end{cases}$$

Second Order Linear Non Homogenous Differential Equations –

Particular Solution For Non Homogeneous Equation Class C

- The particular solution y_p for the nonhomogeneous equation

$$ay'' + by' + cy = g(x)$$

- Class C

$$g(x) = \begin{cases} e^{\alpha x}\left(\sin \beta x \quad \text{or} \quad \cos \beta x\right)P_n(x) \\ e^{(\alpha + i\beta)x}(a_0 x^n + a_1 x^{n-1} + ... + a_n) \end{cases}$$

$$y_p = \begin{cases} e^{\alpha x}\begin{bmatrix} \sin \beta x\left(A_0 x^n + A_1 x^{n-1} + ... + A_n\right) + \\ \cos \beta x\left(B_0 x^n + B_1 x^{n-1} + ... + B_n\right) \end{bmatrix}; & ch(\alpha \pm i\beta) \neq 0 \\ xe^{\alpha x}\begin{bmatrix} \sin \beta x\left(A_0 x^n + A_1 x^{n-1} + ... + A_n\right) + \\ \cos \beta x\left(B_0 x^n + B_1 x^{n-1} + ... + B_n\right) \end{bmatrix}; & ch(\alpha \pm i\beta) = 0 \end{cases}$$

Second Order Linear Non Homogenous Differential Equations –

Particular Solution For Non Homogeneous Equation Summary

- The particular solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_n$	$t^s \left(A_0 t^n + A_1 t^{n-1} + \ldots + A_n\right)$
$P_n(t) e^{\alpha t}$	$t^s \left(A_0 t^n + A_1 t^{n-1} + \ldots + A_n\right) e^{\alpha t}$
$P_n(t) e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s \left[\left(A_0 t^n + A_1 t^{n-1} + \ldots + A_n\right) e^{\alpha t} \cos \beta t + \left(B_0 t^n + B_1 t^{n-1} + \ldots + B_n\right) e^{\alpha t} \sin \beta t\right]$

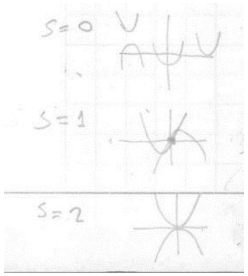
s is the smallest non-negative integer (s=0, 1, or 2) that will ensure that no term in *Yi(t)* is a solution of the corresponding homogeneous equation

s is the number of time

0 is the root of the characteristic equation

α is the root of the characteristic equation

α+iβ is the root of the characteristic equation



Second Order Linear Non Homogenous Differential Equations –

Particular Solution For Non Homogeneous Equation Examples

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Example 1

$$y'' - 3y' - 4y = 3e^{2t}$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\frac{3 \pm \sqrt{9 - 4(-4)}}{2} = \frac{3}{2} \pm \frac{5}{2}$$

$g(x)$	Form of y_p
7. e^{5x}	Ae^{5x}

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Example 1

$$y'' - 3y' - 4y = 3e^{2t}$$

$$\begin{cases} Y(t) = Ae^{2t} \\ Y'(t) = 2Ae^{2t} \\ Y''(t) = 4Ae^{2t} \end{cases}$$

$$\underbrace{(4A - 6A - 4A)}_{-6A}e^{2t} = 3e^{2t}$$

$$A = -\frac{1}{2}$$

$$Y_p(t) = -\frac{1}{2}e^{2t}$$

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Example 2

$$y'' - 3y' - 4y = 2\sin t$$

$g(x)$	Form of y_p
5. $\sin 4x$	$A\cos 4x + B\sin 4x$

Assume
$$\begin{cases} Y(t) = A\sin t \\ Y'(t) = A\cos t \\ Y''(t) = -A\sin t \end{cases}$$

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$
$$(2+5A)\sin t + 3A\cos t = 0$$

There is no choice for constant A that makes the equation true for all t

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Example 2

$$y'' - 3y' - 4y = 2\sin t$$

Assume $$\begin{cases} Y(t) = A\sin t + B\cos t \\ Y'(t) = A\cos t - B\sin t \\ Y''(t) = -A\sin t - B\cos t \end{cases}$$

$$(-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t$$

$$\begin{cases} -5A + 3B = 2 \\ -3A - 5B = 0 \end{cases} \qquad A = -\tfrac{5}{17} \quad B = \tfrac{3}{17}$$

$$Y_p(t) = -\tfrac{5}{17}\sin t + \tfrac{3}{17}\cos t$$

Second Order Linear Non Homogenous Differential Equations –

Method of Undermined Coefficients – Example 3

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$g(x)$	Form of y_p
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$

$$\begin{cases} Y(t) = Ae^t \cos 2t + Be^t \sin 2t \\ Y'(t) = (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t \\ Y''(t) = (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \end{cases}$$

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0 \end{cases} \qquad A = 10/13; \quad B = 2/13$$

$$Y_p(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t$$

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters

Advantage – General method

Diff. eq. $$y'' + p(t)y' + q(t)y = g(t)$$

For the Homogeneous diff. eq.

$$y'' + p(t)y' + q(t)y = 0$$

the general solution is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

so far we solved it for homogeneous diff eq. with constant coefficients.
(Chapter 5 – non constant – series solution)

Second Order Linear Non Homogenous Differential Equations –
Method of Variation of Parameters

Replace the constant $c_1 \& c_2$ by function $u_1(t), u_2(t)$

$$c_1 \rightarrow u_1(t)$$
$$c_2 \rightarrow u_2(t)$$

$$(*)\ y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Find $u_1(t), u_2(t)$ such that is the solution to the nonhomogeneous diff. eq. rather than the homogeneous eq.

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$
$$y_p'' = u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2 y_2'' + u_2' y_2' + u_2' y_2' + u_2'' y_2$$

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters

$$y_p'' + p(x)y_p' + q(x)y_p = \underset{①}{u_1y_1''} + \underset{②}{u_1'y_1'} + \underset{⑤}{u_1'y_1'} + \underset{②}{u_1''y_1} + \underset{①}{u_2y_2''} + \underset{⑤}{u_2'y_2'} + \underset{③}{u_2'y_2'} + \underset{③}{u_2''y_2}$$

$$+\, p(x)[\underset{①}{u_1y_1'} + \underset{④}{u_1'y_1} + \underset{①}{u_2y_2'} + \underset{④}{u_2'y_2}]$$

$$+\, q(x)[\underset{①}{u_1y_1} + \underset{①}{u_2y_2}]$$

$$= \underbrace{u_1\underbrace{[y_1'' + py_1' + qy_1]}_{=0} + u_2\underbrace{[y_2'' + py_2' + qy_2]}_{=0}}_{①} + \underbrace{u_1''y_1 + u_1'y_1'}_{②} + \underbrace{u_2''y_2 + u_2'y_2'}_{③} + \underbrace{p[u_1'y_1 + u_2'y_2]}_{④} + \underbrace{u_1'y_1' + u_2'y_2'}_{⑤}$$

$$\frac{d}{dx}[u_1'y_1] + \frac{d}{dx}[u_2'y_2] + p[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2' = g(t)$$

- Seek to determine 2 unknown function $u_1(t), u_2(t)$
- Impose a condition $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$
- The two Eqs.

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{cases}$$

y_1, y_2, y_1', y_2', known

u_1', u_2' unknown

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters

$$\frac{d}{dx}[u_1' y_1] + \frac{d}{dx}[u_2' y_2] + p[u_1' y_1 + u_2' y_2] + u_1' y_1' + u_2' y_2' = g(t)$$

$$\frac{d}{dx}[u_1' y_1 + u_2' y_2] + p[u_1' y_1 + u_2' y_2] + u_1' y_1' + u_2' y_2' = g(t)$$

- Seek to determine 2 unknown function $u_1(t), u_2(t)$

- Impose a condition $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$ Reducing the diff. equation to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- The two Eqs.

$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{cases}$$

$y_1, y_2, y_1', y_2',$ known

u_1', u_2' unknown

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}; \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$u_1' = -\frac{y_2 g}{W(y_1, y_2)}; \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)}$$

$$u_1 = -\int \frac{y_2 g}{W(y_1, y_2)} dt + c_1; \qquad u_2 = \int \frac{y_1 g}{W(y_1, y_2)} dt + c_2$$

Based on (*) $y_p = u_1(t) y_1(t) + u_2(t) y_2(t)$

$$Y_p(t) = -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dt + c_1 + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dt + c_2$$

Theorem (3.6.1)

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions p, q and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt$$

and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters – Example

$$y'' - y = \frac{1}{x}$$

- Solution to the **homogeneous** diff Eq.

$$\lambda^2 - 1 = 0 \ \rightarrow \ \lambda_1 = -1; \ \lambda_2 = 1$$

$$y_C = c_1 e^{x} + c_2 e^{-x}$$

- Solution to the **nonhomogeneous** diff Eq.

$$W\left(e^{x}, e^{-x}\right) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{0} - e^{0} = -2$$

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters – Example

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ \frac{1}{x} & -e^{-x} \end{vmatrix}}{-2} = -\frac{e^{-x}(1/x)}{-2} \rightarrow u_1 = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t} dt$$

$$u_2' = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{1}{x} \end{vmatrix}}{-2} = \frac{e^{x}(1/x)}{-2} \rightarrow u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t} dt$$

$$y_p = Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt$$

$$y_p = Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

$$y_p = Y(t) = \frac{1}{2}e^{x}\int_{x_0}^{x} \frac{e^{-t}}{t} dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t} dt,$$

Second Order Linear Non Homogenous Differential Equations –

Method of Variation of Parameters – Example

- General Solution to the **nonhomogeneous** diff Eq.

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{x} \int_{x_0}^{x} \frac{e^{-t}}{t} dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^{t}}{t} dt$$