Class Notes 4:

Second Order Differential Equation – Homogeneous

82A – Engineering Mathematics

Second Order Linear Differential Equations

Introduction

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \qquad \begin{cases} y(t_0) = y_0 \\ y_0'(t_0) = y_0' \end{cases}$$

Linear

Non Linear

Homogenous

$$y'' + p(t)y' + q(t)y = \boxed{0}$$

Solution: Complementary Solution
Method: Series Solution

Non-Homogenous

$$y'' + p(t)y' + q(t)y = \boxed{g(t)}$$

Solution: Particular Solution
Method: Series Solution

Homogenous

$$ay'' + by' + cy = 0$$

Solution: Complementary Solution

Non-Homogenous

$$ay'' + by' + cy = g(t)$$

Solution: Particular Solution
Method 1: Undermined Coefficient
Method 2: Variation of Parameters

Structure of the General Solution

$$y'' + p(x)y' + q(x)y = \begin{cases} g(x), & \text{Non-homogeneous} \\ 0, & \text{Homogeneous} \end{cases}$$

- Solution:

$$y = y_c(x) + y_p(x)$$

where

$y_c(x)$: solution of the <u>homogeneous equation</u> (complementary solution)

$y_p(x)$: <u>any</u> solution of the <u>non-homogeneous equation</u> (particular solution)

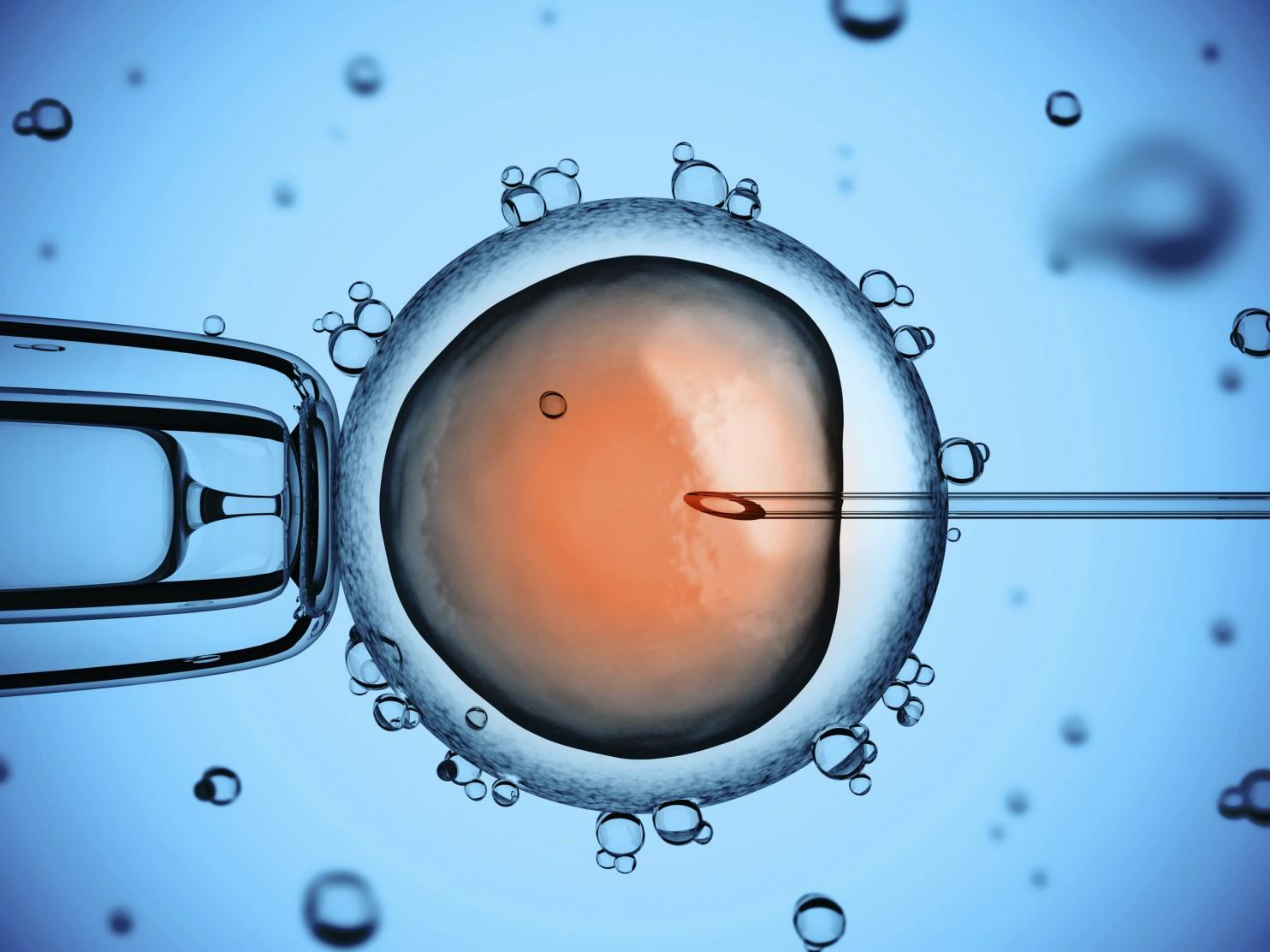
Lumped Parameters Models

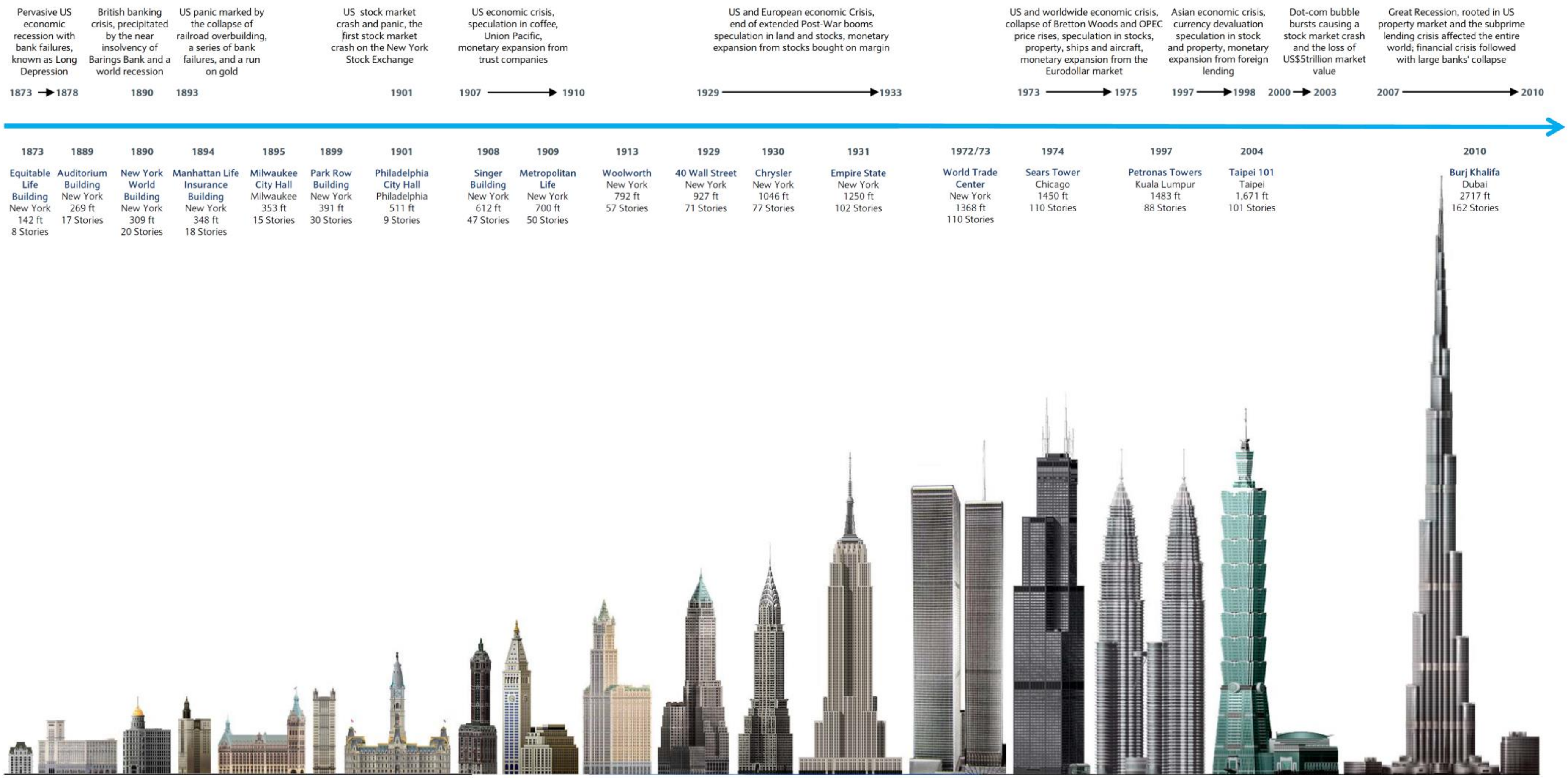
Introduction

Lumped Parameter Model – Introduction

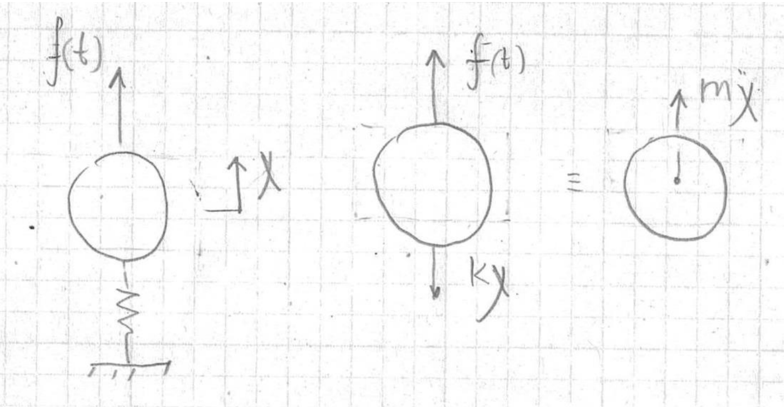

787
DREAMLINER
BOEING


HARLEY-DAVIDSON
XR1200
3664T2




Pervasive US economic recession with bank failures, known as Long Depression
1873 → 1878
British banking crisis, precipitated by the near insolvency of Barings Bank and a world recession
1890
US panic marked by the collapse of railroad overbuilding, a series of bank failures, and a run on gold
1893
US stock market crash and panic, the first stock market crash on the New York Stock Exchange
1901
US economic crisis, speculation in coffee, Union Pacific, monetary expansion from trust companies
1907 → 1910
US and European economic Crisis, end of extended Post-War booms speculation in land and stocks, monetary expansion from stocks bought on margin
1929 → 1933
US and worldwide economic crisis, collapse of Bretton Woods and OPEC price rises, speculation in stocks, property, ships and aircraft, monetary expansion from the Eurodollar market
1973 → 1975
Asian economic crisis, currency devaluation speculation in stock and property, monetary expansion from foreign lending
1997 → 1998
Dot-com bubble bursts causing a stock market crash and the loss of US$5trillion market value
2000 → 2003
Great Recession, rooted in US property market and the subprime lending crisis affected the entire world; financial crisis followed with large banks' collapse
2007 → 2010
1873
Equitable Life Building
New York
142 ft
8 Stories
1889
Auditorium Building
New York
269 ft
17 Stories
1890
New York World Building
New York
309 ft
20 Stories
1894
Manhattan Life Insurance Building
New York
348 ft
18 Stories
1895
Milwaukee City Hall
Milwaukee
353 ft
15 Stories
1899
Park Row Building
New York
391 ft
30 Stories
1901
Philadelphia City Hall
Philadelphia
511 ft
9 Stories
1908
Singer Building
New York
612 ft
47 Stories
1909
Metropolitan Life
New York
700 ft
50 Stories
1913
Woolworth
New York
792 ft
57 Stories
1929
40 Wall Street
New York
927 ft
71 Stories
1930
Chrysler
New York
1046 ft
77 Stories
1931
Empire State
New York
1250 ft
102 Stories
1972/73
World Trade Center
New York
1368 ft
110 Stories
1974
Sears Tower
Chicago
1450 ft
110 Stories
1997
Petronas Towers
Kuala Lumpur
1483 ft
88 Stories
2004
Taipei 101
Taipei
1,671 ft
101 Stories
2010
Burj Khalifa
Dubai
2717 ft
162 Stories

Lumped Parameter Model (Mass Spring)



$$+\uparrow \sum F = f(t) - kx = m\ddot{x}$$

$$m\ddot{x} + kx = f(t)$$

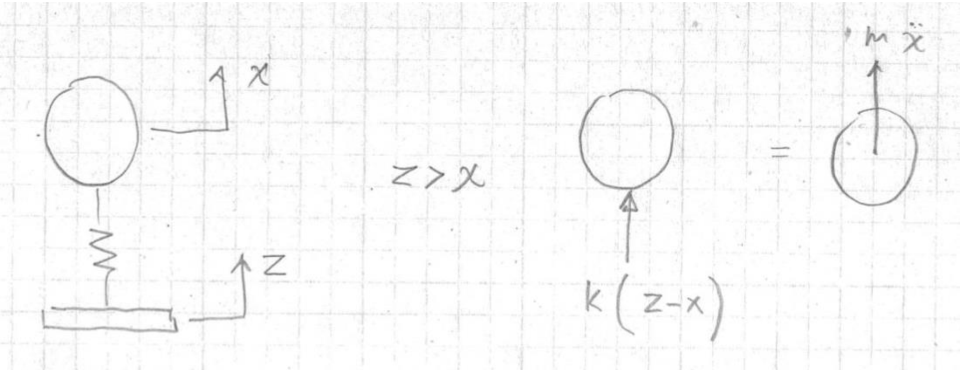
- Initial Condition:

$$\begin{cases} x(t=0) = x_0 = d \\ \dot{x}(t=0) = \dot{x}_0 = v \end{cases}$$

$$\ddot{x} + \frac{k}{m}x = \frac{f(t)}{m} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + \omega^2 x = \begin{cases} \dfrac{f(t)}{m} \\ 0 \end{cases}$$

Lumped Parameter Model (Mass Spring)



$$+\uparrow \sum F = k(z-x) = m\ddot{x}$$

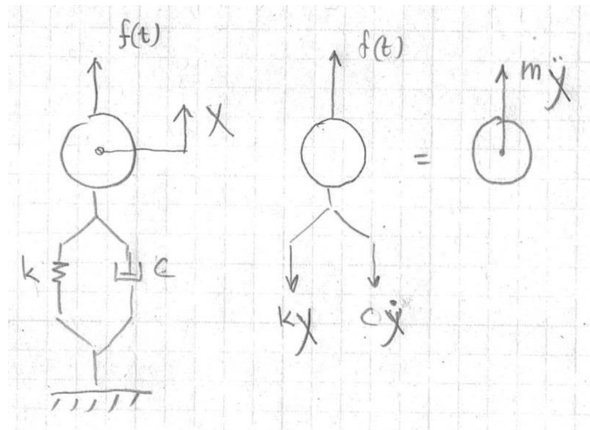
$$m\ddot{x} + kx = kz$$

- Initial Condition: $$\begin{cases} x(t=0) = x_0 = d \\ \dot{x}(t=0) = \dot{x}_0 = v \end{cases}$$

$$\ddot{x} + \frac{k}{m}x = \frac{k}{m}z \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + \omega^2 x = \begin{cases} \omega^2 z \\ \\ 0 \end{cases}$$

Lumped Parameter Model (Mass Spring)



$$+\uparrow\sum F = f(t) - kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\omega_n^2 = \frac{k}{m} \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

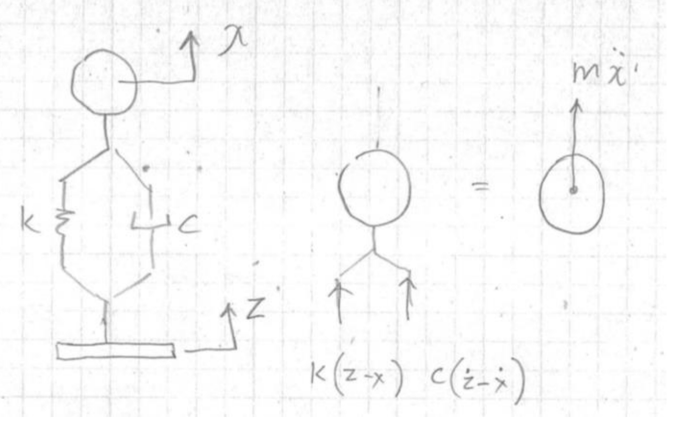
$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\frac{\sqrt{k}}{\sqrt{m}}} = \frac{c}{2\sqrt{mk}}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \begin{cases} \dfrac{f(t)}{m} \\ 0 \end{cases}$$

ζ - Viscous damping factor (non-dimensional)

ω_n - Natural frequency

Lumped Parameter Model (Mass Spring)



$$+\uparrow\sum F = k(z-x) - c(\dot{z}-\dot{x}) = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{z} + kz$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{c}{m}\dot{z} + \frac{k}{m}z$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{z} + \omega_n^2 z$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \begin{cases} 2\zeta\omega_n\dot{z} + \omega_n^2 z \\ \\ 0 \end{cases}$$

Second Order Homogeneous Diff. Eq.

Theorems

Theorem 1 (3.2.1)
Existence and Uniqueness

- **Consider** the initial value problem

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\; y'(t_0) = y'_0$$

- **where** p, q, and g are continuous on an open interval I that contains t_0.
- **Then** there exists a unique solution $y = \phi(t)$ on I.

- Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.

Theorem 2 (3.2.2)
Principle of Superposition

- **If** y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

- **Then** the linear combination $c_1 y_1 + y_2 c_2$ is also a solution, for all constants c_1 and c_2.

The Wronskian Determinant (1/3)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0$$

- From Theorem 3.2.2, we know that $y = c_1 y_1 + c_2 y_2$ is a solution to this equation.

- Next, find coefficients such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

- To do so, we need to solve the following equations:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

The Wronskian Determinant (2/3)

- Solving the equations, we obtain

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

- In terms of determinants:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Wronskian Determinant (3/3)

- In order for these formulas to be valid, the determinant *W* in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

- *W* is called the **Wronskian determinant**, or more simply, the Wronskian of the solutions $y_1$and y_2. We will sometimes use the notation

$$W\left(y_1, y_2\right)\left(t_0\right)$$

Theorem 3 (3.2.3)
Constants Coefficient c_1, c_2

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

with the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

Then it is always possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and only if the Wronskian

$$W = y_1 y'_2 - y'_1 y_2$$

is not zero at the point t_0

Theorem 4 (3.2.4)
General Solution / Fundamental Set of Solutions

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0.$$
Then the family of solutions
$$y = c_1 y_1 + c_2 y_2$$
with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if an only if there is a point t_0 such that $W(y_1,y_2)(t_0) \neq 0$, .

- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Theorem 5 (3.2.5)
Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients *p* and *q* are continuous on some open interval *I*:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

- Let t_0 be a point in *I*, and y_1 and y_2 solutions of the equation with y_1 satisfying initial conditions

$$y_1(t_0) = 1,\ y_1'(t_0) = 0$$

  and y_2 satisfying initial conditions

$$y_2(t_0) = 0,\ y_2'(t_0) = 1$$

- Then y_1, y_2 form a fundamental set of solutions to the given differential equation.

Theorem 6 (3.2.6)
Real & Imaginary Parts – Solution

Consider again the equation (2):

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

where *p* and *q* are continuous real-valued functions.

If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2),

Then its real part *u* and its imaginary part *v* are also solutions of this equation.

$$y = C_1 u(t) + C_2 v(t)$$

Theorem 7 (3.2.7)
Abel's Theorem

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$
- Where p and q are continuous on some open interval I.
- Then the $W(y_1,y_2)(t)$ is given by
$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}$$
where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W(y_1,y_2)(t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Summary

- To find a general solution of the differential equation
$$y'' + p(t)\,y' + q(t)\,y = 0, \;\; \alpha < t < \beta$$
we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W(y_1, y_2)(t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.

Homogeneous Second Order Linear Differential Equations – Constant Coefficient

$$ay'' + by' + cy = 0$$

- Assume: $y = e^{rt}$

sub: $$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$(ar^2 + br + c)e^{rt} = 0$$

since $e^{rt} \neq 0$

Characteristic Equation (Quadratic Equation):

$$ar^2 + br + c = 0$$

Solution for r:

1) Real and different
2) Real and repeated
3) Complex conjugates: $\begin{cases} r_1 = \mathrm{Re} + \mathrm{Im}\ j \\ r_2 = \mathrm{Re} - \mathrm{Im}\ j \end{cases}$

Second Order Homogeneous Diff. Eq.

Mathematical Approach

Homogeneous Equations With Constant Coefficients

Case 1 – Roots – Real & Different

- Case 1: Real and different roots

$$ay'' + by' + cy = 0$$

Assume $y = e^{rt}$

$$(ar^2 + br + c)e^{rt} = 0$$

$$ar^2 + br + c = 0$$

If $b^2 - 4ac > 0$ then $r_1 \neq r_2$ and $r_1 \in \Re, r_2 \in \Re$

$$\begin{cases} y_1 = e^{r_1 t} \\ y_2 = e^{r_2 t} \end{cases}$$

Solution has the form

$$y = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$y(t_0) = y_0$$

$$y'(t_0) = y'_0$$

Homogeneous Equations With Constant Coefficients

Case 1 – Roots – Real & Different

- Substituting $t = t_0; \quad y = y_0$

$$\to \quad y_0 = C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0}$$

- Substituting $t = t_0; \quad y' = y_0'$

$$\to \quad y_0' = C_1 r_1 e^{r_1 t_0} + C_2 r_2 e^{r_2 t_0}$$

- Solve for C_1 and C_2

$$C_1 = \frac{y_0' + y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}; \quad C_2 = \frac{y_0 r_1 + y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

- Notes:

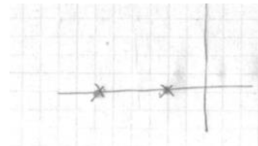
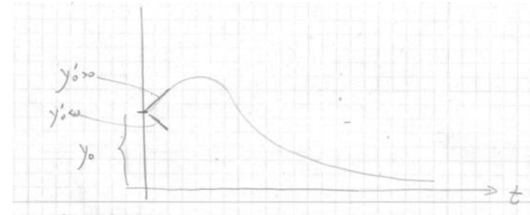
$r_1 \neq r_2 \;\Rightarrow\; r_1 - r_2 \neq 0 \;\Rightarrow\; C_1, C_2$ exist

One possible choice of C_1 and C_2 for some initial conditions

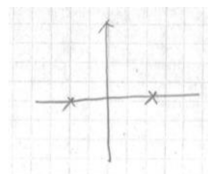
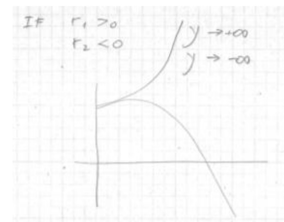
Homogeneous Equations With Constant Coefficients

Case 1 – Roots – Real & Different

- Case 1a $r_1 < 0$ $r_2 < 0$

- Case 1b $r_1 > 0$ $r_2 < 0$

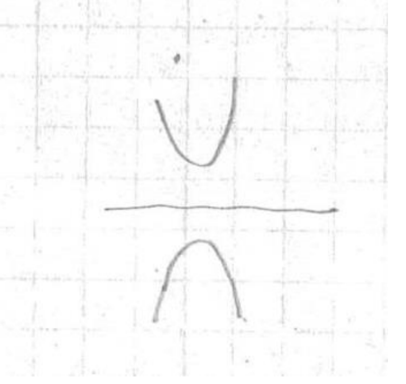
- Case 1c $r_1 = r_2 = 0$

- Case 1d $r_1 > 0$ $r_2 > 0$

Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Homogeneous Equation with Constant Coefficients
- Case 3: Complex conjugate roots



$$ay'' + by' + cy = 0$$

$$y = e^{rt}$$

$$(ar^2 + br + c)e^{rt} = 0$$

$$ar^2 + br + c = 0$$

$$b^2 - 4ac < 0 \quad \rightarrow \quad \begin{cases} r_1 = \lambda + i\mu \\ r_2 = \lambda - i\mu \end{cases}$$

$$\begin{cases} y_1 = e^{(\lambda + i\mu)t} \\ y_2 = e^{(\lambda - i\mu)t} \end{cases}$$

Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Euler's Formula
  - Taylor's series for e^t about t=0

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \qquad -\infty < t < \infty$$

  - Substitute i t for t

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2n!}}_{\substack{\text{Real} \\ \cos(t)}} + i \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}}_{\substack{\text{Imaginary} \\ \sin(t)}}$$

$$\begin{cases} i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \\ i^5 = i \end{cases} \Rightarrow \begin{cases} i^{2n} = (-1)^n \\ i^{2n+1} = (-i)^n \end{cases}$$

Homogeneous Equations With Constant Coefficients

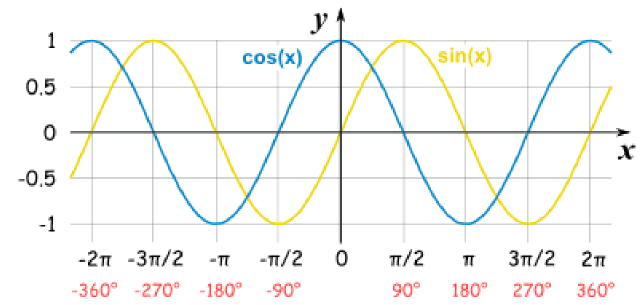
Case 3 – Roots – Complex Conjugates

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

$$\cos(-t) = \cos(t)$$

$$\sin(-t) = -\sin(t)$$



$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)$$

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cdot e^{i\mu t} = e^{\lambda t}(\cos \mu t + i\sin \mu t)$$

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{(\lambda+i\mu)t} + C_2 e^{(\lambda-i\mu)t} =$$

$$e^{\lambda t}(C_1(\cos \mu t + i\sin \mu t) + C_2(\cos \mu t - i\sin \mu t)) =$$

$$e^{\lambda t}((C_1 + C_2)(\cos \mu t) + i(C_1 + C_2)(\sin \mu t))$$

- Or based on the theorem 3.2.6 p 153, both real and imaginary parts are solutions

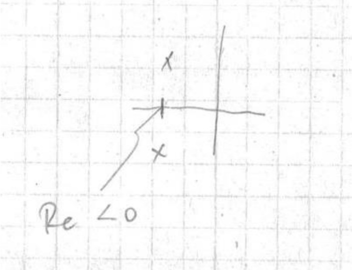
$$y = e^{\lambda t}(\tilde{C}_1 \cos \mu t + \tilde{C}_2 \sin \mu t)$$

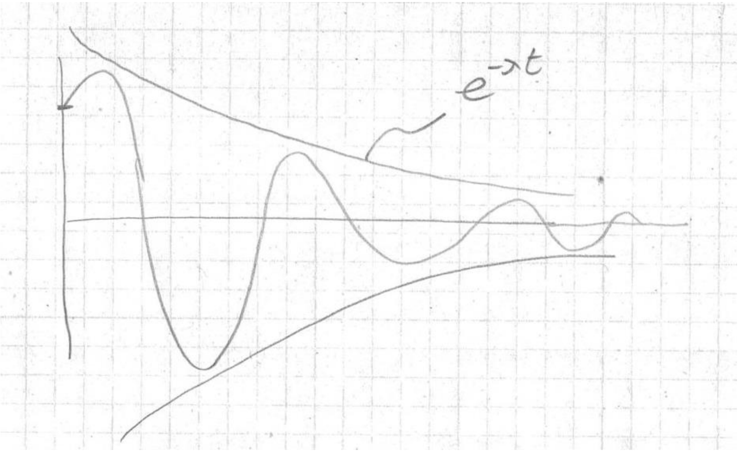
Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Case 3a: For λ<0

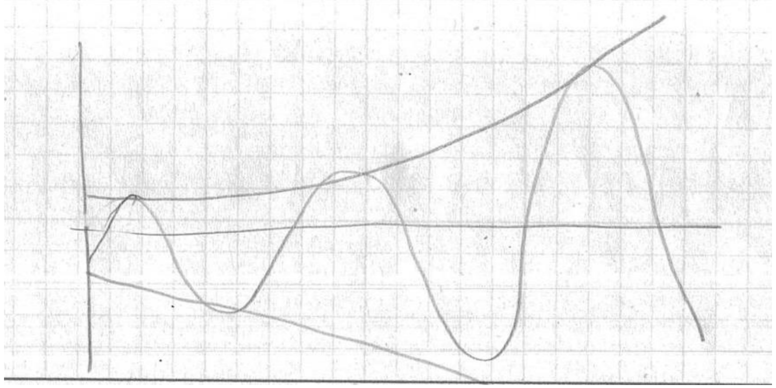
$$y = e^{-\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$$





- Case 3b: For λ>0

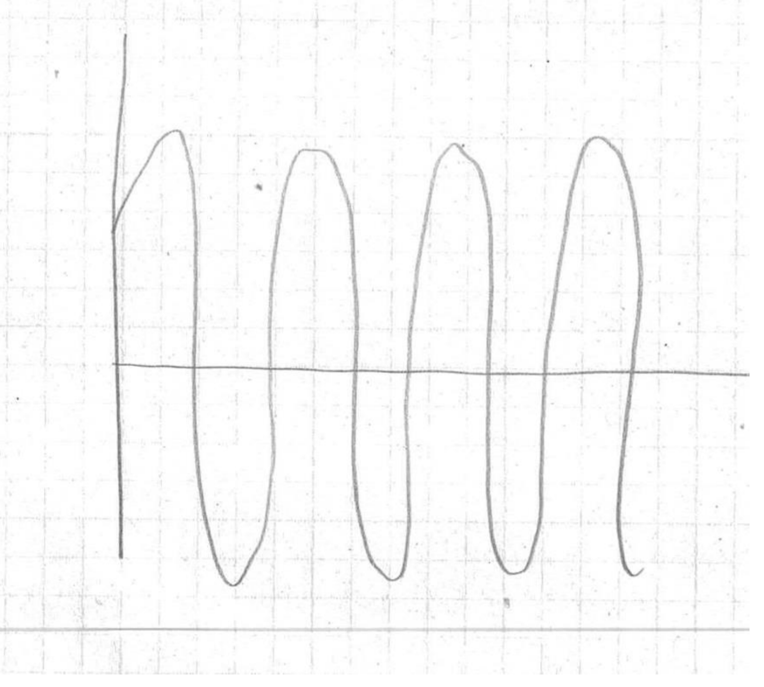
$$y = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$$



Homogeneous Equations With Constant Coefficients

Case 3 – Roots – Complex Conjugates

- Case 3c: For $\lambda=0$



Case 2 – Roots – Repeated

$$ay'' + by' + cy = 0$$

$$y = e^{rt}$$

$$(ar^2 + br + c)e^{rt} = 0$$

$$ar^2 + br + c = 0$$

$$b^2 - 4ac = 0 \quad \rightarrow \quad r_1 = r_2 = -\frac{b}{2a}$$

$$y_1 = e^{-\frac{b}{2a}t}$$

- To find a second solution, we assume that

$$y = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t}$$

Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

- Substitute for y in differential equation $ay'' + by' + cy = 0$

$$y = v(t)y_1(t) = v(t)e^{-\frac{b}{2a}t}$$

$$y' = v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v(t)e^{-\frac{b}{2a}t}$$

$$y'' = v''(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} - \frac{b}{2a}v'(t)e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}v(t)e^{-\frac{b}{2a}t}$$

$$\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + c\,v(t)\right\}e^{-\frac{b}{2a}t} = 0$$

$$a\,v''(t) + \underbrace{(-b+b)}_{0}v'(t) + \underbrace{\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)}_{0}v(t) = 0$$

$$\frac{b^2}{4a} - \frac{b^2}{2a} + c = -\frac{b^2}{4a} + c = \frac{\overbrace{-b^2 + 4ac}^{0}}{4a} = 0$$

$$v''(t) = 0$$

$$v(t) = C_1 + C_2 t$$

$$y = v(t)y_1(t) = C_1 y_1(t) + C_2 t y_1(t)$$

- The solution for the differential equation

$$y = C_1 e^{-\frac{b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}$$

Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

$$y_1(t) = e^{-\frac{b}{2a}t}; \qquad y_2(t) = te^{-\frac{b}{2a}t}$$

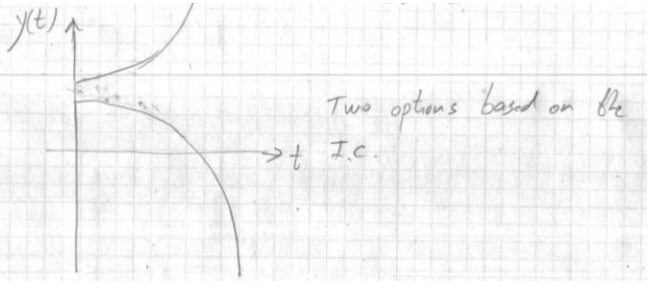
$$W(y_1, y_2) = \begin{vmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a}e^{-\frac{b}{2a}t} & \left(1 - \frac{b}{2a}t\right)e^{-\frac{b}{2a}t} \end{vmatrix} = e^{-\frac{b}{a}t}$$

$$W = e^{-\frac{b}{a}t} \neq 0 \quad \Rightarrow \quad \left.\begin{matrix} y_1 \\ y_2 \end{matrix}\right\}$$ Fundamental set of solutions

Homogeneous Equations With Constant Coefficients

Case 2 – Roots – Repeated

- Case 2a $r_1 = r_2 > 0$



- Cas2 2b $r_1 = r_2 < 0$

Case 2 – Roots – Repeated - Reduction of Order - Generalization

$$y'' + p(t)y' + q(t)y = 0$$

- We know one solution $y_1(t)$ (not everywhere zero)

  To find the second solution let

$$\begin{cases} y = v(t)y_1(t) \\ y' = v'(t)y_1(t) + v(t)y_1'(t) \\ y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{cases}$$

Substitute for y, y', y'' in the differential equation

$$y_1 v'' + \left(2y_1' + py_1\right)v' + \underbrace{\left(y_1'' + py_1' + qy_1\right)}_{y_1 \text{ is a solution} \Rightarrow 0} v = 0$$

Case 2 – Roots – Repeated - Reduction of Order - Generalization

$$y_1 v'' + (2y_1' + p y_1) v' = 0$$

$$u = v'$$

$$y_1 u' + (2y_1' + p y_1) u = 0$$

- First order differential equation

  Find a solution for u

$$v = \int u$$

See example 3, p172

Homogeneous Equations With Constant Coefficients

Summary

- Differential Equation $ay'' + by' + cy = 0$
- Characteristic Equation $ar^2 + br + c = 0$
- Case 1 – Roots – Real and different $r_1 \neq r_2$

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Case 3 – Roots – Complex conjugates $\lambda \pm i\mu$

$$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$$

$$y = e^{\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t)$$

- Case 2 – Roots – Real and repeated $r_1 = r_2$

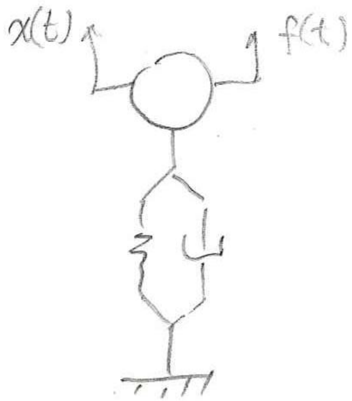
$$y = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$$

Second Order Homogeneous Diff. Eq.

Engineering Approach (Free Response)

Undamped & Damped

SECOND ORDER SYSTEM (FREE RESPONCE)



$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad \begin{cases} x(t_0) = x_0 \\ \dot{x}(t_0) = \dot{x}_0 = v_0 \end{cases}$$

$f(t) = 0 \rightarrow$ free responce

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\omega_n = \sqrt{\frac{k}{m}} \rightarrow$ Natural frequancey

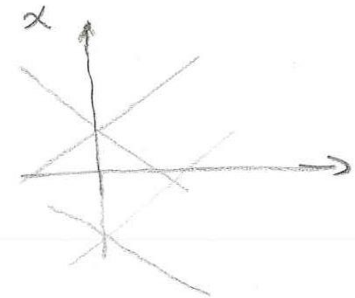
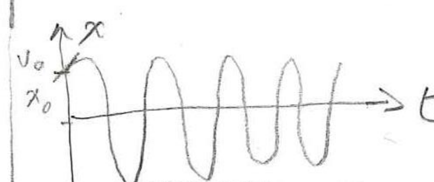
$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{\frac{k}{m}}} = \frac{c}{2\sqrt{mk}}$$

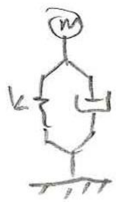
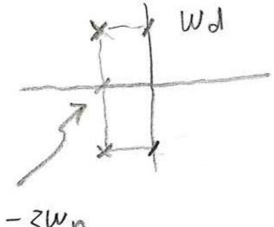
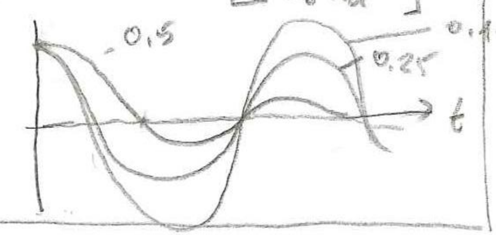
Assume Solution $\quad x(t) = Ce^{\lambda t}$

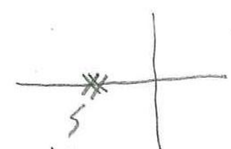
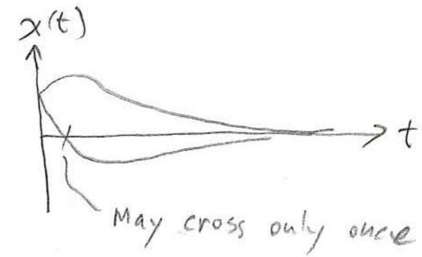
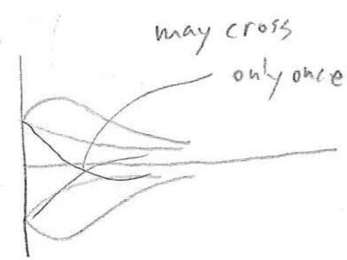
characteristic Eq $\quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2} = \frac{-2\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)}{2} =$$

$$= \omega_n\left(\zeta \pm i\sqrt{1 - \zeta^2}\right)$$

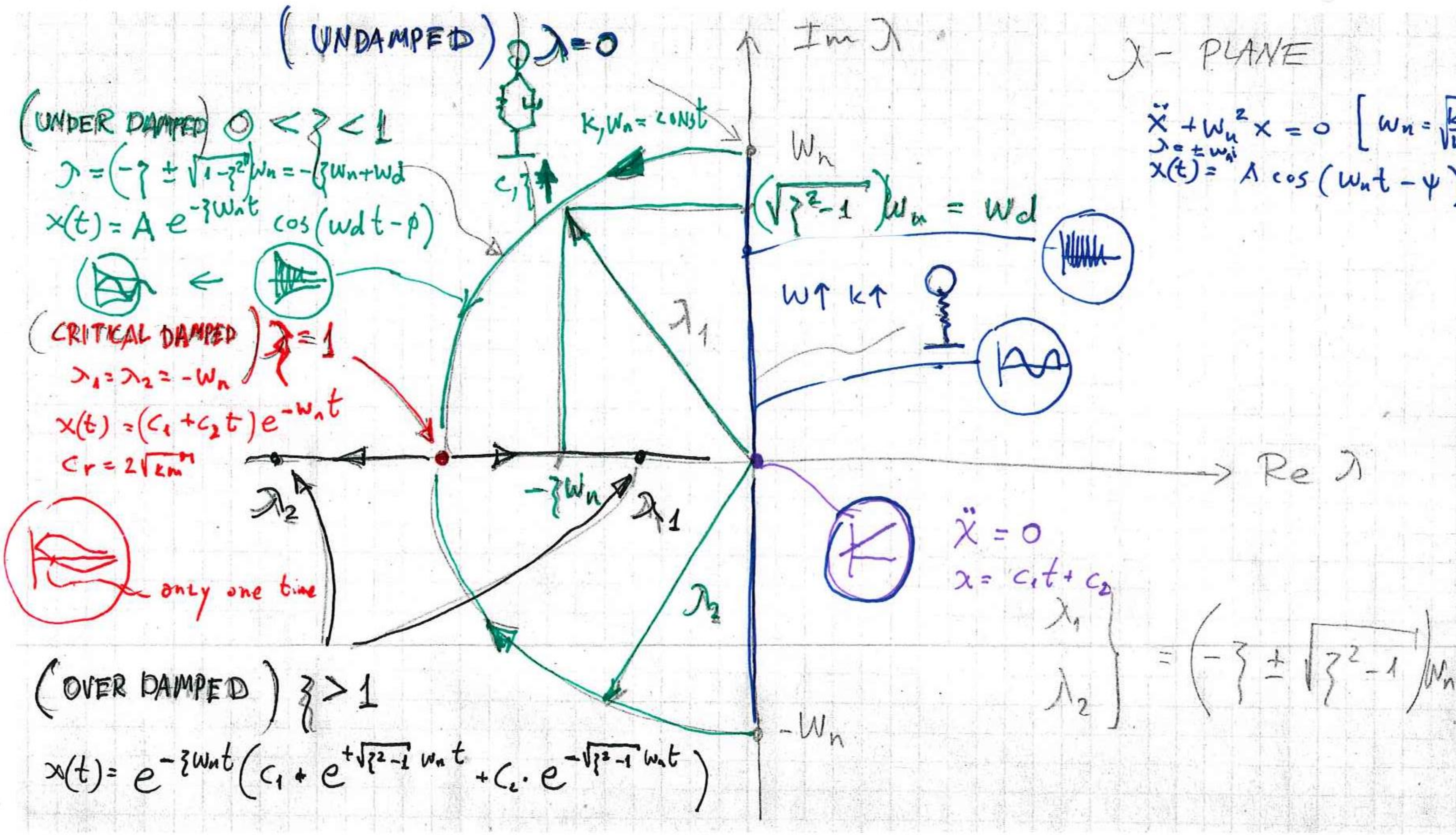
MODEL	PARAMETERS	Diff Eq.	CHARACTERISTIC EQ.	Solution
	m $k = c = 0$	$m\ddot{x} = 0$ $\ddot{x} = 0$	$\lambda^2 = 0$ $\lambda_1 = \lambda_2 = 0$ 	$x = c_1 t + c_2$ 
	m, k $c = 0$	$m\ddot{x} + kx = 0$ $\ddot{x} + w_n x = 0$ $w_n = \sqrt{\frac{k}{m}}$	$\lambda^2 + w^2 = 0$ $\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \pm w_n i$ 	$x = A\cos(w_n t - \psi)$ $x = B_1 \sin(w_n t) + B_2 \cos(w_n t)$ $\left\{\begin{matrix} A = \sqrt{(x_0)^2 + \left(\frac{v_0}{w_n}\right)^2} \\ \psi = \tan^{-1}\left(\frac{v_0}{w_n x_0}\right)\end{matrix}\right.$ 

MODEL	PARAMETERS	Diff Eq.	CHARACTERISTIC EQ.	SOLUTION
	m, k, c	$m\ddot{x} + c\dot{x} + kx = 0$ $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ $\omega_n = \sqrt{\frac{k}{m}}$; $\zeta = \frac{c}{2\sqrt{mk}}$	$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ $\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \begin{cases}\omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right)\\ \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\end{cases}$	
			$0 < \zeta < 1$ Under damped (CASE 3) $\lambda_{1,2} = -\zeta\omega_n + i\omega_d$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$ 	$x(t) = e^{-\zeta\omega_n t}\left(A_1 \cos\omega_d t + A_2 \sin\omega_d t\right)$ $\begin{cases}A_1 = x_0\\ A_2 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}\end{cases}$ $x(t) = A e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$ $\begin{cases}A = \frac{x_0}{\cos(\phi)}\\ \phi = \tan^{-1}\left[\frac{v_0 - x_0\zeta\omega_n}{x_0\omega_d}\right]\end{cases}$ 

MODEL	PARAMETERS	DIFF EQ.	CHARACTERISTIC EQ	SOLUTION
			$\zeta = 1$ Critical Damping (CASE 2) $\lambda_1 = \lambda_2 = -\omega_n$ $\zeta = \frac{c}{2m\omega_n} = 1$ $c_{cr} = 2\sqrt{km}$  	$x(t) = e^{-\omega_n t}\left[c_1 + c_2 t\right]$ $c_1 = x_0(1 - \omega_n t)$ $c_2 = \dot{x}_0 = v_0$ 
			$\zeta > 1$ Overdamped  	$x(t) = e^{-\zeta\omega_n t}\left(c_1 e^{+\sqrt{\zeta^2-1}\,\omega_n t}\right.$ $\left.c_2 e^{-\sqrt{\zeta^2-1}\,\omega_n t}\right.$ $= C_1 = \frac{x_0\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n - \dot{x}_0}{-\left(2\sqrt{\zeta^2-1}\right)\omega_n}$ $C_2 = \frac{\dot{x}_0 - x_0\left(-\zeta + \sqrt{\zeta^2+1}\right)\omega_n}{-\left(2\sqrt{\zeta^2-1}\right)\omega_n}$

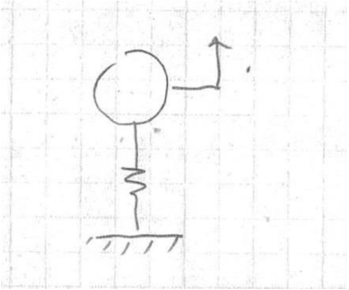
Free Response of Damped Second Order System –

Introduction – Cases – The "S" Plane



Free Response of Underdamped Second Order System

Harmonic Oscillator



Free response $\rightarrow f(t) = 0$

$$\ddot{x} + \omega_n^2 x = 0$$

- The general solution can be written in the form

$$x(t) = Ae^{\lambda t}$$

- Introducing the solution into the equation

$$A\lambda^2 e^{\lambda t} + \omega^2 A e^{\lambda t} = 0$$

$$\underbrace{\left(\lambda^2 + \omega^2\right)}_{=0}\underbrace{Ae^{\lambda t}}_{\neq 0} = 0$$

Free Response of Underdamped Second Order System

Harmonic Oscillator

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow \quad \lambda^2 = -\omega^2$$

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \pm\omega_n i$$

- In general the solution becomes

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

$$\left.\begin{matrix}A_1 \\ A_2\end{matrix}\right\} \text{constants of integration}$$

- Since x(t) must be real $\rightarrow \left.\begin{matrix}A_1 \\ A_2\end{matrix}\right\}$ complex conjugates

$$\left.\begin{matrix}A_1 = \dfrac{1}{2} A e^{-i\psi} \\ A_2 = \dfrac{1}{2} A e^{i\psi}\end{matrix}\right\} \text{where } A \text{ \& } \psi \text{ are real}$$

Free Response of Underdamped Second Order System

Harmonic Oscillator

$$x(t) = \frac{1}{2}A\left[e^{i(w_n t-\psi)} + e^{-i(w_n t-\psi)}\right]$$

$$x(t) = \frac{1}{2}A\left[\cos(w_n - \psi) + i\sin(\cancel{w_n t - \psi}) + \underbrace{\cos(-(w_n t - \psi))}_{=\cos(w_n t - \psi)} + \underbrace{i\sin(-(w_n t - \psi))}_{=-i\sin(\cancel{w_n t - \psi})}\right]$$

$$x(t) = A\cos(w_n t - \psi)$$

Based on

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$x(t) = A\left[\cos w_n t\cos\psi + \sin w_n t\sin\psi\right]$$

Free Response of Underdamped Second Order System

Harmonic Oscillator

$$x(t) = A\left[\cos w_n t \cos\psi + \sin w_n t \sin\psi\right]$$

The solution can also be expressed as

$$x(t) = B_1 \sin w_n t + B_2 \cos w_n t$$

$$\left.\begin{array}{l} B1 \\ B2 \end{array}\right\} \text{constants of integration}$$

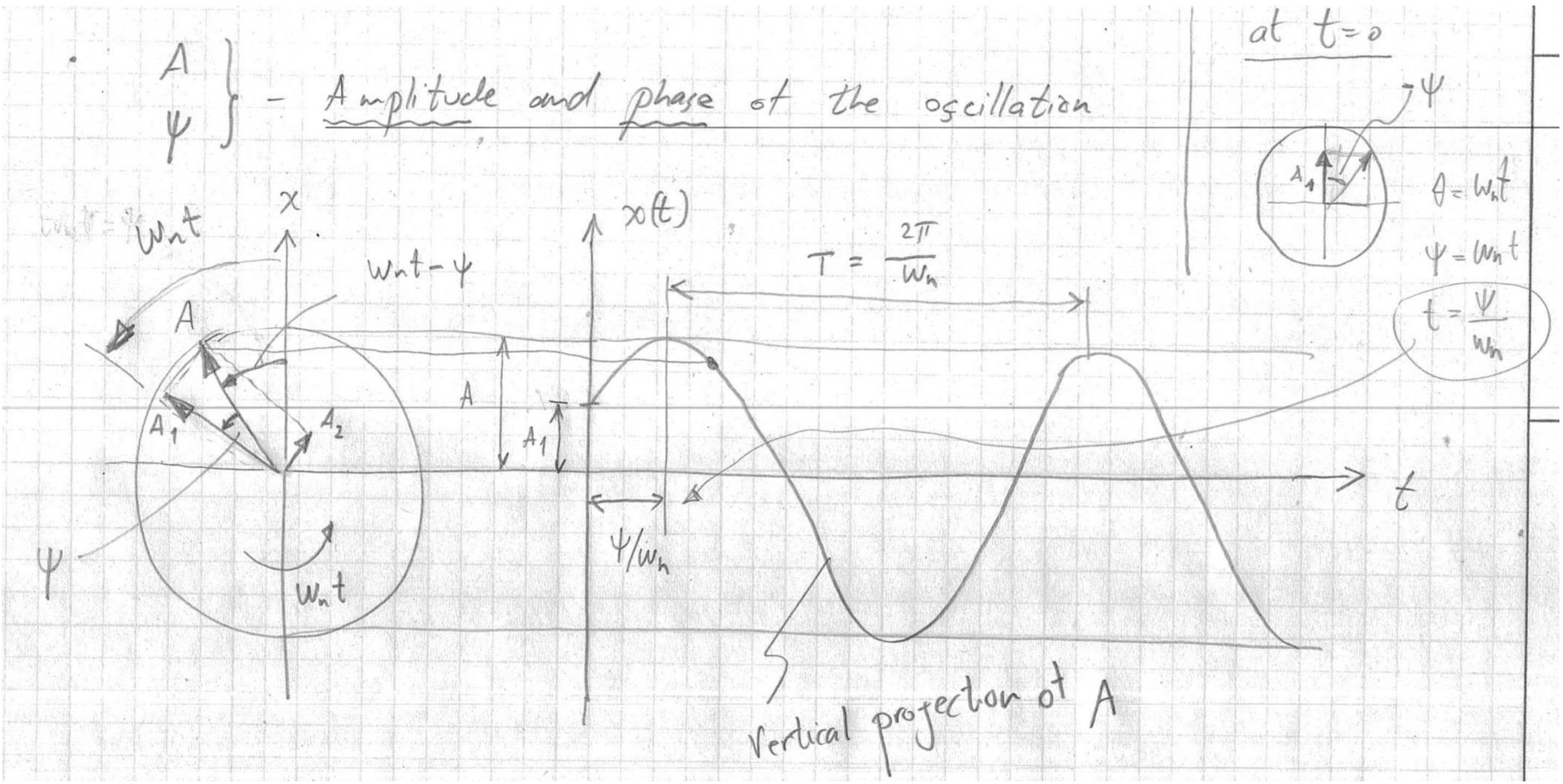
$$x(t) = \overbrace{A\cos\psi}^{B_2} \cos w_n t + \overbrace{A\sin\psi}^{B_1} \sin w_n t$$

- sin / cos - harmonic functions
- Solution – simple harmonic oscillation
- System powered by this type of diff. eq. are called <u>harmonic oscillators</u>

Free Response of Underdamped Second Order System

Harmonic Oscillator

$$x(t) = A\cos(w_n t - \psi)$$



Free Response of Underdamped Second Order System

Harmonic Oscillator

- T - Time period (of oscillation) [sec]
- w_n - Natural frequency [RAD/sec]
- f_n - Natural frequency [Hz]

$$w_n = 2\pi\ f \rightarrow f_n = \frac{w_n}{2\pi}$$

$$T = \frac{1}{f_n} = \frac{2\pi}{w_n}$$

Free Response of Underdamped Second Order System

Harmonic Oscillator

- For spring-mass system

$$T = 2\pi\sqrt{\frac{m}{k}} \ ; \ w_n = \sqrt{\frac{k}{m}} \quad k\uparrow \ \rightarrow \ w_n \uparrow$$

- For pendulum

$$T = 2\pi\sqrt{\frac{L}{g}} \ ; \ w_n = \sqrt{\frac{g}{L}} \quad L\uparrow \ \rightarrow \ w_n \downarrow$$

- Note – that T or w_n are not function of m

Free Response of Underdamped Second Order System

Harmonic Oscillator – Concluding Remarks

Notes

- No matter how the motion is initiated, free oscillation always occurs at the freq. w_n
- $w_n = f(k, m) \quad w_n = f(g, L)$
- w_n is independent of external forces; that is the reason why w_n is called natural frequency.
- Mathematical idealization → perpetuate → *Ad intitum*
- Every real system possesses some measure of damping
- Pendulum damping
  - Air resistance
  - Friction point of support
- When damping is small – harmonic oscillation
- For short period of time t << T small damping does not have any effect over that interval

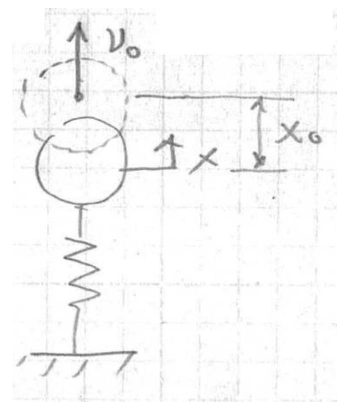
Free Response of Underdamped Second Order System

Harmonic Oscillator – Initial Conditions

IC to determine A, ψ (two conditions)

$$x(0) = x_0 \ (+/-)$$

$$\dot{x}(0) = v_0 \ (+/-)$$



$$x(t) = A\cos(w_n t - \psi)$$

$$\begin{cases} x(0) = A\cos\psi = x_0 & \rightarrow \ \cos\psi = \dfrac{x_0}{A} \\ \dot{x}(0) = w_n A\sin\psi = v_0 & \rightarrow \ \sin\psi = \dfrac{v_0}{w_n A} \end{cases}$$

Free Response of Underdamped Second Order System

Harmonic Oscillator – Initial Conditions

$$\begin{cases} x(0) = A\cos\psi = x_0 & \rightarrow \cos\psi = \dfrac{x_0}{A} \\ \dot{x}(0) = w_n A\sin\psi = v_0 & \rightarrow \sin\psi = \dfrac{v_0}{w_n A} \end{cases}$$

Determine A

$$1 = \sqrt{(\sin\psi)^2 + (\cos\psi)^2} = \sqrt{\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{w_n A}\right)^2}$$

$$A = \sqrt{\left(x_0\right)^2 + \left(\frac{v_0}{w_n}\right)^2}$$

Free Response of Underdamped Second Order System

Harmonic Oscillator – Initial Conditions

Determine ψ

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{v_0}{w_n A}\frac{A}{x_0}$$

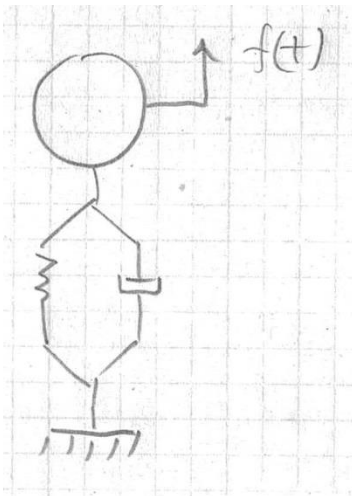
$$\psi = \tan^{-1}\left(\frac{v_0}{w_n x_0}\right)$$

Final Solution

$$x(t) = B_1 \sin(w_n t) + B_2 \cos(w_n t) \quad \begin{cases} B_1 = A\sin\psi \\ B_2 = A\cos\psi \end{cases}$$

$$x(t) = x_0 \cos(w_n t) + \frac{v_0}{w_n}\sin(w_n t)$$

Free Response of Damped Second Order System –
Introduction



Free response $\rightarrow \quad f(t) = 0$

$$\ddot{x}(t) + 2\zeta w_n \dot{x}(t) + w_n^2 x(t) = 0$$

$$w_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2mw_n}$$

$w_n -$ natural frequency of the system

$\zeta \;-$ viscous damping factor

Assume solution $\quad x(t) = ce^{\lambda t}$

characteristic Eq. $\quad \lambda^2 + 2\zeta w_n \lambda + w_n^2 = 0$

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \frac{-2\zeta w_n \pm \sqrt{4\zeta^2 w_n^2 - 4w_n^2}}{2} = \frac{-2w_n(\zeta \pm \sqrt{\zeta^2 - 1})}{2} = \begin{cases} \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) w_n \\ \left(-\zeta \pm i\sqrt{1 - \zeta}\right) w_n \end{cases}$$

Free Response of Damped Second Order System –

Introduction – Cases

CASE 1
(over damped) $\zeta > 1 \rightarrow$ real, negative, distinct

CASE 2
(critical damped) $\zeta = 1 \rightarrow$ real, negative, equal

$$\lambda_1 = \lambda_2 = -w_n$$

CASE 3
(under damped) $0 < \zeta < 1 \rightarrow$ complex, conjugates, negative real

$$\left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)w_n = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)w_n$$

Free Response of Damped Second Order System –

Case 1 – Overdamped

For $\zeta > 1$ (Case 1) - overdamped

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n t} + c_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n t}$$

$$= e^{-\zeta w_n t} \left(c_1 e^{+\sqrt{\zeta^2 - 1} w_n t} + c_2 e^{-\sqrt{\zeta^2 - 1} w_n t} \right)$$

since $\zeta > 1 \rightarrow \zeta > \sqrt{\zeta^2 - 1}$

the response x(t) decays exponentially with time

- Aperiodic motion – approaches zero without oscillation

Applying initial condition $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases}$

Free Response of Damped Second Order System –

Case 1 – Overdamped

$$x(t=0) = c_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)w_n 0} + c_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)w_n 0} = c_1 + c_2 = x_0$$

$$\dot{x}(t=0) = \left(-\zeta+\sqrt{\zeta^2-1}\right)w_n c_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)w_n 0} + \left(-\zeta-\sqrt{\zeta^2-1}\right)w_n c_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)w_n 0} = \dot{x}_0$$

$$\begin{cases} c_1 + c_2 = x_0 \\ \left(-\zeta+\sqrt{\zeta^2-1}\right)w_n c_1 + \left(-\zeta-\sqrt{\zeta^2-1}\right)w_n c_2 = \dot{x}_0 \end{cases}$$

Free Response of Damped Second Order System –
Case 1 – Overdamped

$$
\begin{cases}
c_1 + c_2 = x_0 \\
\left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n c_1 + \left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n c_2 = \dot{x}_0
\end{cases}
$$

$$
\Delta = \begin{vmatrix} 1 & 1 \\ \left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n & \left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n \end{vmatrix} = \left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n + \left(+\zeta - \sqrt{\zeta^2 - 1}\right) w_n
$$

$$
= \left(-2\sqrt{\zeta^2 - 1}\right) w_n
$$

$$
c_1 = \frac{\begin{vmatrix} x_0 & 1 \\ \dot{x}_0 & \left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n \end{vmatrix}}{\Delta} = \frac{x_0 \left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n - \dot{x}_0}{-\left(2\sqrt{\zeta^2 - 1}\right) w_n}
$$

$$
c_2 = \frac{\begin{vmatrix} 1 & x_0 \\ \left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n & \dot{x}_0 \end{vmatrix}}{\Delta} = \frac{\dot{x}_0 - x_0 \left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n}{-\left(2\sqrt{\zeta^2 - 1}\right) w_n}
$$

Free Response of Damped Second Order System –

Case 1 – Overdamped

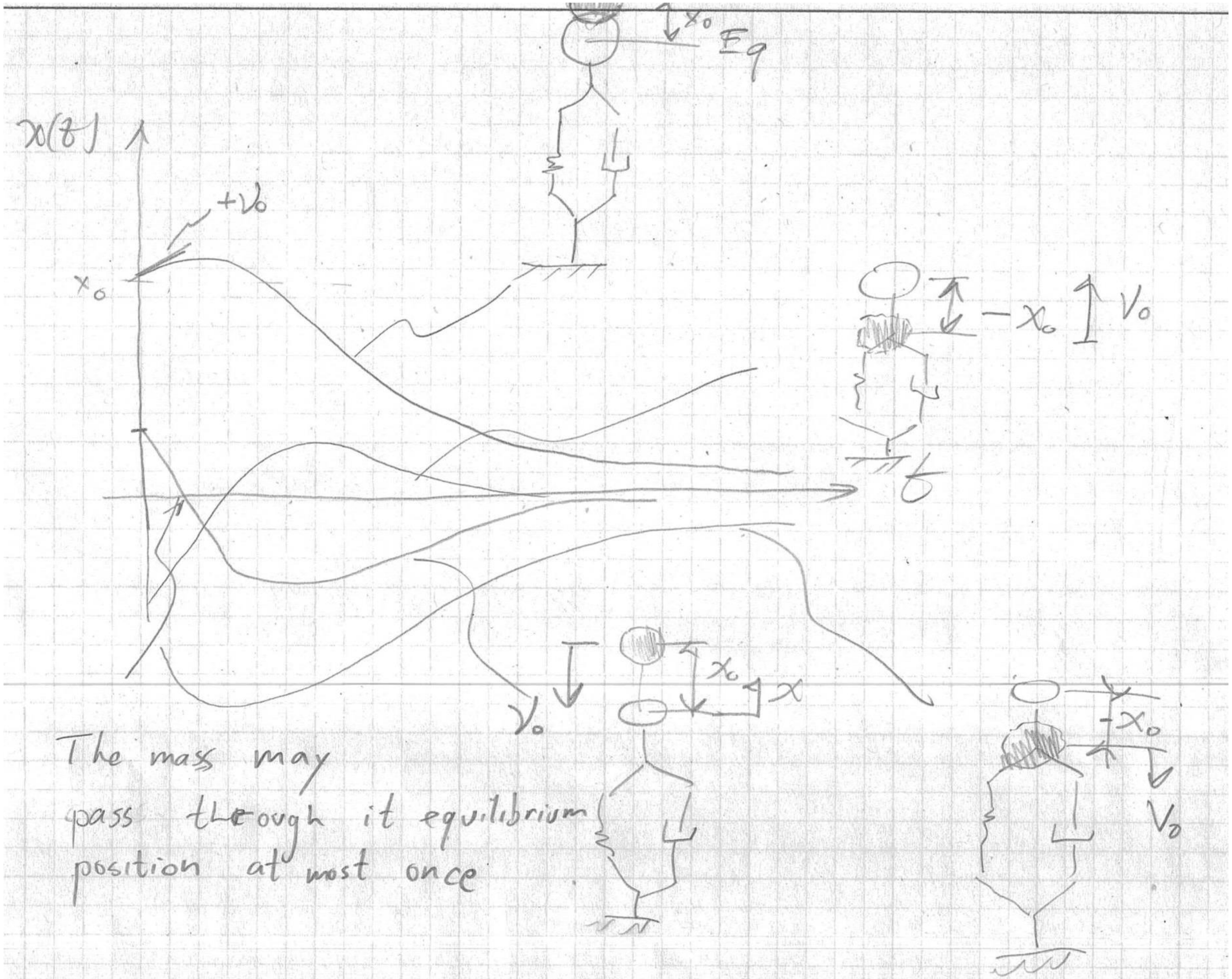
$$w_d = w_n\sqrt{1-\zeta^2}$$

$$c_1 = \frac{-x_0\zeta w_d - x_0 w_d - \dot{x}_0}{-2w_d} = \frac{\dot{x}_0}{2w_d} + \frac{x_0\left(\zeta w_n + w_d\right)}{2w_d}$$

$$c_2 = \frac{\dot{x}_0 + x_0\zeta w_n - x_0 w_d}{-2w_d} = -\frac{\dot{x}_0}{2w_d} + \frac{x_0\left(\zeta w_n - w_d\right)}{2w_d}$$

Free Response of Damped Second Order System –

Case 1 – Overdamped



Free Response of Damped Second Order System –

Case 2 – Critical Damping

For $\zeta = 1$ (Case 1) critical damped

$$\lambda_1 = \lambda_2 = -w_n$$

$$y(t) = (c_1 + c_2 t)e^{-w_n t}$$

For $\zeta = 1$ $\zeta = \dfrac{c}{2mw_n} = 1 \rightarrow c = 2mw_n = 2m\sqrt{\dfrac{k}{m}}$

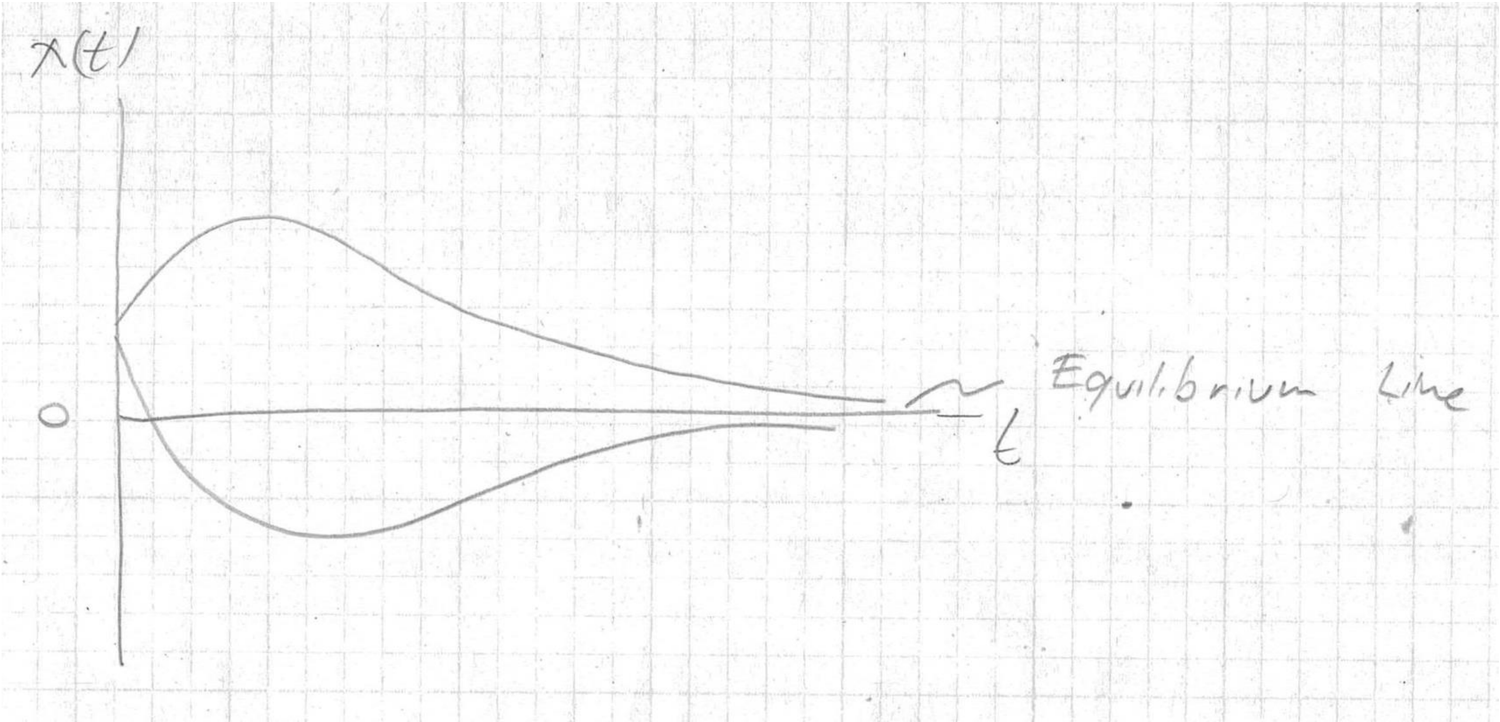
$$c_{cr} = 2\sqrt{km}$$

Applying initial condition $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases}$

$$y(t) = e^{-w_n t}\left[\underbrace{\boxed{x_0(1+w_n t)}}_{c_1} + \underbrace{\boxed{\dot{x}_0}}_{c_2} t\right]$$

Free Response of Damped Second Order System –

Case 2 – Critical Damping



Free Response of Damped Second Order System –

Case 3 – Underdamped

For $0 < \zeta < 1$ (Case 3) underdamped

$$\lambda = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)w_n \quad \begin{cases} -\zeta w_n + i w_d \\ -\zeta w_n - i w_d \end{cases}$$

$$w_d = w_n\sqrt{1-\zeta^2}$$ damped frequency

$$w_d < w_n$$

$$x(t) = e^{-\zeta w_n t}\left(A_1 \cos w_d t + A_2 \sin w_d t\right)$$

Free Response of Damped Second Order System –
Case 3 – Underdamped

Solving for Initial Conditions

$$x(t) = e^{-\zeta w_n t}\left(A_1 \cos w_d t + A_2 \sin w_d t\right)$$

$$x(t=0) = \cancelto{1}{e^{-\zeta w_n 0}}\left(A_1 \cancelto{1}{\cos w_d 0} + A_2 \cancelto{0}{\sin w_d 0}\right) = x_0 \rightarrow A_1 = x_0$$

$$\dot{x}(t=0) = -\zeta w_n \cancelto{1}{e^{-\zeta w_n 0}}\left(A_1 \cancelto{1}{\cos w_d 0} + A_2 \cancelto{0}{\sin w_d 0}\right) + \cancelto{1}{e^{-\zeta w_n 0}}\left(-A_1 w_d \cancelto{0}{\sin w_d 0} + A_2 w_d \cancelto{1}{\cos w_d 0}\right)$$

$$= -\zeta w_n A_1 + A_2 w_d = \dot{x}_0$$

$$A_1 = x_0$$

$$A_2 = \frac{\dot{x}_0 + \zeta w_n A_1}{w_d} = \frac{\dot{x}_0 + \zeta w_n x_0}{w_d}$$

Free Response of Damped Second Order System –
Case 3 – Underdamped

Given the solution

$$x(t) = e^{-\zeta w_n t}\left(A_1 \cos w_d t + A_2 \sin w_d t\right)$$

Another form of solution would be

$$x = Ae^{-\zeta w_n t} \cos(w_d t - \phi)$$

A, ϕ unknown constants

Free Response of Damped Second Order System –

Case 3 – Underdamped

Solving for Initial Conditions

$$x(t_0 = 0) = x_0 \quad \rightarrow \quad x_0 = Ae^{-\zeta w_n 0}\cos(w_d 0 - \phi)$$

$$x_0 = A\cos(\phi) \quad \rightarrow \quad A = \frac{x_0}{\cos(\phi)}$$

$$\dot{x}(t_0 = 0) = v \quad \rightarrow \quad \dot{x} = A\left(-\zeta w_n\right)e^{-\zeta w_n t}\cos(w_d t - \phi) - Ae^{-\zeta w_n t} w_d \sin(w_d t - \phi)$$

$$v_0 = A\left(-\zeta w_n\right)\cos(\phi) + Aw_d \sin(\phi)$$

$$v_0 = \frac{x_0}{\cancel{\cos(\phi)}}\left(\zeta w_n\right)\cancel{\cos(\phi)} + \frac{x_0}{\cos(\phi)} w_d \sin(\phi)$$

Free Response of Damped Second Order System –

Case 3 – Underdamped

$$v_0 = x_0 \zeta w_n + x_0 w_d \frac{\sin\phi}{\cos\phi}$$

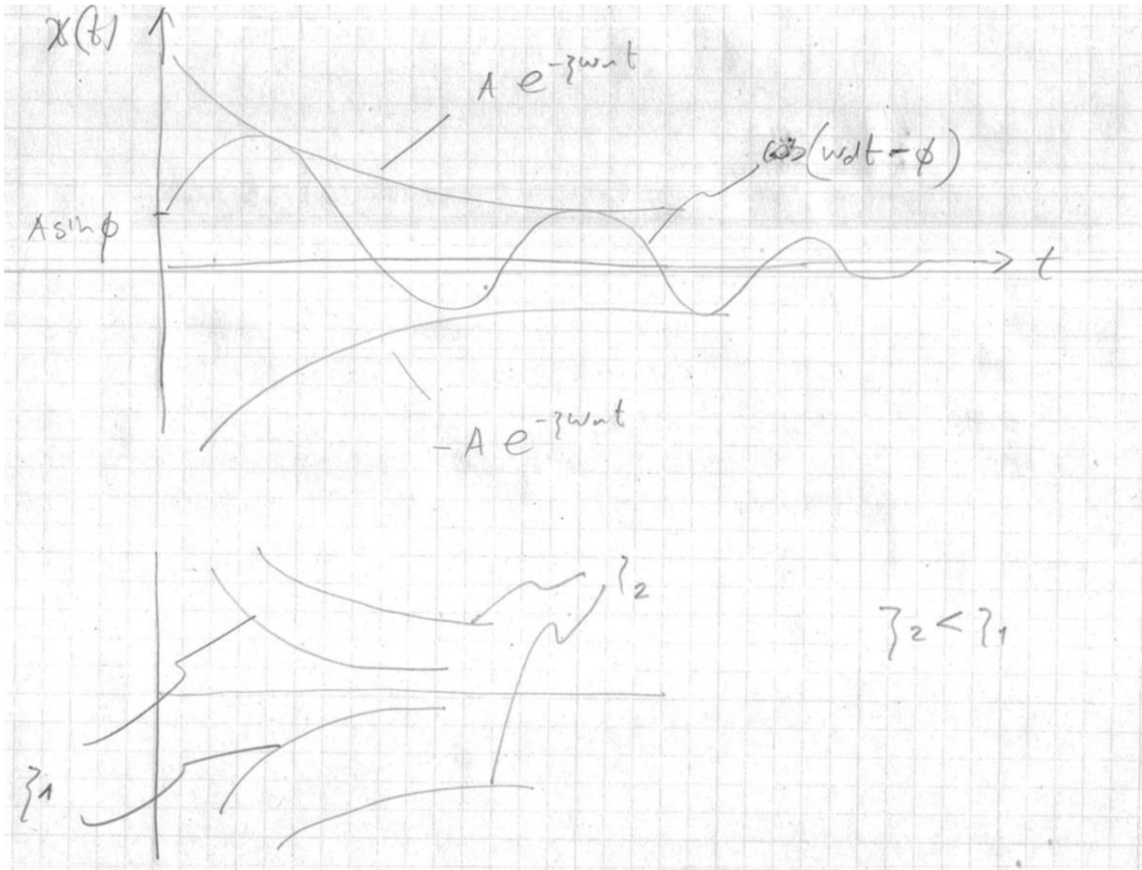
$$\frac{v_0 - x_0 \zeta w_n}{x_0 w_d} = \frac{\sin\phi}{\cos\phi}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{v_0 - x_0 \zeta w_n}{x_0 w_d}$$

$$\begin{cases} \phi = \tan^{-1}\left[\dfrac{v_0 - x_0 \zeta w_n}{x_0 w_d}\right] \\ A = \dfrac{x_0}{\cos\phi} \end{cases}$$

Free Response of Damped Second Order System –
Case 3 – Underdamped

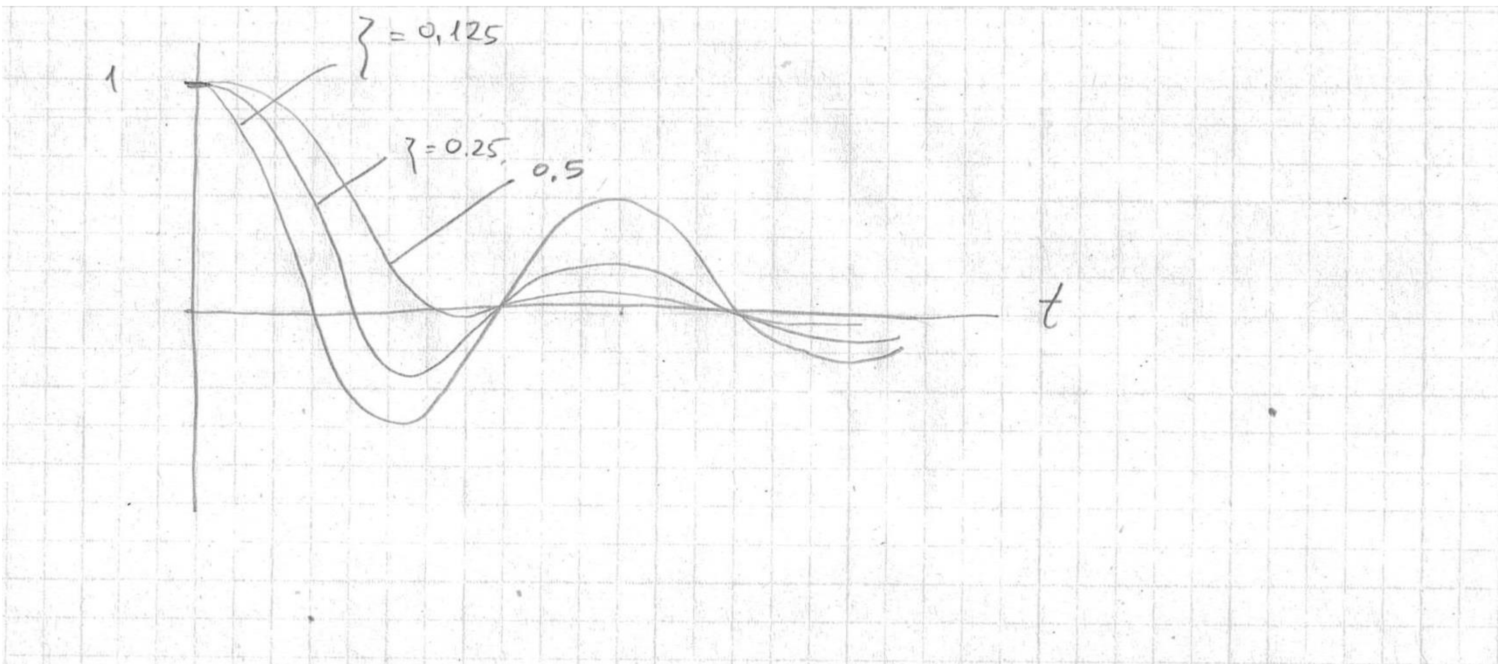
$$x = Ae^{-\zeta w_n t}\cos(w_d t - \phi)$$



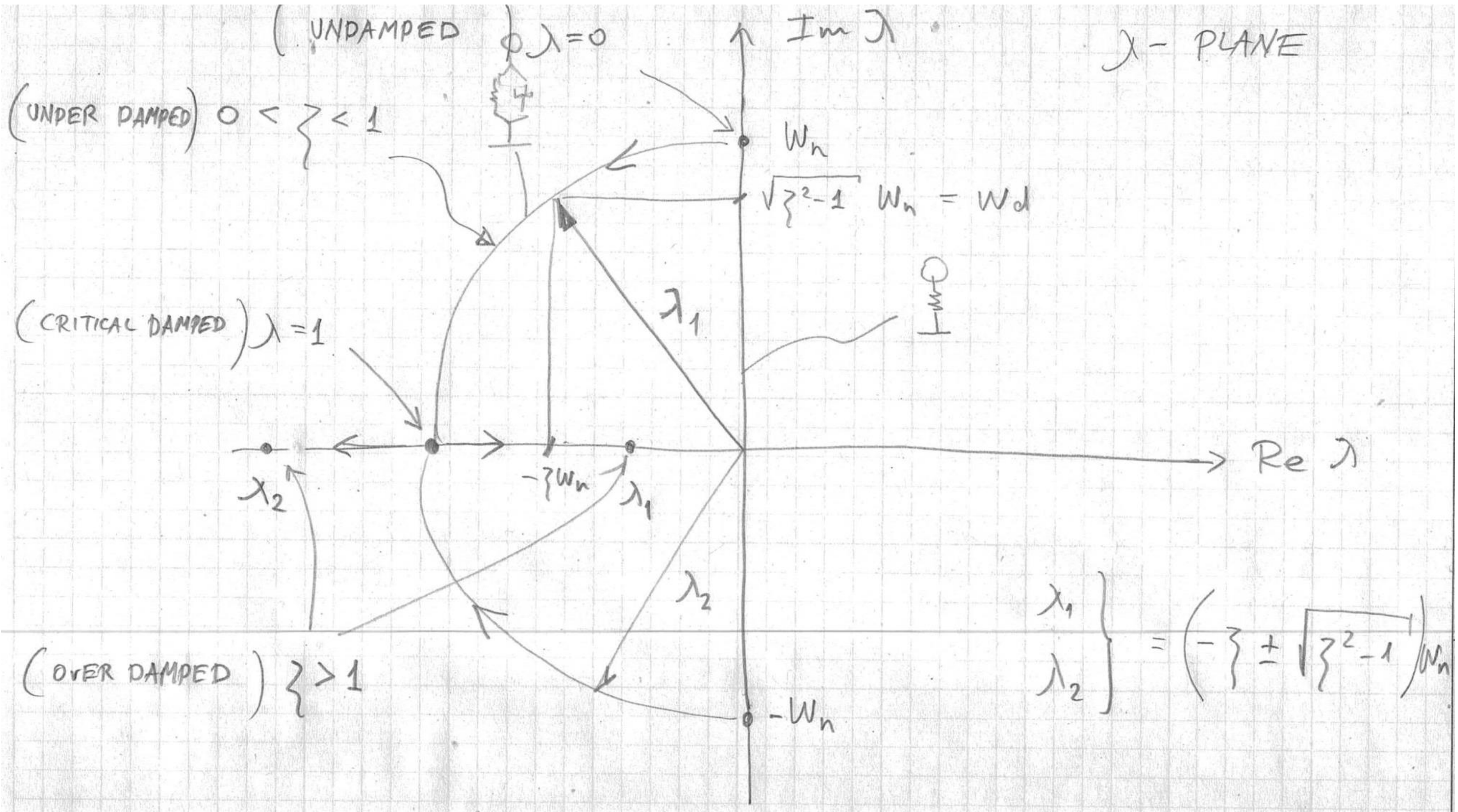
Free Response of Damped Second Order System –
Case 3 – Underdamped

$$x(t) = e^{-\zeta w_n t}\left(c_1 \cos\left(w_n\sqrt{1-\zeta^2}\right)t + c_2 \sin\left(w_n\sqrt{1-\zeta^2}\right)t\right)$$

$$w_n = 1;\ \ x(0) = 1;\ \ \dot{x}(0) = 0$$



Free Response of Damped Second Order System –
Introduction – Cases – The "S" Plane

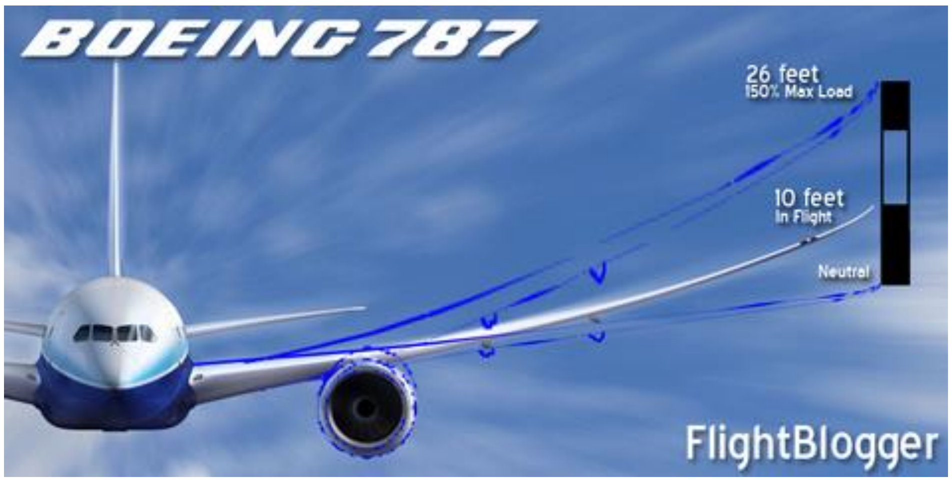


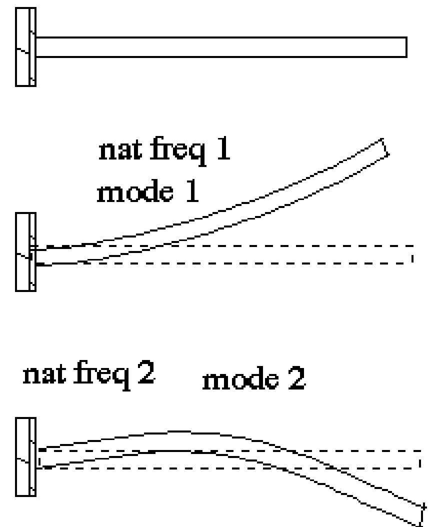
Free Response of Second Order System –
Logarithmic Decrement

- System identification
- Determined coefficients of the model experimentally

Free Response of Second Order System –

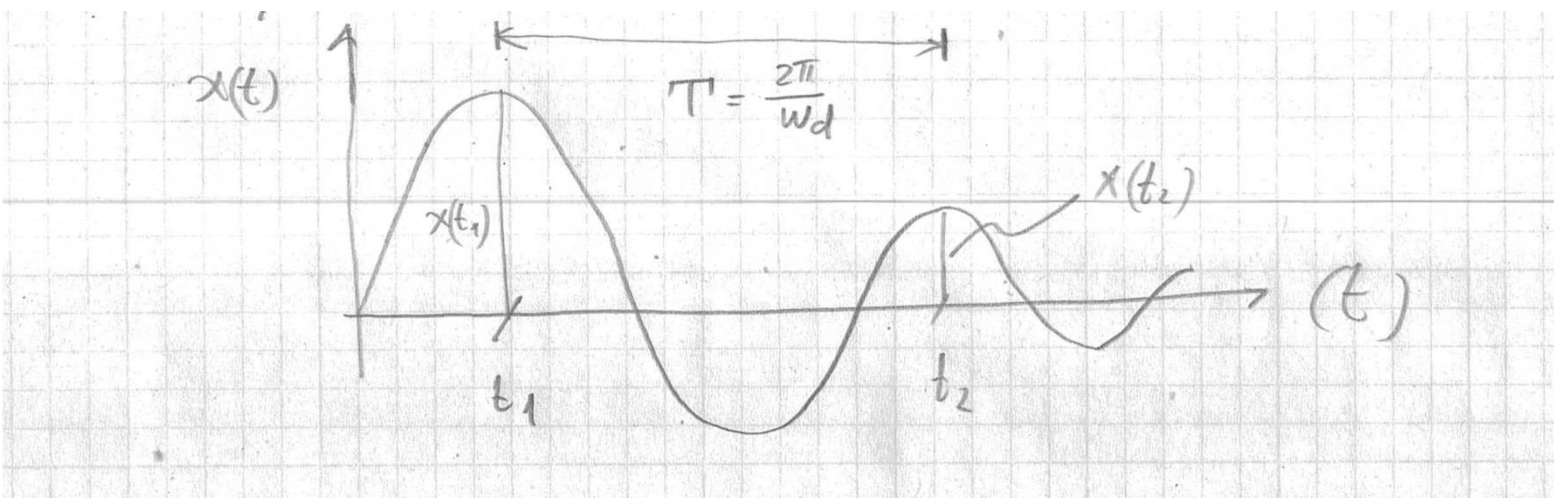
Logarithmic Decrement – Stiffness





Free Response of Second Order System –
Logarithmic Decrement – Damping

- Assumptions
  - Viscous damping
  - Under damped system
  - System identification



Free Response of Second Order System –

Logarithmic Decrement

For $t = t_1$ $\quad x(t_1) = Ae^{-\zeta w_n t_1}\cos(w_d t_1 - \psi)$

$t = t_2$ $\quad x(t_2) = A\boxed{e^{-\zeta w_n t_2}}\cos(w_d t_2 - \psi)$

$$= A\boxed{e^{-\zeta w_n (t_1 + T)}}\cos(w_d (t_1 + T) - \psi)$$

$$\boxed{e^{-\zeta w_n (t_2)} = e^{-\zeta w_n (t_1 + T)}} = e^{-\zeta w_n t_1} e^{-\zeta w_n T} = e^{-\zeta w_n t_1} e^{-2\pi\zeta \frac{w_n}{w_d}} = e^{-\zeta w_n t_1} e^{\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$T = \frac{2\pi}{w_d}$$

$$w_d = w_n\sqrt{1-\zeta^2}$$

$$\frac{w_n}{w_d} = \frac{1}{\sqrt{1-\zeta^2}}$$

Free Response of Second Order System –
Logarithmic Decrement

For

$$t = t_1 \qquad x(t_1) = Ae^{-\zeta w_n t_1}\cos(w_d t_1 - \psi)$$

$$t = t_2 \qquad x(t_2) = Ae^{-\zeta w_n t_2}\boxed{\cos(w_d t_2 - \psi)}$$

$$= Ae^{-\zeta w_n (t_1 + T)}\boxed{\cos(w_d (t_1 + T) - \psi)}$$

$$\boxed{\cos[w_d (t_1) - \psi] = \cos[w_d (t_1 + T) - \psi]} = \cos(w_d t_1 - \psi)\cos(w_d T) - \sin(w_d t_1 - \psi)\sin(w_d T)$$

$$= \cos(w_d t_1 - \psi)\cos(2\pi) - \sin(w_d t_1 - \psi)\sin(2\pi) = \cos(w_d t_1 - \psi)$$

$$w_d T = 2\pi \qquad \Rightarrow \cos(w_d t_2 - \psi) = \cos(w_d t_1 - \psi)$$

Free Response of Second Order System –
Logarithmic Decrement

- The ratio between the two peak values

$$\frac{x(t_1)}{x(t_2)} = \frac{Ae^{-\zeta w_n t_1}\cos(w_d t_1 - \psi)}{Ae^{-\zeta w_n t_2}\cos(w_d t_2 - \psi)} = \frac{e^{-\zeta w_n t_1}}{e^{-\zeta w_n t_1} e^{-2\pi\zeta/\sqrt{1-\zeta^2}}}$$

$$\frac{x(t_1)}{x(t_2)} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\delta = \ln\frac{x(t_1)}{x(t_2)} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

δ – Logarithmic decrement

δ – Measured experimentally – ln of two consecutive peak values(not necessarily the first two)

Free Response of Second Order System –

Logarithmic Decrement

$$\delta^2 = \frac{4\pi^2\zeta^2}{1-\zeta^2}$$

$$\delta^2(1-\zeta^2) = 4\pi^2\zeta^2$$

$$\delta^2 - \zeta^2\delta^2 = 4\pi^2\zeta^2$$

$$\delta^2 = \zeta^2(4\pi^2 + \delta^2)$$

$$\zeta^2 = \frac{\delta^2}{4\pi^2 + \delta^2}$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

For small damping $\delta << 2\pi$

$$\zeta \cong \frac{\delta}{2\pi}$$