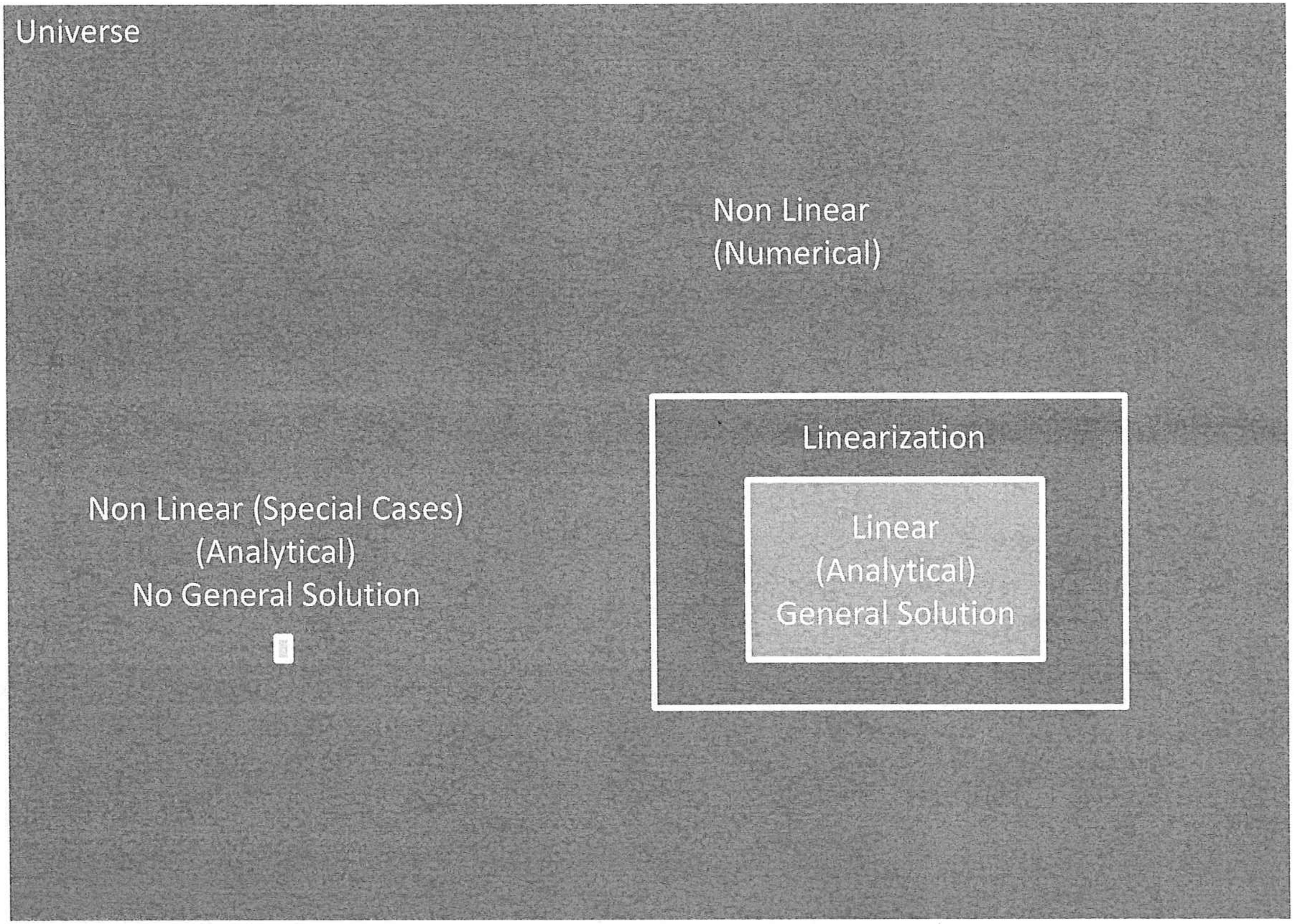
Class Notes 3:

First Order Differential Equation – Non Linear

MAE 82 – Engineering Mathematics


Universe
Non Linear
(Numerical)
Linearization
Non Linear (Special Cases)
(Analytical)
No General Solution
Linear
(Analytical)
General Solution

Introduction – First Order Non Linear Differential Equations

- First order differential equations
  - Linear - General Solution
  - Non Linear - No general solution

- Analytical solution exist only for special classes of non linear equations

Introduction – Schematics

First Order Differential Equations

Linear Equation	Non Linear Equation
Type 1 $$\frac{dy}{dx} + ay = g(t)$$ Solution: Integration Factors	Type 1 – Separable Eqs. $$M(x)dx + N(y)dy = 0$$ Solution: Integration Factors
Type 2 $$\frac{dy}{dx} + p(t)y = g(t)$$ Solution: Integration Factors	Type 2 – Exact Eqs. $$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ Solution: Integration Factors
Type 3 (Autonomous) $$\frac{dy}{dx} = f(t)$$ Solution: Integration Factors	Type 3 – Convert Non Exact to Exact Eqs. Solution 1: Integration Factors Solution 2: Change of Variables

Introduction – Relationships Among Classes of Equations

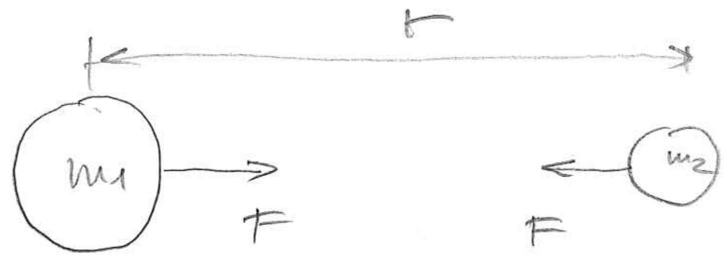
Eq. that can be made exact

Exact $M(x,y)dx + N(x,y)dy = 0$ $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Separable $M(x)dx + N(y)dy = 0$

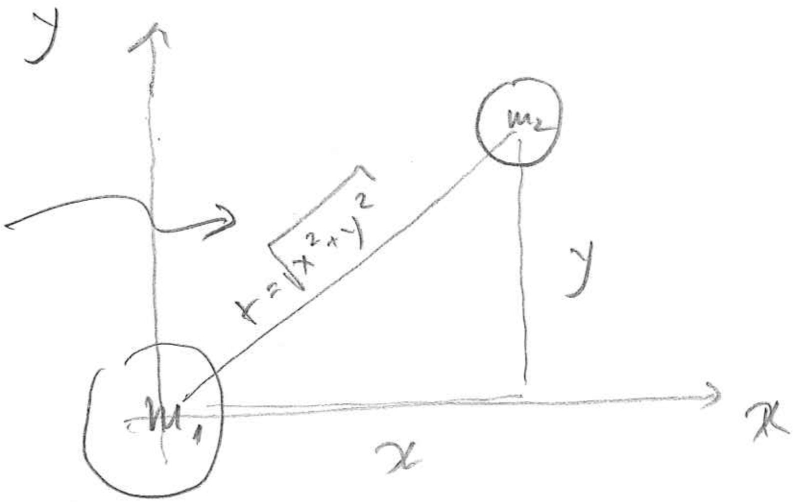
Autonomous $\dfrac{dy}{dx} = f(t)$

EXACT EQ. - INTRO



$$\vec{F} = G \frac{m_1 m_2}{r^3} \vec{r}$$

$\vec{r} = (x i ; y J)$



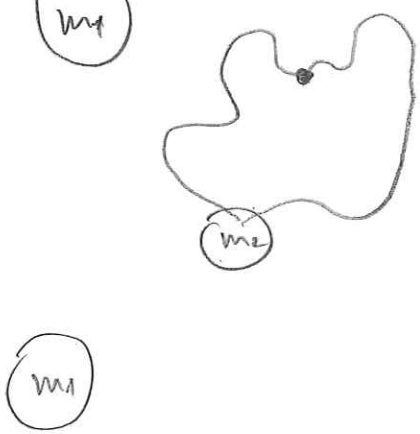
$$\vec{F}(x,y) = G m_1 m_2 \left[\frac{x}{(\sqrt{x^2+y^2})^3} i + \frac{y}{(\sqrt{x^2+y^2})^3} J \right]$$

$$= G m_1 m_2 \left[\frac{x}{(x^2+y^2)^{3/2}} i + \frac{y}{(x^2+y^2)^{3/2}} J \right]$$

- Gravity → Conservative Force
  → Leads to Potential Function

Potentia Function

① No work is performed on a close path



② Work is independent of the path

- potential energy satisfies

$$\psi = U = -\frac{G m_1 m_2}{(x^2+y^2)^{1/2}}$$

- Implicit differentiation

$$d\psi(x,y) = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

Divide by dx

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

- If the potential function ψ satisfies

$$\psi(x,y) = C$$

CONST Level of potential field

- Than

$$\frac{d\,\psi(x,y)}{dx} = 0$$

This gives rise to an Exact Diff Eq.

Defenition: Suppose there is a function $\psi(x,y)$ with

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

The first order differential equation given by

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

OR

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Is an exact diff. eq. with the implicit solution satisfying

$$\psi(x,y) = C$$

As an example consider the diff. eq.

$$2y\,y' + 2x = 0 \qquad \left[\, 2x\,dx + 2y\,dy = 0 \,\right]$$

$$\int 2x\,dx = -\int 2y\,dy$$

which is separable, so it is exact.

- A potential function is

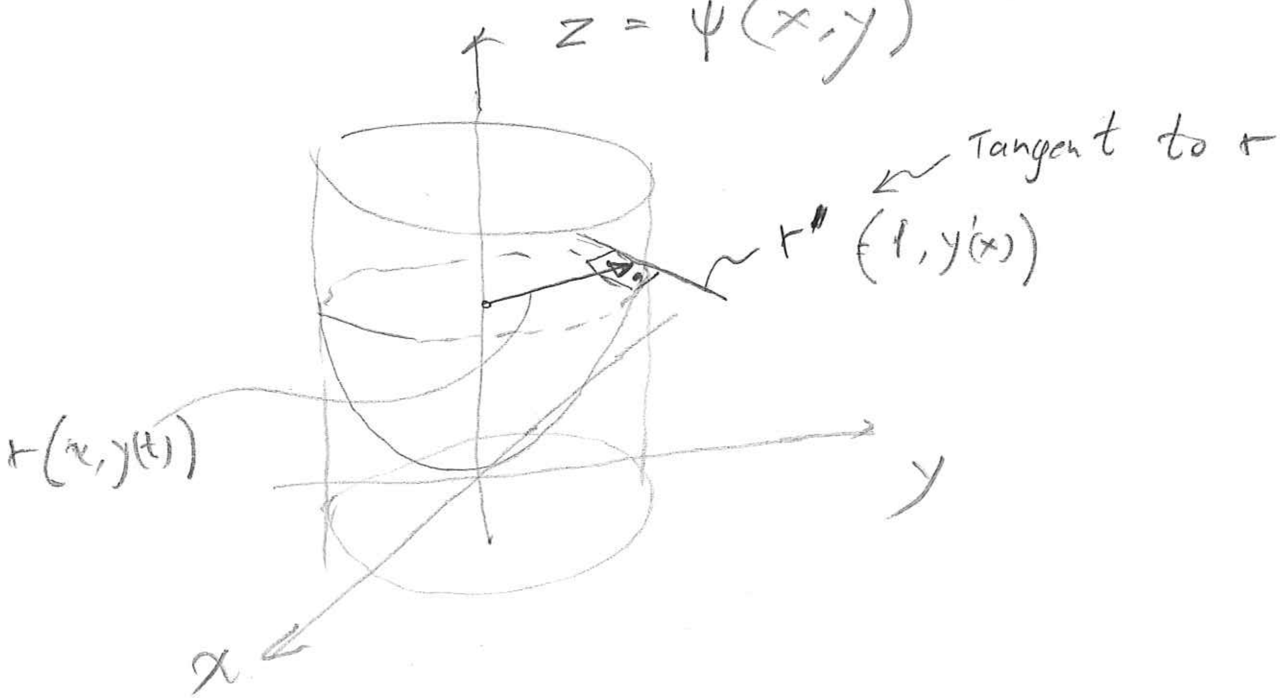
$$\psi = x^2 + y^2$$

a paraboliod.

- Solutions y are defined by the equation

$$x^2 + y^2 = c$$

- An exact equation and its solutions can be pictured on a graph of a potential function



- The solutions of the diff eq. define level of curves of the potential function which are level curves of ψ for $c > 0$

$$y(x) = \pm\sqrt{c - x^2}$$

The graph of the solution is shown on the x-y-plane

- The vector $r(x) = f(x, y(x))$ point to the solution's graph while the derivative $r'(x) = \frac{dr}{dx} = f(1, y'(x))$ is tangent to the level curve

- The gradient vector by definition is

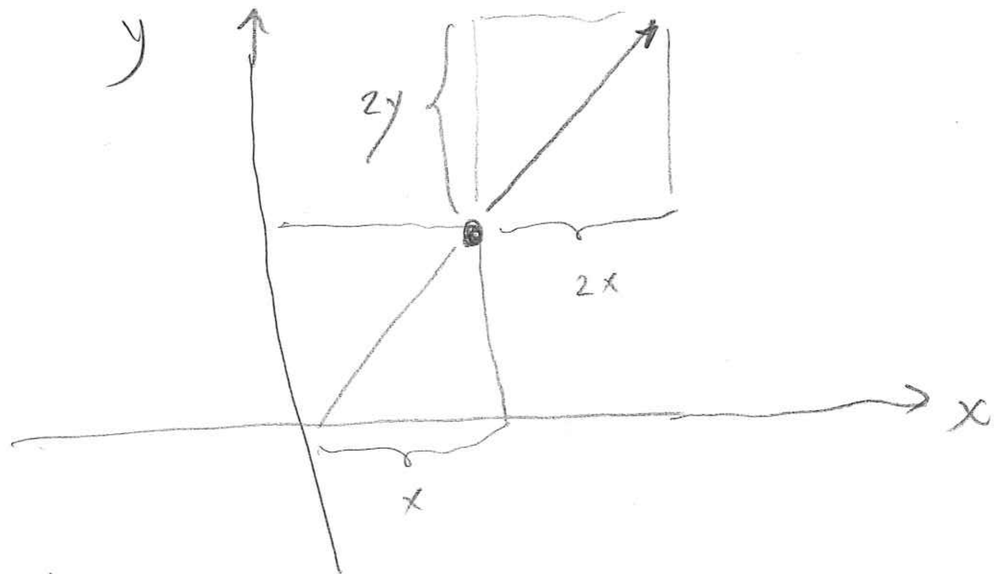
$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

in a 2D the gradient vector of ψ is

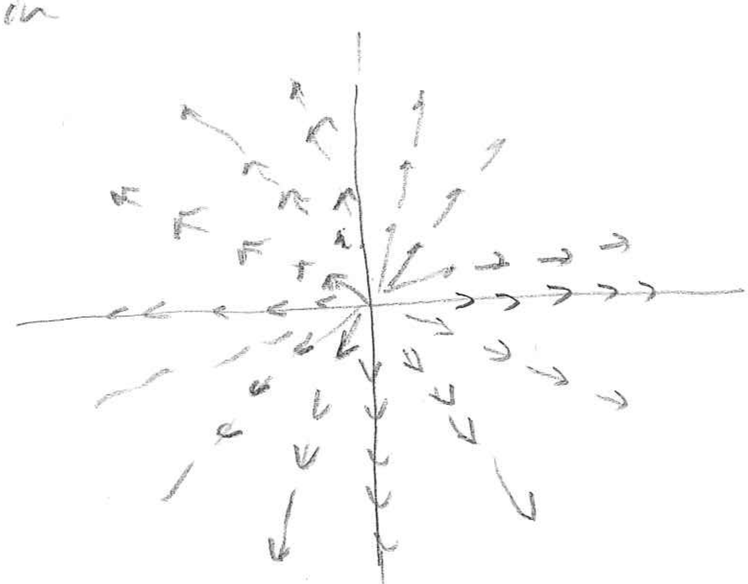
$$\nabla \psi = \begin{bmatrix} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial y} \end{bmatrix} = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

- Note that the gradient vector $\nabla\psi$ is perpendicular to the level curve

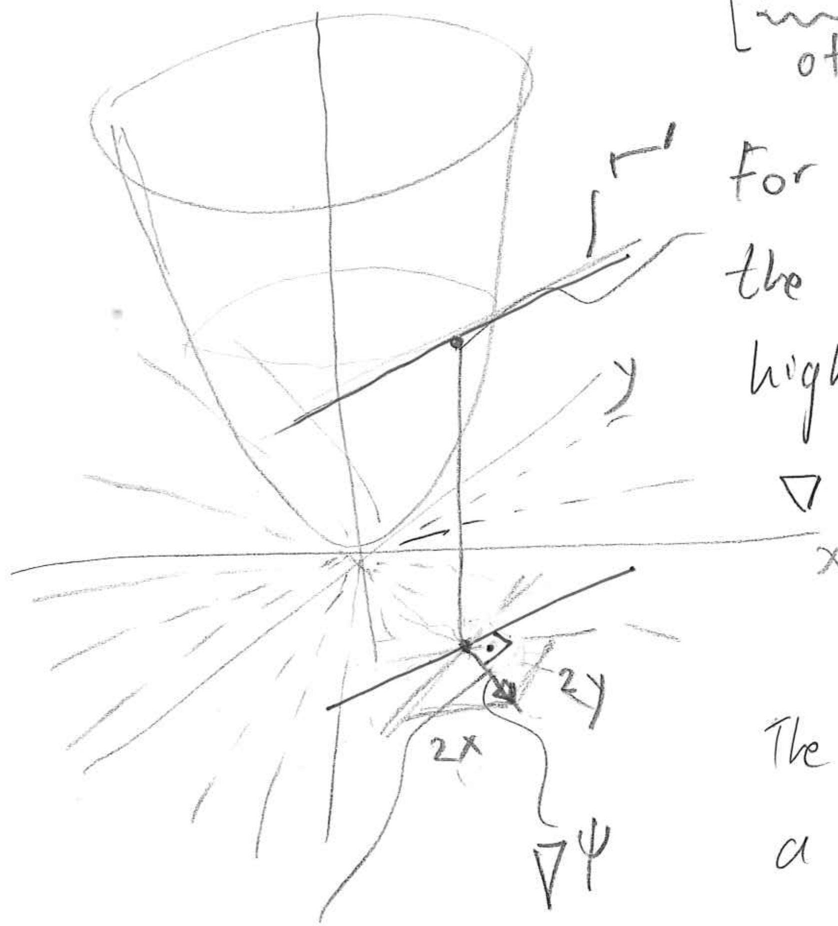
- for a selected point (x, y) the gradient is $(2x, 2y)$



- Scaling down the vector of the vector field is resulted in



[Hiking up the face of the cone]



For any given point the direction of the highest ascent is

$$\nabla \Psi = \begin{Bmatrix} 2x \\ 2y \end{Bmatrix}$$

The vector field is a set of vector the showing us which direction we need to follow to maximized our climb "up hill"

The length of the directional field shows us the steepest of the function

- The gradient vector $\nabla \psi = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ is perpendicular to the level curve and to the tangent of the level curve r'

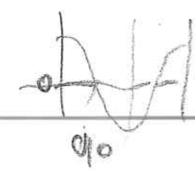
$$r' \perp \nabla \psi \Rightarrow r' \cdot \nabla \psi = 0$$

$\uparrow$

dot product $|r||\psi| \cancelto{0}{\cos\theta}$

$\uparrow$

$\theta = 90$



FIRST ORDER NON LINEAR DIFFERATIAL EQUATIONS

TYPE 1 - SEPERABLE EQUATION

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

- Condition (seperable)

$$\begin{cases} M = M(x) & \text{Only a function of } x \\ N = N(y) & \text{Only a function of } y \end{cases}$$

$$M(x)\,dx + N(y)\,dy = 0.$$

- Solution Approach: (1) Terms involving each variable may be placed on opposite sides of the equation
  (2) Integrating the functions M and N (Sperevatly)

- Note: Suppress the distinction between independent and dependent variables

TYPE 1 - SEPERABLE EQUATION - EXAMPLE

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

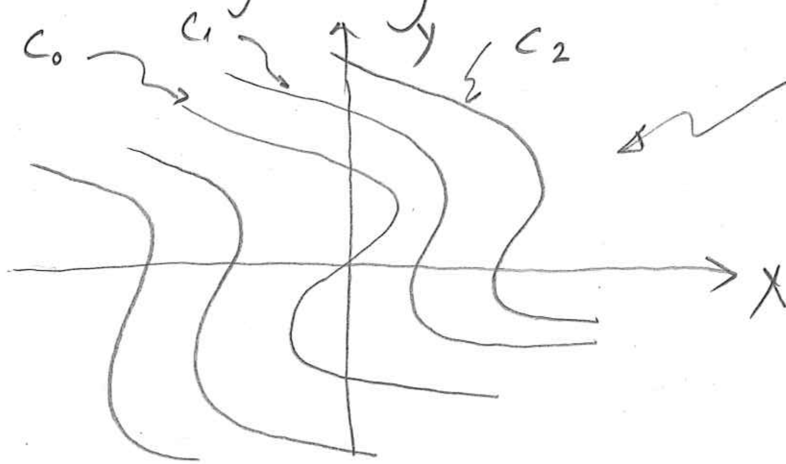
$$\underbrace{x^2}_{M(x)} dx \underbrace{- (1-y^2)}_{N(y)} dy = 0$$

$$x^2 dx = (1-y^2) dy$$

Integration $$\int x^2 \, dx = \int (1-y^2) \, dy$$

$$\frac{1}{3}x^3 + C = y - \frac{1}{3}y^3$$

$$x^3 + y^3 - 3y = \tilde{C}$$



Implicit Graphs

FIRST ORDER NON LINEAR DIFFERANTIAL EQUATIONS

TYPE 2 - EXACT EQUATION

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

- Condition : (Criteria to be Exact)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

OR

$$M_y = N_x$$

- Solution Approach

1. Check if the eq. is Exact ($M_y = N_x$)
2. Solution form $\psi(x,y) = C$
3. $\left.\begin{aligned} \psi_x &= M(x,y) \\ \psi_y &= N(x,y) \end{aligned}\right\}$ Integration

- Solution:

- Assume the solution has the form $\psi(x, y) = C_0$

- Differentiat the solution $d[\psi(x,y)] = \psi_x \, dx + \psi_y \, dy = 0$

- Compare with the original form $M(x,y) \, dx + N(x,y) \, dy = 0$

- Integrating

$$\psi_x = M(x,y)$$

$$\psi(x,y) = \int^{x} M(t,y)\,dt + h(y) \quad (*)$$

- Diffirentiat $(*)$ partially with respect to y using $\psi_y = N(x,y)$

$$\psi_y(x,y) = \underbrace{\int M_y(t,y)\,dt + \frac{d}{dy} h(y)}_{\text{Diff } (*) \to \psi_y} = N(x,y) \quad (+)$$

- Solving for $\frac{d}{dy} h(y)$

$$\frac{d}{dy} h(y) = N(x,y) - \int^{x} M_y(t,y)\,dt$$

- Solving for $h(y)$ by integration

$$h(y) = \int \frac{d}{dy} h(y) = \int^{y} N(x,s)\,ds - \int^{y}\left[\int^{x} M_s(t,s)\,dt\right]ds \quad (0)$$

Plug Eq. (0) into (*)

$$\psi(x,y) = \int^{x} M(t,y)\,dt + \underbrace{\int^{y} N(x,s)\,ds - \int^{y}\left[\int^{x} M_s(t,s)\,dt\right]ds}_{h(t)} = c$$

TYPE 2 - EXACT EQUATION - EXAMPLE

$$(2xy^2 + 2y) + (2x^2y + 2x)\underset{\uparrow \left(\frac{dy}{dx}\right)}{y'} = 0$$

$$\underbrace{(2xy^2 + 2y)}_{M(x,y)}dx + \underbrace{(2x^2y + 2x)}_{N(x,y)}dy = 0$$

- check if the equation is exact

$$\left.\begin{aligned} M_y &= \frac{\partial M}{\partial y} = 4xy + 2 \\ N_x &= \frac{\partial N}{\partial x} = 4xy + 2 \end{aligned}\right\} \Rightarrow \text{Exact Eq.}$$

(*) Solution Method 1

- Assume the solution has the form

$$\psi(x,y) = C$$

$$\frac{\partial \psi}{\partial x} = M(x,y) = 2xy^2 + 2y \quad \text{Integrate with } x \quad \psi = x^2y^2 + 2xy + \boxed{f(y)}$$

$$\frac{\partial \psi}{\partial y} = N(x,y) = 2x^2y + 2x \quad \text{Integrate with } y \quad \psi = x^2y^2 + 2xy + \boxed{g(x)}$$

- Note : $\quad f(y) = g(x) = C$

- Solution $\quad x^2y^2 + 2xy = \tilde{c}$

(*) Solution Method 2 – Solution using the formula

$$\psi(x,y) = \int_0^x M(t,y)\,dt + \int_0^y N(x,s)\,ds - \int_0^y\left[\int_0^x M_s(t,s)\,dt\right]ds = C$$

$$\psi(x,y) = \int_0^x \underbrace{(2xy^2+2y)}_{M}\,dx + \int_0^y \underbrace{(2x^2y+2x)}_{N}\,dy - \int_0^y\int_0^x \underbrace{(4xy+2)}_{M_y}\,dx\,dy$$

$2x^2y+2x$

x^2y^2+2xy

$$= \left[x^2y^2+2xy\right] + \left[\cancel{x^2y^2+2xy}\right] - \left[\cancel{x^2y^2+2xy}\right]$$

$$\psi(x,y) = x^2y^2+2xy = C$$

- See also Example 2 p. 72/73
- Example 3 p. 73

FIRST ORDER NON LINEAR DIFFERENTIAL EQUATIONS

TYPE 3 - CONVERT NON EXACT TO EXACT

SOLUTION METHOD 1 - INTEGRATION FACTORS

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

- Condition: $M_y \neq N_x \rightarrow$ Non Exact

Solution Approach
① Define an integration factor $\mu(x,y)$
② Multiplying the diff. eq. by the integration factor will turn it into an exact eq.

$$\underbrace{\mu(x,y)\,M(x,y)}\,dx + \underbrace{\mu(x,y)\,N(x,y)}\,dy = 0$$

$$[\mu M]_y = [\mu N]_x \Rightarrow \text{Exact}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

- Definition of the integration factor $\mu(x, y)$

CASE 1 — μ is a function of x alone $\rightarrow$ $\mu_y = 0$

$$\underbrace{\cancel{\mu_y}^{\,0} M}_{=0} + \mu M_y = \mu_x N + \mu N_x$$

$$\frac{M_y - N_x}{N} = \frac{1}{\mu}\frac{d\mu}{dx}$$

If : $\dfrac{M_y - N_x}{N} \longrightarrow$ a function of x alone

Then : $\mu(x) = e^{\int \left[\frac{M_y - N_x}{N}\right] dx}$

$\uparrow$

Integration Factor

CASE 2 — μ is a function of y alone $\rightarrow \mu_x = 0$

$$\mu_y M + \mu M_y = \underbrace{\overset{0}{\mu_x} N}_{=0} + \mu N_x$$

$$\frac{N_x - M_y}{M} = \frac{1}{\mu}\frac{d\mu}{dy}$$

If: $\dfrac{N_x - M_y}{M} \longrightarrow$ a function of y alone

Then $\mu(\cancel{x}) = e^{\int \left[\frac{N_x - M_y}{M}\right] dy}$

Integration factor

TYPE 3 - SOLUTION METHOD 1 - INTEGRATION FACTORS - EXAMPLE

$$\underbrace{\left(3x^2y + 2xy + y^3\right)}_{M(x,y)} dx + \underbrace{\left(x^2 + y^2\right)}_{N(x,y)} dy = 0$$

- Check if the diff. eq. is exact

$$\left.\begin{aligned} M_y &= 3x^2 + 2x + 3y^2 \\ N_x &= 2x \end{aligned}\right\} \quad M_y \neq N_x \quad \text{Not Exact}$$

- Find what is the case (Case 1 OR Case 2)

CASE 1 : $\dfrac{M_y - N_x}{N} = \dfrac{\left(3x^2 + \cancel{2x} + 3y^2\right) - \left(\cancel{2x}\right)}{x^2 + y^2} = \dfrac{3\left(x^2 + y^2\right)}{x^2 + y^2} = 3$

CASE 2 : $\dfrac{N_x - M_y}{M} = \dfrac{\left(\cancel{2x}\right) - \left(3x^2 + \cancel{2x} + 3y^2\right)}{3x^2y + 2xy + y^3} = -\dfrac{3\left(x^2 + y^2\right)}{3x^2y + 2xy + y^3} = f(x,y)$

- Conclusion $\longrightarrow$ CASE 1 $\quad \frac{1}{\mu}\frac{d\mu}{dx} = 3$

- The integration factor

$$\mu = e^{\int \underbrace{\left[\frac{M_y - N_x}{N}\right]}_{=3} dx} = e^{\int 3dx} = e^{3x}$$

New $M \rightarrow \tilde{M} = \mu M = e^{3x}(3x^2y + 2xy + y^3)$

New $N \rightarrow \tilde{N} = \mu N = e^{3x}(x^2 + y^2)$

- check that the new diff is Exact

$$\tilde{M}(x,y)\,dx + \tilde{N}(x,y)\,dy = 0 \Rightarrow \text{Exact}$$

- Calculate

$$\left.\begin{aligned} \tilde{M}_y &= e^{3x}(3x^2 + 2x + 3y^2) \\ \tilde{N}_x &= 3e^{3x}x^2 + 2xe^{3x} + 3e^{3x}y^2 \end{aligned}\right\} \quad \tilde{M}_y \overset{\checkmark}{=} \tilde{N}_x$$

use the equation of the solution $\psi(x,y)$ of the Exact diff. eq.

$$\psi(x,y) = \int^{x} \tilde{M}(t,y)\,dt + \int^{y} \tilde{N}(x,s)\,ds - \int^{y}\left[\int^{x} \tilde{M}_y(t,s)\,dt\right]ds$$

$$\psi(x,y) = \int \underbrace{e^{3x}(3x^2y+2x+y^3)}_{\tilde{M}}\,dx + \int^{y} \underbrace{e^{3x}(x^2+y^2)}_{\tilde{N}}\,dy - \int^{y}\int^{x} \underbrace{e^{3x}(3x^2+2x+3y^2)}_{\tilde{M}_y}\,dx\,dy$$

– Skiping the integration steps

$$\psi(x,y) = e^{3x}(3x^2y + y^3) = C$$

FIRST ORDER NON LINEAR DIFFERATIAL EQUATIONS

TYPE 3 - CONVERT NON EXACT TO EXACT

SOLUTION METHOD 2 - SUBSTITUTION / CHANG OF VARIABLES

- Homogeneou Function - Defenition

If a function possesses the property

$$f(tx, ty) = t^{\alpha} f(x, y)$$

for a real number a

Then f is said to be homogeneous function of degree α

Homogeneous / No Homogeneous Function - Example

Example 1 $\quad f(x,y) = x^3 + y^3$

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^2+y^2) = t^3 f(x,y)$$

$\Rightarrow$ $f(x,y)$ is a homogeneous function of degree 3

Example 2 $\quad f(x,y) = x^3 + y^3 + 1$

$$f(tx, ty) = (tx)^3 + (ty)^3 + 1 = t^3(x^3+y^3) + 1 \neq t^3 f(x,y)$$

= $f(x,y)$ is a non homogeneous function

- A first order differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be homogeneous if both both coefficients function M and N are homogeneous functions of the same degree

$$\left\{\begin{array}{l} M(tx,ty) = t^{\alpha} M(x,y) \\ N(tx,ty) = t^{\alpha} N(x,y) \end{array}\right]$$

Homogeneous functions of the same degree

If M and N are homogeneous function of degree α

Then
$$\begin{cases} M(x,y) = x^{\alpha} M(1,u) \\ N(x,y) = x^{\alpha} N(1,u) \\ \text{where } u = \dfrac{y}{x} \end{cases} \quad \text{AND} \quad \begin{cases} M(x,y) = y^{\alpha} M(v,1) \\ N(x,y) = y^{\alpha} N(v,1) \\ \text{where } v = \dfrac{x}{y} \end{cases}$$

using substition $\begin{cases} y = ux \\ x = vy \end{cases}$ where v, u independent veriables

The diff eq.

$$M(x,y)dx + N(x,y)dy = 0$$

can be rewritten as

$$\cancel{x^{\alpha}} M(1,u)dx + \cancel{x^{\alpha}} N(1,u)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

where $u = \frac{y}{x}$ or $y = ux$

Define the differential

$$dy = u\,dx + x\,du$$

and substitute it into the diff eq. we obtain a separable differential equation in the variable u, and x

$$M(1,u)\,dx + N(1,u)\underbrace{\left[u\,dx + x\,du\right]}_{dy} = 0$$

$$\underbrace{\left[M(1,u) + u\,N(1,u)\right]dx + x\,N(1,u)\,du = 0}_{\text{seperable}}$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{M(1,u) + u\,N(1,u)} = 0$$

TYPE 3 - SUBSTITUTION / CHANGE OF VARIABLE - EXAMPLE

$$\underbrace{(x^2+y^2)}_{M}dx + \underbrace{(x^2-xy)}_{N}dy = 0$$

- check Exact

$$\left.\begin{aligned} M_y &= 2y \\ N_x &= 2x - y \end{aligned}\right\} \quad M_y \neq N_x \Rightarrow \text{Not Exact}$$

- check Homogeneous

$$\begin{cases} M(tx, ty) = (tx)^2 + (ty)^2 \Rightarrow t^2(x^2+y^2) = t^2 M(x,y) \\ N(tx, ty) = (tx)^2 - tx\,ty \Rightarrow t^2(x^2-xy) = t^2 N(x,y) \end{cases}$$

$\Rightarrow$ M, N Homogeneous functions of degree 2

Let $y = ux$

Then $dy = u\,dx + x\,du$

After substitution the given differential equation becomes

$$\underbrace{[x^2 + u^2x^2]}_{M(u,\,ux)}dx + \underbrace{[x^2 - ux^2]}_{N(x,\,ux)}\underbrace{u\,dx + x\,du}_{dy} = 0$$

$$x^2(1+u)\,dx + x^3(1-u)\,du = 0$$

$$x^2(1+u)\,dx = -x^3(1-u)\,du$$

$$\frac{x^2}{x^3}\,dx + \frac{1-u}{1+u}\,du = 0$$

$$\frac{1}{x}\,dx + \frac{1-u}{1+u}\,du = 0$$

$$\frac{dx}{x} + \left[-1 + \frac{2}{1+u}\right] du = 0$$

Integration

$$\ln|x| - u + 2\ln|1+u| = \ln c$$

Resubstitute $\quad u = \frac{y}{x}$

$$\ln|x| - \frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| = \ln c$$

$$\underbrace{\ln\left(\left|\frac{x+y}{x}\right|\right)^2 + \ln|x|} - \ln|c| = \frac{y}{x}$$

$$\underbrace{\ln x\left|\frac{x+y}{x}\right|^2 \qquad - \ln|c|} = \frac{y}{x}$$

$$\ln\left|\frac{\frac{\cancel{x}((x+y)^2)}{\cancel{x}^{\cancel{2}}}}{\frac{c}{1}}\right| = \frac{y}{x}$$

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x}$$

$$(x^2+y^2)^2 = cx\,e^{\frac{y}{x}}$$

$$\frac{(x^2+y)^2}{x\,e^{\frac{y}{x}}} = C$$