Class Notes 4:

Second Order Differential Equation – Homogeneous

82A – Engineering Mathematics
Second Order Linear Differential Equations

**Introduction**

\[ \frac{d^2 y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad \begin{cases} y(t_0) = y_0 \\ y'_0(t_0) = y'_0 \end{cases} \]

- **Linear**
- **Non Linear**

- **Homogenous**
  \[ y'' + p(t)y' + q(t)y = 0 \]
  Solution: Complementary Solution
  Method: Series Solution

- **Non-Homogenous**
  \[ y'' + p(t)y' + q(t)y = g(t) \]
  Solution: Particular Solution
  Method: Series Solution

- **Homogenous**
  \[ ay'' + by' + cy = 0 \]
  Solution: Complementary Solution

- **Non-Homogenous**
  \[ ay'' + by' + cy = g(t) \]
  Solution: Particular Solution
  Method 1: Undermined Coefficient
  Method 2: Variation of Parameters
Structure of the General Solution

\[ y'' + p(x)y' + q(x)y = \begin{cases} g(x), & \text{Non-homogeneous} \\ 0, & \text{Homogeneous} \end{cases} \]

- Solution:

\[
y = y_c(x) + y_p(x)
\]

where

- \( y_c(x) \): solution of the **homogeneous equation** (complementary solution)
- \( y_p(x) \): any solution of the **non-homogeneous equation** (particular solution)
Lumped Parameters Models

Introduction
Lumped Parameter Model – Introduction
Lumped Parameter Model (Mass Spring)

\[ f(t) + \sum F = f(t) - kx = m\ddot{x} \]

\[ m\ddot{x} + kx = f(t) \]

- Initial Condition:

\[ \begin{align*}
x(t = 0) &= x_0 = d \\
\dot{x}(t = 0) &= \dot{x}_0 = v
\end{align*} \]

\[ \ddot{x} + \frac{k}{m} x = \frac{f(t)}{m} \quad \omega = \sqrt{\frac{k}{m}} \]

\[ \ddot{x} + \omega^2 x = \begin{cases} 
\frac{f(t)}{m} \\
0
\end{cases} \]
Lumped Parameter Model (Mass Spring)

\[ \dot{x} + \frac{k}{m} x = \frac{k}{m} z \]

\[ \ddot{x} + \frac{k}{m} x = \frac{k}{m} z \]

\[ \ddot{x} + \omega^2 x = \begin{cases} \omega^2 z & \text{for } \omega^2 \neq 0 \\ 0 & \text{for } \omega^2 = 0 \end{cases} \]

Initial Condition:

\[ \begin{cases} x(t = 0) = x_0 = d \\ \dot{x}(t = 0) = \dot{x}_0 = v \end{cases} \]

where \( \omega = \sqrt{\frac{k}{m}} \).
Lumped Parameter Model (Mass Spring)

\[ + \uparrow \sum F = f(t) - kx - c\dot{x} = m\ddot{x} \]

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]

\[ \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = \frac{f(t)}{m} \]

\[ \omega_n^2 = \frac{k}{m} \Rightarrow \omega_n = \sqrt{\frac{k}{m}} \]

\[ 2\zeta \omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{k}} = \frac{c}{2\sqrt{mk}} \]

\[ \dot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \begin{cases} \frac{f(t)}{m} & \text{\(f(t)\)} \\ 0 & \text{\(0\)} \end{cases} \]

\[ \zeta \text{ - Viscous damping factor (non-dimensional)} \]

\[ \omega_n \text{ - Natural frequency} \]
Lumped Parameter Model (Mass Spring)

\[ + \sum F = k(z - x) - c(\dot{z} - \dot{x}) = m\ddot{x} \]

\[ m\ddot{x} + c\dot{x} + kx = c\dot{z} + kz \]

\[ \ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = \frac{c}{m} \dot{z} + \frac{k}{m} z \]

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 2\zeta\omega_n \dot{z} + \omega_n^2 z \]

\[ \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \begin{cases} 2\zeta\omega_n \dot{z} + \omega_n^2 z \\ 0 \end{cases} \]
Second Order Homogeneous Diff. Eq.
Theorems
Theorem 1 (3.2.1)
Existence and Uniqueness

- **Consider** the initial value problem

\[ y'' + p(t) y' + q(t) y = g(t) \]
\[ y(t_0) = y_0, \ y'(t_0) = y'_0 \]

- **where** \( p, q, \) and \( g \) are continuous on an open interval \( I \) that contains \( t_0 \).
- **Then** there exists a unique solution \( y = \phi(t) \) on \( I \).

- **Note:** While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.
Theorem 2 (3.2.2)
Principle of Superposition

- If \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) \, y' + q(t) \, y = 0
\]

- Then the linear combination \( c_1 y_1 + y_2 c_2 \) is also a solution, for all constants \( c_1 \) and \( c_2 \).
The Wronskian Determinant (1/3)

• Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) y = 0
\]

• From Theorem 3.2.2, we know that \( y = c_1 y_1 + c_2 y_2 \) is a solution to this equation.

• Next, find coefficients such that \( y = c_1 y_1 + c_2 y_2 \) satisfies the initial conditions

\[
y(t_0) = y_0, \quad y'(t_0) = y_0'
\]

• To do so, we need to solve the following equations:

\[
c_1 y_1(t_0) + c_2 y_2(t_0) = y_0
\]

\[
c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'
\]
The Wronskian Determinant (2/3)

• Solving the equations, we obtain

\[ c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \]
\[ c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0 \]

• In terms of determinants:

\[ c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)} \]
\[ c_2 = \frac{-y_0 y'_1(t_0) + y'_0 y_1(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)} \]
The Wronskian Determinant (3/3)

- In order for these formulas to be valid, the determinant $W$ in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{W}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} = y_1(t_0)y'_2(t_0) - y'_1(t_0)y_2(t_0)$$

- $W$ is called the **Wronskian determinant**, or more simply, the Wronskian of the solutions $y_1$ and $y_2$. We will sometimes use the notation

$$W(y_1, y_2)(t_0)$$
Theorem 3 (3.2.3)  
Constants Coefficient  \( c_1, \ c_2 \)

- Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) \ y = 0
\]

with the initial conditions

\[
y(t_0) = y_0, \ y'(t_0) = y'_0
\]

Then it is always possible to choose constants \( c_1, \ c_2 \) so that

\[
y = c_1 y_1(t) + c_2 y_2(t)
\]

satisfies the differential equation and initial conditions if and only if the Wronskian

\[
W = y_1 y'_2 - y'_1 y_2
\]

is not zero at the point \( t_0 \)
Theorem 4 (3.2.4)
General Solution / Fundamental Set of Solutions

• Suppose \( y_1 \) and \( y_2 \) are solutions to the equation

\[
L[y] = y'' + p(t) y' + q(t) y = 0.
\]

Then the family of solutions

\[
y = c_1 y_1 + c_2 y_2
\]

with arbitrary coefficients \( c_1, c_2 \) includes every solution to the differential equation if and only if there is a point \( t_0 \) such that

\[
W(y_1, y_2)(t_0) \neq 0.
\]

• The expression \( y = c_1 y_1 + c_2 y_2 \) is called the general solution of the differential equation above, and in this case \( y_1 \) and \( y_2 \) are said to form a fundamental set of solutions to the differential equation.
Theorem 5 (3.2.5)
Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients $p$ and $q$ are continuous on some open interval $I$:

$$L[y] = y'' + p(t) y' + q(t) y = 0$$

- Let $t_0$ be a point in $I$, and $y_1$ and $y_2$ solutions of the equation with $y_1$ satisfying initial conditions

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0$$

and $y_2$ satisfying initial conditions

$$y_2(t_0) = 0, \quad y_2'(t_0) = 1$$

- Then $y_1, y_2$ form a fundamental set of solutions to the given differential equation.
Consider again the equation (2):

\[ L[y] = y'' + p(t) y' + q(t) y = 0 \]

where \( p \) and \( q \) are continuous real-valued functions.

If \( y = u(t) + iv(t) \) is a complex-valued solution of Eq. (2),

Then its real part \( u \) and its imaginary part \( v \) are also solutions of this equation.

\[ y = C_1 u(t) + C_2 v(t) \]
Theorem 7 (3.2.7)
Abel’s Theorem

• Suppose $y_1$ and $y_2$ are solutions to the equation
  \[ L[y] = y'' + p(t) y' + q(t) y = 0 \]
  Where $p$ and $q$ are continuous on some open interval $I$.

• Then the $W(y_1, y_2)(t)$ is given by
  \[
  W(y_1, y_2)(t) = c e^{-\int p(t) dt}
  \]
  where $c$ is a constant that depends on $y_1$ and $y_2$ but not on $t$.

• Note that $W(y_1, y_2)(t)$ is either zero for all $t$ in $I$ (if $c = 0$) or else is never zero in $I$ (if $c \neq 0$).
Summary

• To find a general solution of the differential equation
  \[ y'' + p(t) y' + q(t) y = 0, \quad \alpha < t < \beta \]
  we first find two solutions \( y_1 \) and \( y_2 \).
• Then make sure there is a point \( t_0 \) in the interval such that \( W(y_1, y_2)(t_0) \neq 0 \).
• It follows that \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the equation, with general solution \( y = c_1 y_1 + c_2 y_2 \).
• If initial conditions are prescribed at a point \( t_0 \) in the interval where \( W \neq 0 \), then \( c_1 \) and \( c_2 \) can be chosen to satisfy those conditions.
Homogeneous Second Order Linear Differential Equations – Constant Coefficient

\[ ay'' + by' + cy = 0 \]

• Assume:
  \[ y = e^{rt} \]

sub:
  \[ ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \]

\[ (ar^2 + br + c)e^{rt} = 0 \]

since
  \[ e^{rt} \neq 0 \]

Characteristic Equation (Quadratic Equation):

\[ ar^2 + br + c = 0 \]

Solution for \( r \):

1) Real and different
2) Real and repeated
3) Complex conjugates:

\[ \begin{align*}
  r_1 &= \text{Re} + \text{Im} \, j \\
  r_2 &= \text{Re} - \text{Im} \, j
\end{align*} \]
Second Order Homogeneous Diff. Eq.
Mathematical Approach
Homogeneous Equations With Constant Coefficients

Case 1 – Roots – Real & Different

- Case 1: Real and different roots
  \[ ay'' + by' + cy = 0 \]

Assume \( y = e^{rt} \)

\[ (ar^2 + br + c)e^{rt} = 0 \]

\[ ar^2 + br + c = 0 \]

If \( b^2 - 4ac > 0 \) then \( r_1 \neq r_2 \) and \( r_1 \in \mathbb{R}, r_2 \in \mathbb{R} \)

\[ \begin{cases} y_1 = e^{r_1t} \\ y_2 = e^{r_2t} \end{cases} \]

Solution has the form

\[ y = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{r_1t} + C_2 e^{r_2t} \]

\[ y(t_0) = y_0 \]

\[ y'(t_0) = y'_0 \]
Homogeneous Equations With Constant Coefficients

Case 1 – Roots – Real & Different

- Substituting \( t = t_0; \ y = y_0 \)
  \[ \rightarrow y_0 = C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0} \]
- Substituting \( t = t_0; \ y' = y'_0 \)
  \[ \rightarrow y'_0 = C_1 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0} \]
- Solve for \( C_1 \) and \( C_2 \)
  \[ C_1 = \frac{y'_0 + y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}; \quad C_2 = \frac{y_0 r_1 + y'_0}{r_1 - r_2} e^{-r_2 t_0} \]
- Notes:
  \[ r_1 \neq r_2 \Rightarrow r_1 - r_2 \neq 0 \Rightarrow C_1, C_2 \text{ exist} \]

One possible choice of \( C_1 \) and \( C_2 \) for some initial conditions
Homogeneous Equations With Constant Coefficients

**Case 1 – Roots – Real & Different**

- **Case 1a** \( r_1 < 0 \)
  \( r_2 < 0 \)

- **Case 1b** \( r_1 > 0 \)
  \( r_2 < 0 \)

- **Case 1c** \( r_1 = r_2 = 0 \)

- **Case 1d** \( r_1 > 0 \)
  \( r_2 > 0 \)
Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Homogeneous Equation with Constant Coefficients
- Case 3: Complex conjugate roots

\[ ay'' + by' + cy = 0 \]
\[ y = e^{rt} \]
\[ (ar^2 + br + c)e^{rt} = 0 \]
\[ ar^2 + br + c = 0 \]
\[ b^2 - 4ac < 0 \quad \rightarrow \quad \begin{cases} r_1 = \lambda + i\mu \\ r_2 = \lambda - i\mu \end{cases} \]
\[ \begin{cases} y_1 = e^{(\lambda + i\mu)t} \\ y_2 = e^{(\lambda - i\mu)t} \end{cases} \]
Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

• Euler’s Formula

  • Taylor’s series for \( e^t \) about \( t=0 \)

  \[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad -\infty < t < \infty
\]

  • Substitute \( it \) for \( t \)

  \[
e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}
\]

    - Real part: \( \cos(t) \)
    - Imaginary part: \( \sin(t) \)

\[
\begin{cases}
i^2 = -1 \\
i^3 = -i \\
i^4 = 1 \\
i^5 = i
\end{cases} \implies \begin{cases}i^{2n} = (-1)^n \\
i^{2n+1} = (-i)^n
\end{cases}
\]
Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

\[ e^{it} = \cos(t) + i \sin(t) \]
\[ e^{-it} = \cos(t) - i \sin(t) \]
\[ \cos(-t) = \cos(t) \]
\[ \sin(-t) = -\sin(t) \]
\[ e^{it} = \cos(t) + i \sin(t) \]
\[ e^{i\mu t} = \cos(\mu t) + i \sin(\mu t) \]
\[ e^{(\lambda + i\mu) t} = e^{\lambda t} \cdot e^{i\mu t} = e^{\lambda t} (\cos(\mu t) + i \sin(\mu t)) \]
\[ y = C_1 y_1 + C_2 y_2 = C_1 e^{(\lambda + i\mu) t} + C_2 e^{(\lambda - i\mu) t} = e^{\lambda t} (C_1 (\cos(\mu t) + i \sin(\mu t)) + C_2 (\cos(\mu t) - i \sin(\mu t))) = e^{\lambda t} ((C_1 + C_2)(\cos(\mu t) + i(C_1 + C_2)(\sin(\mu t))) \]

- Or based on the theorem 3.2.6 p 153, both real and imaginary parts are solutions

\[ y = e^{\lambda t} (\tilde{C}_1 \cos(\mu t) + \tilde{C}_2 \sin(\mu t)) \]
Homogeneous Equations With Constant Coefficients

**Case 3 – Roots – Complex Conjugates**

- **Case 3a:** For $\lambda < 0$
  \[ y = e^{-\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t) \]

- **Case 3b:** For $\lambda > 0$
  \[ y = e^{\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t) \]
Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Case 3c: For $\lambda=0$
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

\[ ay'' + by' + cy = 0 \]
\[ y = e^{rt} \]
\[ (ar^2 + br + c)e^{rt} = 0 \]
\[ ar^2 + br + c = 0 \]
\[ b^2 - 4ac = 0 \quad \rightarrow \quad r_1 = r_2 = -\frac{b}{2a} \]
\[ y_1 = e^{-\frac{b}{2a}t} \]

- To find a second solution, we assume that

\[ y = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t} \]
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

- Substitute for \( y \) in differential equation \( ay'' + by' + cy = 0 \)

\[
y = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t}
\]

\[
y' = v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t}
\]

\[
y'' = v''(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}
\]

\[
\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + c v(t)\right\}e^{-\frac{b}{2a}t} = 0
\]

\[
a v''(t) + (-b + b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0
\]

\[
v''(t) = 0
\]

\[
v(t) = C_1 + C_2t
\]

\[
y = v(t)y_1(t) = C_1y_1(t) + C_2ty_1(t)
\]

- The solution for the differential equation

\[
y = C_1e^{-\frac{b}{2a}t} + C_2te^{-\frac{b}{2a}t}
\]
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

\[ y_1(t) = e^{-\frac{b}{2a}t} \; ; \; \quad y_2(t) = te^{-\frac{b}{2a}t} \]

\[ W(y_1, y_2) = \begin{vmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a} e^{-\frac{b}{2a}t} & \left(1 - \frac{b}{2a}t\right) e^{-\frac{b}{2a}t} \end{vmatrix} = e^{-\frac{b}{a}} \]

\[ W = e^{-\frac{b}{a}} \neq 0 \quad \Rightarrow \quad \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \] Fundamental set of solutions
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

- Case 2a \( r_1 = r_2 > 0 \)

- Case 2b \( r_1 = r_2 < 0 \)
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated - Reduction of Order - Generalization

\[ y'' + p(t)y' + q(t)y = 0 \]

- We know one solution \( y_1(t) \) (not everywhere zero)
  
  To find the second solution let

  \[
  \begin{cases}
  y = v(t)y_1(t) \\
  y' = v'(t)y_1(t) + v(t)y_1'(t) \\
  y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)
  \end{cases}
  \]

  Substitute for \( y, y', y'' \) in the differential equation

  \[ y_1v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0 \]

  \( y_1 \) is a solution \( \Rightarrow 0 \)
Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated - Reduction of Order - Generalization

\[ y_1 \nu'' + (2y_1' + py_1)\nu' = 0 \]

\[ u = \nu' \]

\[ y_1u' + (2y_1' + py_1)u = 0 \]

- First order differential equation
  Find a solution for \( u \)

\[ \nu = \int u \]

See example 3, p172
### Homogeneous Equations With Constant Coefficients

**Summary**

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<td><strong>Case 3</strong> – Roots – Complex conjugates $\lambda \pm i\mu$</td>
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<td>$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$</td>
<td>$y = e^{\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t)$</td>
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<tr>
<td><strong>Case 2</strong> – Roots – Real and repeated $r_1 = r_2$</td>
<td></td>
<td>$y = C_1 e^{r_1 t} + C_2 te^{r_2 t}$</td>
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Second Order Homogeneous Diff. Eq.

Engineering Approach (Free Response)
Undamped & Damped
**SECOND ORDER SYSTEM (FREE RESPONSE)**

\[ m\ddot{x} + c\dot{x} + kx = f(t) \]

\[ \begin{cases} x(t_0) = x_0 \\ \dot{x}(t_0) = \dot{x}_0 = U_0 \end{cases} \]

\[ f(t) = 0 \rightarrow \text{free response} \]

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 0 \]

\[ \omega_n = \sqrt{\frac{k}{m}} \rightarrow \text{Natural frequency} \]

\[ \zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{k/m}} = \frac{c}{2\sqrt{mk}} \]

**Assume Solution**

\[ x(t) = Ce^{\lambda t} \]

**Characteristic Eq**

\[ \lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2 = 0 \]

\[ \lambda_1, \lambda_2 = \frac{-2\zeta \omega_n \pm \sqrt{4\zeta^2 \omega_n^2 - 4\omega_n^2}}{2} \]

\[ \begin{aligned} \lambda_1 & = -2\zeta \omega_n + \sqrt{4\zeta^2 \omega_n^2 - 4\omega_n^2} \\ \lambda_2 & = -2\zeta \omega_n - \sqrt{4\zeta^2 \omega_n^2 - 4\omega_n^2} \end{aligned} \]

\[ \begin{aligned} \lambda_1 & = -\omega_n (\zeta + \sqrt{\zeta^2 - 1}) \\ \lambda_2 & = -\omega_n (\zeta - \sqrt{\zeta^2 - 1}) \end{aligned} \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]
<table>
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<td>m</td>
<td>m, k, c = 0</td>
<td>$m\dddot{x} = 0$</td>
<td>$\lambda^2 = 0$</td>
<td>$x = c_1 t + c_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\dddot{x} = 0$</td>
<td>$\lambda_1 = \lambda_2 = 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Diagram:**
- Graph showing the characteristic equation with roots $\lambda_1$ and $\lambda_2$.
- Graph showing the solution $x = c_1 t + c_2$.

**Additional Equations:**
- $\dddot{x} = \dot{\lambda} x$
- $\lambda = \lambda_1$ or $\lambda = \lambda_2$
- $x = e^{\lambda t}$
- For $\lambda = \lambda_1$ or $\lambda = \lambda_2$, the solutions are:
  - $x = c_1 e^{\lambda_1 t}$
  - $x = c_2 e^{\lambda_2 t}$
- $x = A\cos(\omega t - \Psi) + B_1\sin(\omega t) + B_2\cos(\omega t)$
  - $A = \sqrt{(x_0)^2 + (\frac{v_0}{\omega})^2}$
  - $\Psi = \tan^{-1}\left(\frac{v_0}{\omega x_0}\right)$
<table>
<thead>
<tr>
<th>MODEL</th>
<th>PARAMETERS</th>
<th>DIFF EQ.</th>
<th>CHARACTERISTIC EQ.</th>
<th>SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m, k, c$</td>
<td>$m \dddot{x} + c \ddot{x} + kx = 0$</td>
<td>$\lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2 = 0$</td>
<td>\begin{align*} \lambda_1 &amp; = \omega_n \left( \zeta \pm \sqrt{\zeta^2 - 1} \right) \ \lambda_2 &amp; = \omega_n \left( \zeta \pm i \sqrt{1 - \zeta^2} \right) \end{align*}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\dddot{x} + 2\zeta \omega_n \ddot{x} + \omega_n^2 x = 0$</td>
<td>$x(t) = e^{-3\omega_n t} \left( A_1 \cos \omega_d t + A_2 \sin \omega_d t \right)$</td>
<td></td>
</tr>
</tbody>
</table>

**CASE 3**

- Underdamped

\[ x(t) = e^{-3\omega_n t} \left( A_1 \cos \omega_d t + A_2 \sin \omega_d t \right) \]

\[
\begin{align*}
A_1 & = x_0 \\
A_2 & = \frac{u_0 + i \omega_n x_0}{\omega_d} \\
\omega_d & = \omega_n \sqrt{1 - \zeta^2} \\
x(t) & = A e^{-3\omega_n t} \cos(\omega_d t + \phi) \\
A & = \frac{x_0}{\cos(\phi)} \\
\phi & = \tan^{-1} \left[ \frac{u_0 - x_0}{\omega_d x_0} \right]
\end{align*}
\]
<table>
<thead>
<tr>
<th>MODEL</th>
<th>PARAMETERS</th>
<th>DIFF EQ.</th>
<th>CHARACTERISTIC EQ</th>
<th>SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\zeta = 1$</td>
<td>Critical Damping</td>
<td>$x(t) = e^{-\omega nt} \left[ c_1 + c_2 t \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_1 = \lambda_2 = -\omega n$</td>
<td>(CASE 2)</td>
<td>$c_1 = x_0 \left( 1 - \omega nt \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\zeta = \frac{c}{2mw_n} = 1$</td>
<td>$c_2 = \ddot{x}_0 = v_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$c_{cr} = 2\sqrt{km}$</td>
<td>$x(t)$</td>
<td>May cross only once</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-\omega n$</td>
<td>$\zeta &gt; 1$</td>
<td>Overdamped</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\zeta &gt; 1$</td>
<td></td>
<td>$x(t) = e^{-\omega nt} \left( c_1 e^{1/2 \omega nt} + c_2 e^{-1/2 \omega nt} \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_1 = -\omega n - \sqrt{\omega^2 - \zeta^2} \omega n$</td>
<td>$c_1 = \frac{x_0 \left( -1 - \sqrt{\omega^2 - 1} \right) \omega n - \dot{x}_0}{-\left(2 \sqrt{\omega^2 - 1} \right) \omega n}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -\omega n + \sqrt{\omega^2 - \zeta^2} \omega n$</td>
<td>$c_2 = \frac{x_0 - x_0 \left( -3 + \sqrt{\omega^2 - 1} \right) \omega n}{-\left(2 \sqrt{\omega^2 - 1} \right) \omega n}$</td>
<td></td>
</tr>
</tbody>
</table>
Free Response of Damped Second Order System –
Introduction – Cases – The “S” Plane
Free Response of Underdamped Second Order System

Harmonic Oscillator

Free response \( \rightarrow f(t) = 0 \)

\[ \ddot{x} + \omega_n^2 x = 0 \]

- The general solution can be written in the form

\[ x(t) = Ae^{\lambda t} \]

- Introducing the solution into the equation

\[ A\lambda^2 e^{\lambda t} + \omega^2 Ae^{\lambda t} = 0 \]

\[ \left( \lambda^2 + \omega^2 \right) Ae^{\lambda t} = 0 \]

\[ = 0 \quad \neq 0 \]
Free Response of Underdamped Second Order System

**Harmonic Oscillator**

\[
\lambda^2 + \omega^2 = 0 \quad \Rightarrow \quad \lambda^2 = -\omega^2
\]

\[
\begin{align*}
\lambda_1 & = \pm \omega_n i \\
\lambda_2 & = \pm \omega_n i
\end{align*}
\]

- In general the solution becomes

\[
x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{i \omega_n t} + A_2 e^{-i \omega_n t}
\]

\[
\begin{align*}
A_1 \\
A_2
\end{align*}
\]

constants of integration

- Since \( x(t) \) must be real \( \rightarrow \)

\[
\begin{align*}
A_1 & = \frac{1}{2} A e^{-i \psi} \\
A_2 & = \frac{1}{2} A e^{i \psi}
\end{align*}
\]

where A & \( \psi \) are real
Free Response of Underdamped Second Order System

\textbf{Harmonic Oscillator}

\[ x(t) = \frac{1}{2} A \left[ e^{i(w_n t - \psi)} + e^{-i(w_n t - \psi)} \right] \]

\[ x(t) = \frac{1}{2} A \left[ \cos(w_n t - \psi) + i \sin(w_n t - \psi) + \cos(-(w_n t - \psi)) + i \sin(-(w_n t - \psi)) \right] \]

\[ = \cos(w_n t - \psi) + -i \sin(w_n t - \psi) \]

\[ x(t) = A \cos(w_n t - \psi) \]

Based on
\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \]

\[ x(t) = A \left[ \cos w_n t \cos \psi + \sin w_n t \sin \psi \right] \]
Free Response of Underdamped Second Order System

Harmonic Oscillator

\[ x(t) = A[\cos w_n t \cos \psi + \sin w_n t \sin \psi] \]

The solution can also be expressed as

\[ x(t) = B_1 \sin w_n t + B_2 \cos w_n t \]

\[ \begin{align*}
    B_1 & \quad \text{constants of integration} \\
    B_2 & \quad \text{constants of integration}
\end{align*} \]

\[ x(t) = \frac{B_2}{A} \cos \psi \cos w_n t + \frac{B_1}{A} \sin \psi \sin w_n t \]

- sin / cos - harmonic functions
- Solution – simple harmonic oscillation
- System powered by this type of diff. eq. are called harmonic oscillators
Free Response of Underdamped Second Order System

Harmonic Oscillator

\[ x(t) = A \cos(w_n t - \psi) \]
Free Response of Underdamped Second Order System

Harmonic Oscillator

- $T$  - Time period (of oscillation) [sec]
- $w_n$ - Natural frequency [RAD/sec]
- $f_n$ - Natural frequency [Hz]

\[
    w_n = 2\pi f \rightarrow f_n = \frac{w_n}{2\pi}
\]

\[
    T = \frac{1}{f_n} = \frac{2\pi}{w_n}
\]
Free Response of Underdamped Second Order System

**Harmonic Oscillator**

- For spring-mass system

\[ T = 2\pi \sqrt{\frac{m}{k}} \; ; \; w_n = \sqrt{\frac{k}{m}} \quad k \uparrow \rightarrow w_n \uparrow \]

- For pendulum

\[ T = 2\pi \sqrt{\frac{L}{g}} \; ; \; w_n = \sqrt{\frac{g}{L}} \quad L \uparrow \rightarrow w_n \downarrow \]

- Note – that \( T \) or \( w_n \) are not function of \( m \)
No matter how the motion is initiated, free oscillation always occurs at the freq. $w_n$

$w_n = f(k, m) \quad w_n = f(g, L)$

$w_n$ is independent of external forces; that is the reason why $w_n$ is called **natural frequency**.

Mathematical idealization → perpetuate → *Ad infinitum*

Every real system possesses some measure of **damping**

Pendulum damping

Air resistance

Friction point of support

When damping is small – harmonic oscillation

For short period of time $t << T$ small damping does not have any effect over that interval
Free Response of Underdamped Second Order System

Harmonic Oscillator – Initial Conditions

IC to determine $A, \psi$ (two conditions)

\[
x(0) = x_0 (+/-)
\]

\[
\dot{x}(0) = v_0 (+/-)
\]

\[
x(t) = A \cos(w_n t - \psi)
\]

\[
\begin{align*}
x(0) &= A \cos \psi = x_0 \quad \rightarrow \quad \cos \psi = \frac{x_0}{A} \\
\dot{x}(0) &= w_n A \sin \psi = v_0 \quad \rightarrow \quad \sin \psi = \frac{v_0}{w_n A}
\end{align*}
\]
Free Response of Underdamped Second Order System

**Harmonic Oscillator – Initial Conditions**

\[
\begin{align*}
  x(0) &= A \cos \psi = x_0 \quad \rightarrow \quad \cos \psi = \frac{x_0}{A} \\
  \dot{x}(0) &= w_n A \sin \psi = v_0 \quad \rightarrow \quad \sin \psi = \frac{v_0}{w_n A}
\end{align*}
\]

Determine \( A \)

\[
1 = \sqrt{(\sin \psi)^2 + (\cos \psi)^2} = \sqrt{\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{w_n A}\right)^2}
\]

\[
A = \sqrt{\left(x_0\right)^2 + \left(\frac{v_0}{w_n}\right)^2}
\]
Free Response of Underdamped Second Order System

Harmonic Oscillator – Initial Conditions

Determine $\psi$

$$\tan \psi = \frac{\sin \psi}{\cos \psi} = \frac{v_0}{w_n A x_0}$$

$$\psi = \tan^{-1} \left( \frac{v_0}{w_n x_0} \right)$$

Final Solution

$$x(t) = B_1 \sin(w_n t) + B_2 \cos(w_n t) \quad \begin{cases} 
B_1 = A \sin \psi \\
B_2 = A \cos \psi 
\end{cases}$$

$$x(t) = x_0 \cos(w_n t) + \frac{v_0}{w_n} \sin(w_n t)$$
Free Response of Damped Second Order System –

**Introduction**

Free response \( f(t) = 0 \)

\[
\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0
\]

\[
\omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2m\omega_n}
\]

\( \omega_n \) – natural frequency of the system

\( \zeta \) – viscous damping factor

Assume solution \( x(t) = ce^{\lambda t} \)

characteristic Eq. \( \lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \)

\[
\lambda_1, \lambda_2 = -2\zeta\omega_n \pm \frac{\sqrt{4\zeta^2 \omega_n^2 - 4\omega_n^2}}{2} = -2\omega_n (\zeta \pm \sqrt{\zeta^2 - 1}) = \begin{cases} 
-\zeta \pm \sqrt{\zeta^2 - 1} \omega_n \\
-\zeta \pm i\sqrt{1-\zeta} \omega_n 
\end{cases}
\]
Free Response of Damped Second Order System –
Introduction – Cases

CASE 1
(over damped)
\( \zeta > 1 \) \( \rightarrow \) real, negative, distinct

CASE 2
(critical damped)
\( \zeta = 1 \) \( \rightarrow \) real, negative, equal
\[ \lambda_1 = \lambda_2 = -w_n \]

CASE 3
(under damped)
\( 0 < \zeta < 1 \) \( \rightarrow \) complex, conjugates, negative real
\[ \lambda_1 = \lambda_2 = \left( -\zeta \pm i\sqrt{1 - \zeta^2} \right)w_n = \left( -\zeta \pm \sqrt{\zeta^2 - 1} \right)w_n \]
Free Response of Damped Second Order System –

Case 1 – Overdamped

For \( \zeta > 1 \) (Case 1) - overdamped

\[
\begin{align*}
x(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
&= c_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + c_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \\
&= e^{-\zeta\omega_n t} \left(c_1 e^{+\sqrt{\zeta^2 - 1}\omega_n t} + c_2 e^{-\sqrt{\zeta^2 - 1}\omega_n t}\right)
\end{align*}
\]

since \( \zeta > 1 \) \( \rightarrow \) \( \zeta > \sqrt{\zeta^2 - 1} \)

the response \( x(t) \) decays exponentially with time

- Aperiodic motion – approaches zero without oscillation

Applying initial condition

\[
\begin{align*}
x(t = 0) &= x_0 \\
\dot{x}(t = 0) &= \dot{x}_0
\end{align*}
\]
Free Response of Damped Second Order System –

**Case 1 – Overdamped**

\[
x(t = 0) = c_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})w_n 0} + c_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})w_n 0} = c_1 + c_2 = x_0
\]

\[
\dot{x}(t = 0) = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})w_n 0} + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})w_n 0} = \dot{x}_0
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]

\[
\begin{cases}
\left(-\zeta + \sqrt{\zeta^2 - 1}\right)w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right)w_n c_2 = \dot{x}_0 \\
\end{cases}
\]
Free Response of Damped Second Order System – Case 1 – Overdamped

\[
\begin{aligned}
\begin{cases}
(c_1 + c_2) \left( -\zeta + \sqrt{\zeta^2 - 1} \right) w_n c_1 + \left( -\zeta - \sqrt{\zeta^2 - 1} \right) w_n c_2 = x_0 \\
\end{cases}
\end{aligned}
\]

\[
\Delta = \left| \begin{array}{cc}
\left( -\zeta + \sqrt{\zeta^2 - 1} \right) w_n & \left( -\zeta - \sqrt{\zeta^2 - 1} \right) w_n \\
\end{array} \right| = \left( -\zeta - \sqrt{\zeta^2 - 1} \right) w_n + \left( +\zeta - \sqrt{\zeta^2 - 1} \right) w_n
\]

\[
\begin{aligned}
c_1 &= \frac{x_0}{\Delta} \left( -\zeta - \sqrt{\zeta^2 - 1} \right) w_n - \dot{x}_0 \\
c_2 &= \frac{\dot{x}_0}{\Delta} \left( -\zeta + \sqrt{\zeta^2 - 1} \right) w_n - x_0
\end{aligned}
\]
Free Response of Damped Second Order System –

**Case 1 – Overdamped**

\[ w_d = w_n \sqrt{1 - \zeta^2} \]

\[
c_1 = \frac{-x_0 \zeta w_d - x_0 w_d - \dot{x}_0}{-2w_d} = \frac{\dot{x}_0}{2w_d} + \frac{x_0 (\zeta w_n + w_d)}{2w_d}
\]

\[
c_2 = \frac{\dot{x}_0 + x_0 \zeta w_n - x_0 w_d}{-2w_d} = -\frac{\dot{x}_0}{2w_d} + \frac{x_0 (\zeta w_n - w_d)}{2w_d}
\]
Free Response of Damped Second Order System –

**Case 1 – Overdamped**

The mass may pass through its equilibrium position at most once.
Free Response of Damped Second Order System –

**Case 2 – Critical Damping**

For \( \zeta = 1 \) (Case 1) critical damped

\[
\lambda_1 = \lambda_2 = -w_n
\]

\[
y(t) = (c_1 + c_2 t)e^{-w_n t}
\]

For \( \zeta = 1 \)

\[
\zeta = \frac{c}{2mw_n} = 1 \rightarrow c = 2mw_n = 2m\sqrt{\frac{k}{m}}
\]

\[
c_{cr} = 2\sqrt{km}
\]

Applying initial condition

\[
\begin{cases}
x(t = 0) = x_0 \\
\dot{x}(t = 0) = \dot{x}_0
\end{cases}
\]

\[
y(t) = e^{-w_n t} \left[ c_1 x_0 (1 + w_n t) + c_2 \dot{x}_0 t \right]
\]

\[
c_1 \quad c_2
\]
Free Response of Damped Second Order System –
Case 2 – Critical Damping
Free Response of Damped Second Order System –

Case 3 – Underdamped

For \(0 < \zeta < 1\) (Case 3) underdamped

\[
\lambda = \left( -\zeta \pm i\sqrt{1-\zeta^2} \right) w_n
\]

\[
\begin{align*}
\lambda & = -\zeta w_n + iw_d \\
\lambda & = -\zeta w_n - iw_d \\
w_d &= w_n \sqrt{1-\zeta^2} \quad \text{damped frequency} \\
w_d &< w_n
\end{align*}
\]

\[
x(t) = e^{-\zeta w_n t} \left( A_1 \cos w_d t + A_2 \sin w_d t \right)
\]
Free Response of Damped Second Order System –
Case 3 – Underdamped

Solving for Initial Conditions

\( x(t) = e^{-\zeta w_d t} \left( A_1 \cos w_d t + A_2 \sin w_d t \right) \)

\( x(t = 0) = e^{-\zeta w_d 0} \left( A_1 \cos w_d 0 + A_2 \sin w_d 0 \right) = x_0 \rightarrow A_1 = x_0 \)

\( \dot{x}(t = 0) = -\zeta w_n e^{-\zeta w_d 0} \left( A_1 \cos w_d 0 + A_2 \sin w_d 0 \right) + e^{-\zeta w_d 0} \left( -A_1 w_d \sin w_d 0 + A_2 w_d \cos w_d 0 \right) \)

\[ = -\zeta w_n A_1 + A_2 w_d = \dot{x}_0 \]

\( A_1 = x_0 \)

\( A_2 = \frac{\dot{x}_0 + \zeta w_n A_1}{w_d} = \frac{\dot{x}_0 + \zeta w_n x_0}{w_d} \)
Free Response of Damped Second Order System –
**Case 3 – Underdamped**

Given the solution

\[ x(t) = e^{-\zeta \omega_n t} \left( A_1 \cos \omega_d t + A_2 \sin \omega_d t \right) \]

Another form of solution would be

\[ x = A e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \]

\[ A, \phi \] unknown constants
Free Response of Damped Second Order System –

Case 3 – Underdamped

Solving for Initial Conditions

\[ x(t_0 = 0) = x_0 \quad \rightarrow \quad x_0 = Ae^{-\zeta \omega_n t} \cos(w_d t - \phi) \]

\[ x_0 = A \cos(\phi) \quad \rightarrow \quad A = \frac{x_0}{\cos(\phi)} \]

\[ \dot{x}(t_0 = 0) = v \quad \rightarrow \quad \dot{x} = A(-\zeta \omega_n)e^{-\zeta \omega_n t} \cos(w_d t - \phi) - Ae^{-\zeta \omega_n t} w_d \sin(w_d t - \phi) \]

\[ v_0 = A(-\zeta \omega_n) \cos(\phi) + A w_d \sin(\phi) \]

\[ v_0 = \frac{x_0}{\cos(\phi)}(-\zeta \omega_n) \cos(\phi) + \frac{x_0}{\cos(\phi)} w_d \sin(\phi) \]
Free Response of Damped Second Order System –

**Case 3 – Underdamped**

\[
v_0 = x_0 \zeta w_n + x_0 w_d \frac{\sin \phi}{\cos \phi}
\]

\[
\frac{v_0 - x_0 \zeta w_n}{x_0 w_d} = \frac{\sin \phi}{\cos \phi}
\]

\[
\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{v_0 - x_0 \zeta w_n}{x_0 w_d}
\]

\[
\begin{aligned}
\phi &= \tan^{-1} \left[ \frac{v_0 - x_0 \zeta w_n}{x_0 w_d} \right] \\
A &= \frac{x_0}{\cos \phi}
\end{aligned}
\]
Free Response of Damped Second Order System –
**Case 3 – Underdamped**

\[ x = Ae^{-\xi w_n t} \cos(w_d t - \phi) \]
Free Response of Damped Second Order System –
Case 3 – Underdamped

\[ x(t) = e^{-\zeta w_n t} \left( c_1 \cos(w_n \sqrt{1 - \zeta^2} t) + c_2 \sin(w_n \sqrt{1 - \zeta^2} t) \right) \]

\[ w_n = 1; \ x(0) = 1; \ \dot{x}(0) = 0 \]
Free Response of Damped Second Order System –
Introduction – Cases – The “S” Plane
Free Response of Second Order System –
Logarithmic Decrement

- System identification
- Determined coefficients of the model experimentally
Free Response of Second Order System – Logarithmic Decrement – Stiffness
Free Response of Second Order System – Logarithmic Decrement – Damping

• Assumptions
  – Viscous damping
  – Under damped system
  – System identification
Free Response of Second Order System – Logarithmic Decrement

For \[ t = t_1 \quad x(t_1) = Ae^{-\zeta w_n t_1} \cos(w_d t_1 - \psi) \]
\[ t = t_2 \quad x(t_2) = Ae^{-\zeta w_n t_2} \cos(w_d t_2 - \psi) \]
\[ = Ae^{-\zeta w_n (t_1 + T)} \cos(w_d (t_1 + T) - \psi) \]

\[
\begin{align*}
e^{-\zeta w_n (t_2)} &= e^{-\zeta w_n (t_1 + T)} \\
e^{-\zeta w_n t_1} e^{-\zeta w_n T} &= e^{-\zeta w_n t_1} e^{\frac{-2\pi \zeta w_n}{w_d}} = e^{-\zeta w_n t_1} e^{\frac{-2\pi \zeta}{\sqrt{1-\zeta^2}}}
\end{align*}
\]

\[ T = \frac{2\pi}{w_d} \]
\[ w_d = w_n \sqrt{1-\zeta^2} \]
\[ w_n \frac{w_n}{w_d} = \frac{1}{\sqrt{1-\zeta^2}} \]
Free Response of Second Order System – Logarithmic Decrement

For

\[ t = t_1 \quad x(t_1) = A e^{-\zeta w_n t_1} \cos(w_d t_1 - \psi) \]
\[ t = t_2 \quad x(t_2) = A e^{-\zeta w_n t_2} \cos(w_d t_2 - \psi) \]
\[ = A e^{-\zeta w_n (t_1 + T)} \cos(w_d (t_1 + T) - \psi) \]

\[
\cos[w_d (t_1) - \psi] = \cos[w_d (t_1 + T) - \psi] = \cos(w_d t_1 - \psi) \cos(w_d T) - \sin(w_d t_1 - \psi) \sin(w_d T) \\
= \cos(w_d t_1 - \psi) \cos(2\pi) - \sin(w_d t_1 - \psi) \sin(2\pi) = \cos(w_d t_1 - \psi) \\
w_d T = 2\pi \quad \Rightarrow \cos(w_d t_2 - \psi) = \cos(w_d t_1 - \psi) \]
Free Response of Second Order System –
Logarithmic Decrement

- The ratio between the two peak values

\[
\frac{x(t_1)}{x(t_2)} = \frac{Ae^{-\zeta w t_1} \cos(w_d t_1 - \psi)}{Ae^{-\zeta w t_2} \cos(w_d t_2 - \psi)} = \frac{e^{-\zeta w t_1}}{e^{-\zeta w t_1} e^{-2\pi \zeta / \sqrt{1-\zeta^2}}}
\]

\[
\frac{x(t_1)}{x(t_2)} = e^{\frac{2\pi \zeta}{\sqrt{1-\zeta^2}}}
\]

\[
\delta = \ln \frac{x(t_1)}{x(t_2)} = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}
\]

\(\delta\) – Logarithmic decrement

\(\delta\) – Measured experimentally – ln of two consecutive peak values (not necessarily the first two)
Free Response of Second Order System – Logarithmic Decrement

\[
\begin{align*}
\delta^2 &= \frac{4\pi^2 \zeta^2}{1 - \zeta^2} \\
\delta^2 (1 - \zeta^2) &= 4\pi^2 \zeta^2 \\
\delta^2 - \zeta^2 \delta^2 &= 4\pi^2 \zeta^2 \\
\delta^2 &= \zeta^2 (4\pi^2 + \delta^2) \\
\zeta^2 &= \frac{\delta^2}{4\pi^2 + \delta^2} \\
\zeta &= \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}
\end{align*}
\]

For small damping \( \delta \ll 2\pi \)

\( \zeta \approx \frac{\delta}{2\pi} \)