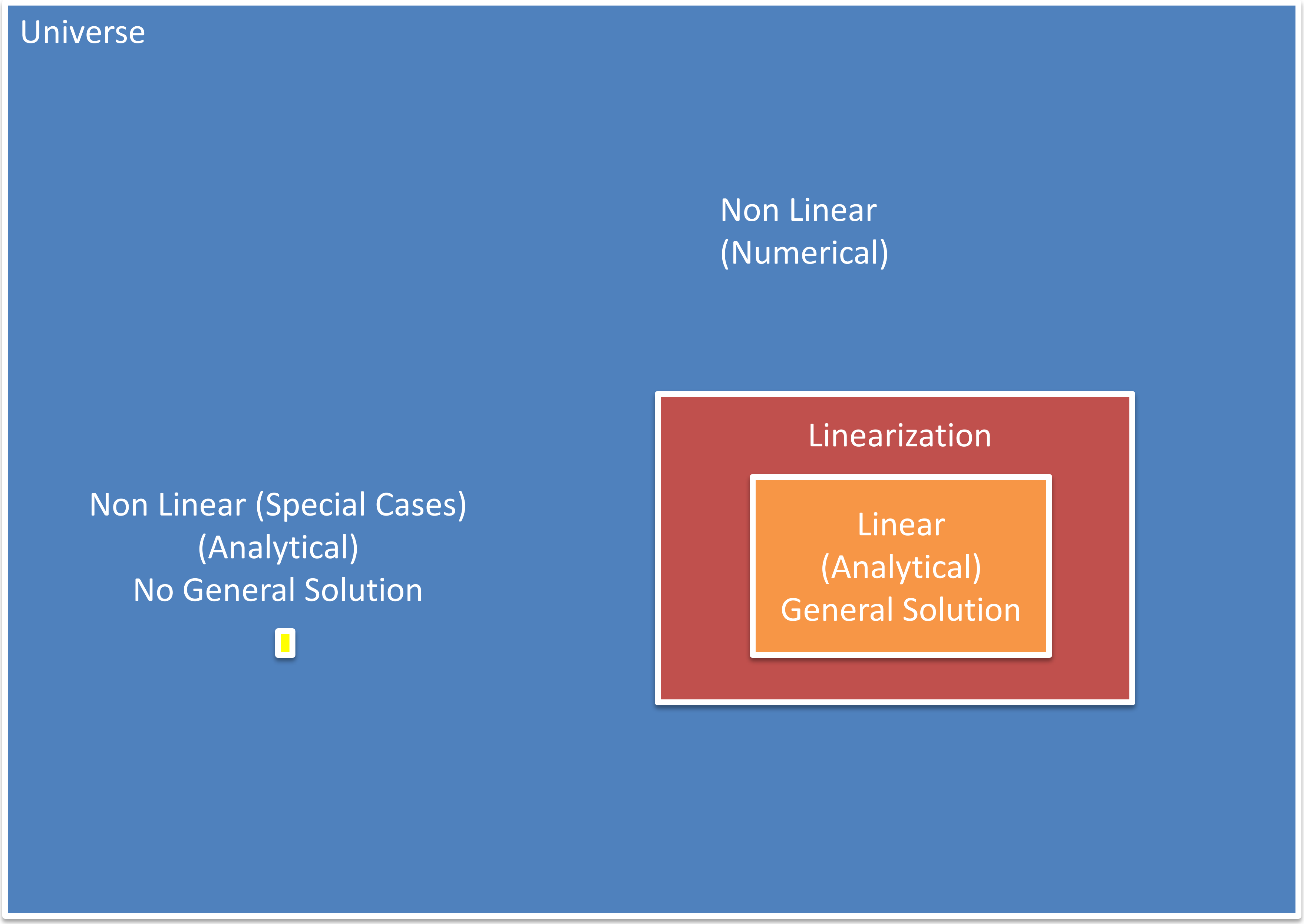
Class Notes 3:

First Order Differential Equation – Non Linear

MAE 82 – Engineering Mathematics


Universe
Non Linear
(Numerical)
Linearization
Linear
(Analytical)
General Solution
Non Linear (Special Cases)
(Analytical)
No General Solution

Introduction – First Order Non Linear Differential Equations

- First order differential equations
  - Linear - General Solution
  - Non Linear - No general solution

- Analytical solution exist only for special classes of non linear equations

Introduction – Schematics

First Order Differential Equations

Linear Equation	Non Linear Equation
Type 1 $$\frac{dy}{dx} + ay = g(t)$$ Solution: Integration Factors	Type 1 – Separable Eqs. $$M(x)dx + N(y)dy = 0$$ Solution: Integration Factors
Type 2 $$\frac{dy}{dx} + p(t)y = g(t)$$ Solution: Integration Factors	Type 2 – Exact Eqs. $$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ Solution: Integration Factors
Type 3 (Autonomous) $$\frac{dy}{dx} = f(t)$$ Solution: Integration Factors	Type 3 – Convert Non Exact to Exact Eqs. Solution 1: Integration Factors Solution 2: Change of Variables

Introduction – Relationships Among Classes of Equations

Eq. that can be made exact

Exact $M(x,y)dx + N(x,y)dy = 0$ $\quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Separable $M(x)dx + N(y)dy = 0$

Autonomous $\dfrac{dy}{dx} = f(t)$

First Order Non Linear Differential Equations

Type 1 - Separable Equations

- General form of 1st order DE
  - Note that (t) was substituted with (x)

$$\frac{dy}{dx} = f(x, y)$$

  - Rewrite the general form of the first order DE as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Which is possible to do it by setting

$$\begin{cases} M(x, y) = -f(x, y) \\ N(x, y) = 1 \end{cases}$$

First Order Non Linear Differential Equations

Type 1 - Separable Equations

If it happens that $\longrightarrow$ $\begin{cases} M = M(x) \\ N = N(y) \end{cases}$

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$M(x)dx + N(y)dy = 0$$

- **Separable Equation** - terms involving each variable may be placed on opposite sides of the equation.
- **The Differential Form** - suppress the distinction between independent and dependent variables.
- **Solution** - Integrating the function M and N.

First Order Non Linear Differential Equations

Type 1 - Separable Equations – Example

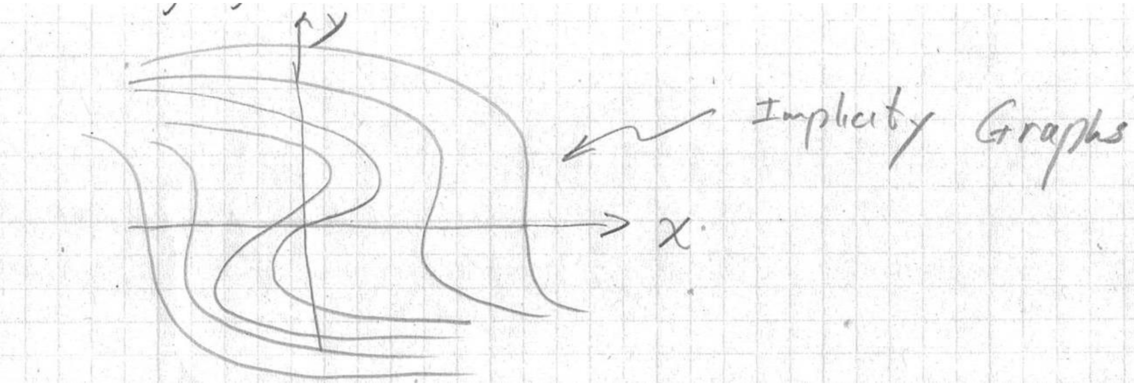
$$\frac{dy}{dx}=\frac{x^2}{1-y^2}$$

$$(1-y^2)dy=(x^2)dx$$

$$\int(1-y^2)dy=\int x^2dx$$

$$y-\frac{1}{3}y^3=\frac{1}{3}x^3+C$$

$$3y-y^3=x^3+C$$



First Order Non Linear Differential Equations

Type 2 - Exact Equation – Definition

- z is a function of two variables

$$z = f(x, y)$$

- Differential of a function of two variables
- Where the variables have continuous first partial derivative in region R of the xy- plane.
- The differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- In the special case when $f(x, y) = c$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation - Definition

or

$$\left.\begin{array}{c} M(x,y)dx + N(x,y)dy = 0 \\ M(x,y) + N(x,y)y' = 0 \end{array}\right\}(*)$$

$$y(x_0) = x_0$$

- If a function $\psi(x,y)$ exists such that

$$\begin{cases} M(x,y) = \dfrac{\partial \psi}{\partial x} \\ N(x,y) = \dfrac{\partial \psi}{\partial y} \end{cases}$$

- Then the left hand side of the equation (*) becomes an <u>exact differentiation</u> and (*) can be reduced to

$$d\left[\psi(x,y)\right] = 0$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – Definition - Example

$$\overbrace{x^2y^3}^{M(x,y)}dx+\overbrace{x^3y^2}^{N(x,y)}dy=0(*)$$

$\rightarrow$ (*) is an exact eq.

$$\psi(x,y)=\frac{1}{3}x^3y^3$$

$$d(\frac{1}{3}x^3y^3)=x^2y^3dx+x^3y^2dy$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – Solution

- In this case the solution to the differentiation equation is

$$\psi(x, y) = \psi(x_0, y_0) = c$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – Criteria for Exact Differential

- The necessary and sufficient condition for the equation

$$Mdx + Ndy = 0$$

to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ or $M_y = N_x$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – Criteria – Example

$$x^2 - 5xy + y^3 = c$$

$$\overbrace{(2x-5y)}^{M}dx + \overbrace{(-5x+3y^2)}^{N}dy = 0$$

$$\left.\begin{array}{l} \dfrac{\partial M}{\partial y} = -5 \\[2ex] \dfrac{\partial N}{\partial x} = -5 \end{array}\right\} \Rightarrow \text{Exact Eq.}$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – General Solution

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

- For the exact equation (1) we can write the perfect differential

$$d[\psi(x,y)] = \psi_x dx + \psi_y dy = 0 \qquad (2)$$

$$\psi_x = M(x,y); \quad \psi_y = N(x,y)$$

- If $$M_y = N_x$$

- the solution is $$\psi(x,y) = c$$

- Integrating $$\psi_x = M(x,y)$$

$$\psi(x,y) = \int^x M(t,y)dt + h(y) \qquad (3)$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – General Solution

- Differentiating (3) partially with respect to y and using

$$\psi_y = N(x,y)$$

$$\psi_y(x,y) = \int^x M_y(t,y)dt + \frac{d}{dy}h(y) = N(x,y) \quad (4)$$

- Solving for h(y) by Integrating (4) with respect to y

$$h(y) = \int \frac{d}{dy}h(y) = \int^y N(x,s)ds - \int^y \left[\int^x M_s(t,s)dt \right] ds \quad (5)$$

- Plug (5) in (3)

$$\psi(x,y) = \int^x M(t,y)dt + \int^y N(x,s)ds - \int^y \left[\int^x M_s(t,s)dt \right] ds = c$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – General Solution - Example

- Example $$(2xy^2+2y)+(2x^2y+2x)y'=0$$

$$(2xy^2+2y)dx+(2x^2y+2x)dy=0$$

$$M(x,y)=2xy^2+2y; \quad N(x,y)=2x^2y+2x$$

- Check if the differential equation is exact

$$M_y=\frac{\partial M}{\partial y}=4xy+2; \quad N_x=4xy+2$$
$$M_y=N_x \rightarrow \text{exact equation}$$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – General Solution - Example

- **Solution Method 1**

  – Solve $\psi(x, y) = c$

$$\frac{\partial \psi}{\partial x} = M(x, y) = 2xy^2 + 2y \quad \text{Integrate with x} \quad \psi = x^2y^2 + 2xy + \boxed{f(y)}$$

$$\frac{\partial \psi}{\partial y} = N(x, y) = 2x^2y + 2x \quad \text{Integrate with y} \quad \psi = x^2y^2 + 2xy + \boxed{g(x)}$$

- Comparing $f(y) = c, \quad g(x) = c$

- Solution $x^2y^2 + 2xy = c$

First Order Non Linear Differential Equations

Type 2 - Exact Equation – General Solution - Example

- **Solution Method 2 -** Solution using the formula

$$\psi(x,y)=\int^{x} M(t,y)dt+\int^{y} N(x,s)ds-\int^{y}\left[\int^{x} M_s(t,s)dt\right]ds=c$$

$$\psi(x,y)=\int \underbrace{(2xy^2+2y)}_{M}dx+\int \underbrace{(2x^2y+2x)}_{N}dy-\int^{y}\int^{x} \underbrace{4xy+2dxdy}_{M_y}$$

$$= \quad x^2y^2+2xy \; + \cancel{x^2y^2+2xy} \; - \; (\cancel{x^2y^2+2xy})$$

$2x^2y+2x$

x^2y^2+2xy

$$\boxed{\psi(x,y)=x^2y^2+2xy=c}$$

- See also Example 2 p.98, Example 3 p.98

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Integration Factors

- Given

$$M(x, y)dx + N(x, y)dy = 0$$

$$M_y \neq N_x \rightarrow \text{non exact}$$

- Can sometimes be reduced to an exact differential equation by multiplying the given equation by a function

$$\mu(x, y)$$

- Defined as the integration factor of the differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

- Turning it into an exact differential equation

$$[\mu M]_y = [\mu N]_x$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x \quad (1)$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Integration Factors

Case1 : μ is a function of x alone $\rightarrow \mu_y = 0$

$$\underbrace{\cancel{\mu_y M}}_{0} + \mu M_y = \mu_x N + \mu N_x$$

$$(2a) \quad \frac{M_y - N_x}{N} = \frac{1}{\mu}\frac{d\mu}{dx} \rightarrow \text{ a function of x alone}$$

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Integration Factors

Case2 : μ is a function of y alone $\rightarrow$ $\mu_x = 0$

$$\mu_y M + \mu M_y = \cancel{\mu_x N}^{0} + \mu N_x$$

$$(2b) \quad \frac{N_x - M_y}{M} = \frac{1}{\mu}\frac{d\mu}{dy} \rightarrow$$ a function of y alone

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Integration Factors
Special Cases – Example

$$(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$$

$$\begin{cases} M(x,y) = 3x^2y + 2xy + y^3 \\ N(x,y) = x^2 + y^2 \end{cases}$$

$$\begin{cases} M_y = 3x^2 + 2x + 3y^2 \\ N_x = 2x \end{cases} \Rightarrow \text{not exact}$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N} = 3 \rightarrow \text{case 1, function of x alone } \mu(\text{x})$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = 3$$

$$\mu(x) = e^{3x}$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Integration Factors
Special Cases – Example

(New M) $\mu(x)M = e^{3x}(3x^2y + 2xy + y^3) = \tilde{M} \quad \tilde{M}_y = e^{3x}(3x^2 + 2x + 3y^2)$

(New N) $\mu(x)N = e^{3x}(x^2 + y^2) \qquad = \tilde{N}$

$$\psi(x,y) = \int^x \tilde{M}(t,y)dt + \int^y N(x,s)ds - \int^y \left[\int^x M_y(t,s)dt \right] ds$$

$$\psi(x,y) = e^{3x}(3x^2y + y^3) = c$$

Review Ex. 4 P.100

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables

- **Substitution / Change of Variables** - Transforming the differential equation into another differential equation by means of substitution
- Transform the first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

by substituting $$y = g(x,u)$$

where u is regarded as a function of variable x

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{\cancel{dx}}{\cancel{dx}} + \frac{\partial g}{\partial u}\frac{du}{dx} \Rightarrow \frac{dy}{dx} = g_x(x,u) + g_u(x,u)\frac{du}{dx}$$

- The differential equation becomes

$$g_x(x,u) + g_u(x,u)\frac{du}{dx} = f(x, g(x,u))$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Substitution / Change of Variables

- Which solve for $$\frac{du}{dx} = F(x,u)$$

- If we can determine a solution

$$u = \phi(x)$$

than the solution of the original diff eq. is

$$y = g(x, \phi(x))$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables

- **Homogeneous Function (Definition)** - If a function possesses the property

$$f(tx, ty) = t^{\alpha} f(x, y)$$

for some real number α, then f is said to be a homogeneous function of degree α

First Order Non Linear Differential Equations

**Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables**

Homogeneous Function – Example

$$f(x,y) = x^3 + y^3$$

is a homogeneous function of degree 3 since

$$f(tx,ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x,y)$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables

Non - Homogeneous Function – Example

$$f(x,y) = x^3 + y^3 + 1$$

is a non homogeneous function

$$f(tx,ty) = (tx)^3 + (ty)^3 + 1 = t^3(x^3 + y^3) + 1 \neq t^3 f(x,y)$$

$\rightarrow$ Non homogeneous

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Substitution / Change of Variables

- A first order differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be homogeneous if both coefficient function M and N are homogenous function of the same degree

$$M(tx,ty) = t^{\alpha}M(x,y)$$
$$N(tx,ty) = t^{\alpha}N(x,y)$$

Note: the word homogeneous does not mean the same as it may also refer to the linear equation

$$\begin{cases} a_1(x)y' + a_0(x)y = g(x) \\ \quad \text{when } g(x) = 0 \end{cases}$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables

- If M and N are homogeneous functions of degree α, we can also write

$$\begin{cases} M(x,y) = x^{\alpha} M(1,u) \\ N(x,y) = x^{\alpha} N(1,u) \\ \\ where \;\; u = \dfrac{y}{x} \end{cases}$$

and

$$\begin{cases} M(x,y) = y^{\alpha} M(v,1) \\ N(x,y) = y^{\alpha} N(v,1) \\ \\ where \;\; v = \dfrac{x}{y} \end{cases}$$

- using substitution

$$\begin{cases} y = ux \\ x = vy \end{cases}$$

v, u independent variables

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Substitution / Change of Variables

- The equation can be rewritten as

$$x^{\alpha} M(1,u)dx + x^{\alpha} N(1,u)dy = 0$$

or

$$M(1,u)dx + N(1,u)dy = 0$$

where

$$u = \frac{y}{x} \quad \text{or} \quad y = ux$$

- By substituting the differential $dy = udx + xdu$

into the last equation and gathering terms we obtain a separate differential equation in the variable u and x

$$M(1,u)dx + N(1,u)\left[udx + xdu\right] = 0$$

$$\left[M(1,u) + uN(1,u)\right]dx + xN(1,u)du = 0$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables

$$\frac{dx}{x}+\frac{N(1,u)du}{M(1,u)+uN(1,u)}=0$$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. –
Substitution / Change of Variables – Example

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0$$

$$\begin{cases} M(tx,ty) = (tx)^2 + (ty)^2 \Rightarrow t^2(x^2 + y^2) = t^2 M(x,y) \\ N(tx,ty) = (tx)^2 - txty \Rightarrow t^2(x^2 - xy) = t^2 N(x,y) \end{cases}$$

$$M, N \rightarrow \text{Homogeneous function of degree } 2$$

- Let $y = ux$
- Then $dy = udx + xdu$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Substitution / Change of Variables

- After substitution the given equation becomes

$$\underbrace{(x^2+u^2y^2)}_{M(x,ux)}dx+\underbrace{(x^2-ux^2)}_{N(x,ux)}\underbrace{[udx+xdu]}_{dy}=0$$

$$x^2(1+u)dx+x^3(1-u)du=0$$

- Long division $\quad \dfrac{1-u}{1+u}du+\dfrac{dx}{x}=0$
- Integration $\quad \left[-1+\dfrac{2}{1+u}\right]du+\dfrac{dx}{x}=0$

$$-u+2\ln|1+u|+\ln|x|=\ln|c|$$

- Re-substitute $\quad u=\left(\dfrac{y}{x}\right)$

First Order Non Linear Differential Equations

Type 3 - Convert Non Exact to Exact Diff. Eq. – Substitution / Change of Variables

$$-\frac{y}{x}+2\ln\left|1+\frac{y}{x}\right|+\ln|x|=\ln|c|$$

$$\ln\left(\left|\frac{x+y}{x}\right|\right)^2+\ln|x|-\ln|c|=\frac{y}{x}$$

$$\ln x\left|\frac{x+y}{x}\right|^2-\ln|c|=\frac{y}{x}$$

$$\ln\left|\frac{\cancel{x}\dfrac{(x+y)^2}{x^{\cancel{2}}}}{\dfrac{c}{1}}\right|=\frac{y}{x}$$

$$\ln\left|\frac{(x+y)^2}{cx}\right|=\frac{y}{x}$$

$$(x+y)^2=cxe^{\frac{y}{x}}$$