Class Notes 2:

First Order Differential Equation – Linear

MAE 82 – Engineering Mathematics

Introduction – First Order Differential Equations

- First Order Differential Equation

$$\frac{dy}{dt} = f(t, y)$$

- Solution

$$y = \phi(t)$$

- Any differentiable function that satisfy this equation for all *t* in some interval is called a solution

- Objective
  - Determine whether such function exist
  - Develop Methods for funding them
    - Linear equations (section 2.1)
    - Separable equations (section 2.2)
    - Exact equations (section 2.6)

Classification

FIRST ORDER DIFF. EQ.

LINEAR EQ

TYPE 1

CONST

$$\frac{dy}{dt} + q\,y = g(t)$$

SOLUTION : INTEGRATION FACTOR

TYPE 2

NON CONST

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

SOLUTION : VERIATION OF RARMETERS

TYPE 3

$$\frac{dy}{dt} = f(y)$$

AUTONOMOUS

NON LINEAR EQ.

TYPE 1 SEPERABLE EQ.

$$M(x)\,dx + N(y)\,dy = 0$$

SOLUTION : INTEGRATION

TYPE 2 EXACT EQ.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

SOLUTION : INTEGRATIO

TYPE 3 CONVERT NON EXACT TO AN EXACT

SOLUTION: INGERATION FACTORS

General Form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

GIVEN P, Q, G AND $P(t) \neq 0$

VERIABLE COEFFICIENT

$$\frac{dy}{dt} + p(t)y = g(t)$$

CONSTANT COEFFICIENTS

$$\frac{dy}{dt} + ay = b$$

Example 1 – Integration

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

LEFT SIDE → Linear combination of dy/dt and y

CALCULUS → Differentiation a product

$$\frac{d}{dt}\Big[\underbrace{(4+t^2)}_{g}\,\underbrace{y}_{h}\Big] = \underbrace{(4+t^2)}_{g}\,\underbrace{\frac{dy}{dt}}_{h'} + \underbrace{2t}_{g'}\,\underbrace{y}_{h}$$

Rewrite the equation

$$\frac{d}{dt}\Big[(4+t^2)y\Big] = 4t$$

Integrate both sides with respect to t

Example 1 – Integration

$$(4+t^2)y = 2t^2 + C$$

Arbitrary constant of integration

$$y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$$

NOTE – Special Case – The left hand side of the eq. is a derivative of a product of y and some other function

Integration Factors Method

- For a constant a

CONSTANT

$$\frac{dy}{dt} + ay = g(t)$$

- If the diff eq. is multiplied by a certain function $\mu(t)$
- The diff. eq. is converted into one that is immediately integrable by using the product rule of derivations

$$(gh)' = gh' + g'h$$

Integration Factors Method

$$\frac{dy}{dt} + \underset{\text{const}}{a}\, y = g(t)$$

a - Given Constant

$g(t)$ - Given Function

- The integration factor must setisfy

$$\frac{d}{dt}\left[\mu(t)\, y\right] \equiv \overbrace{\boxed{\mu \frac{dy}{dt}} + \frac{d\mu}{dt}\, y}^{\text{By Defenition}} = \overbrace{\boxed{\mu \frac{dy}{dt}} + \mu a y}^{\text{Left hand side of the } \mu(t)}$$

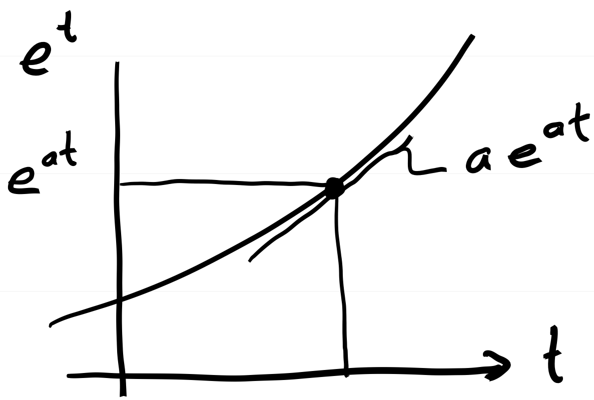
Integration Factors Method

$$\frac{dy}{dt} = a\mu$$

- The integration factor is

$$\mu(t) = e^{at}$$

- What makes e^{at} so special



$$y = e^{at}$$

$$y' = ae^{at}$$

$$y' = ay$$

The slope

Value of the function $\times a$

Integration Factors Method

- Multiply the diff eq. with the integration factor $\mu(t) = e^{at}$

$$\mu(t)\left[\frac{dy}{dt} + ay\right] = \mu(t)\, g(t)$$

$$\underbrace{e^{at}}_{g}\,\underbrace{\frac{dy}{dt}}_{h'} + \underbrace{a e^{at}}_{g'}\,\underbrace{y}_{h} = e^{at} g(t)$$

$$\frac{d}{dt}\Big(\underbrace{e^{at}}_{g}\,\underbrace{y}_{h}\Big) = e^{at} g(t)$$

Integration Factors Method

Integrate both sides of the diff eq.

$$e^{at}y = \int e^{at} g(t) + c$$

$$\boxed{y = e^{-at}\int_{t_0}^{t} e^{as} g(s)\, ds + c\, e^{-at}}$$

Integration Factors Method – Example 3

$$\frac{dy}{dt} - 2y = 4 - t$$

$$\frac{dy}{dt} + a\,y = g(t)$$

$$\mu(t) = e^{-2t}$$

$$\underbrace{e^{-2t}}_{g}\ \underbrace{\frac{dy}{dt}}_{h'} - \underbrace{2e^{-2t}}_{g'}\ \underbrace{y}_{h} = 4e^{-2t} - te^{-2t}$$

Integration Factors Method – Example 3

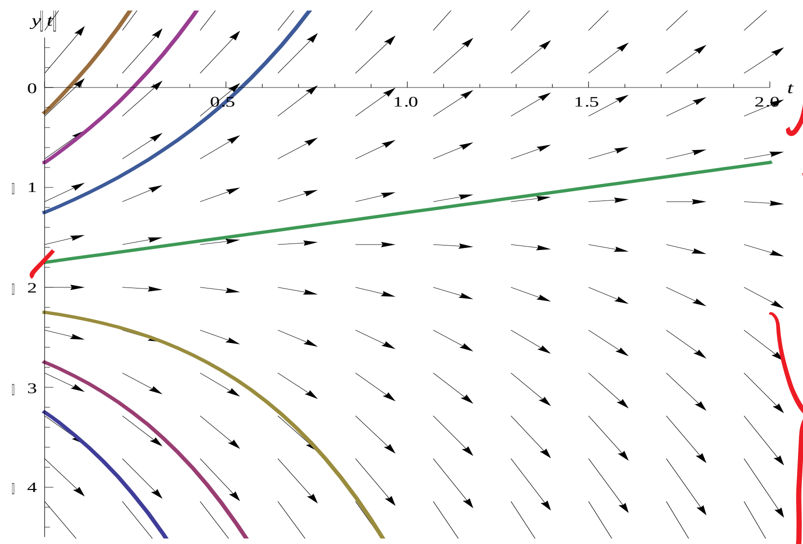
$$e^{-2t}y = \int 4e^{-2t} - te^{-2t}\, dt$$

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c$$

$\therefore$

Integration Factors Method – Example 3

$$y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t}$$



$C > 0$

$C = 0$

$y = -\frac{7}{4} + \frac{1}{2}t$

$C < 0$

$-\frac{7}{4}$

- Review Examples 4 p.37, 5 p.38

Variation of Parameters Method

$$\frac{dy}{dt} + p(t)y = g(t)$$

For function $p(t)$ <u>not</u> constant

- Multiply the diff. eq. by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t)$$

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y = \mu\frac{dy}{dt} + \mu p y$$

Variation of Parameters Method

- Condition to be met

$$\frac{d\mu}{dt} = \mu P$$

- Seperation of veriable

$$\frac{d\mu}{\mu} = p\,dt$$

- Integrating both sides

$$\int \frac{1}{\mu}\,d\mu = \int p\,dt$$

$$\ln|\mu(t)| = \int p(t)\,dt + C$$

- Integration factor

$$\mu(t) = e^{\int p(t)\,dt}$$

Variation of Parameters Method

$$\underbrace{e^{\int p(t)\,dt}}_{g}\underbrace{\frac{dy}{dt}}_{h'} + \underbrace{p(t)\,e^{\int p(t)\,dt}}_{g'}\underbrace{y}_{h} = e^{\int p(t)\,dt}\,g(t)$$

$$\frac{d}{dt}\left[e^{\int p(t)\,dt}\,y\right] = e^{\int p(t)\,dt}\,g(t)$$

$$\boxed{y = e^{-\int p(t)\,dt}\int e^{\int p(t)\,dt}\,g(t)\,dt + c\,e^{-\int p(t)\,dt}}$$

Example - Variation of Parameters Method

$$\frac{dy}{dt}\boxed{-2t}y = 2 \qquad y(0) = 1$$

$p(t)$

$$\mu(t) = e^{\int p(t)dt} = e^{\int -2t\,dt} = e^{-t^2}$$

$-\frac{2t^2}{2}$

$$\underbrace{e^{-t^2}}_{g}\underbrace{\frac{dy}{dt}}_{h'} - \underbrace{2te^{-t^2}}_{g'}\underbrace{y}_{h} = 2e^{-t^2}$$

$$\frac{d}{dt}\left[e^{-t^2}y\right] = 2e^{-t^2}$$

$$y = 2e^{t^2}\boxed{\int_0^t e^{-s^2}ds} + ce^{t^2}$$

$\rightarrow$ KNOWN INTEGRAL

Example - Variation of Parameters Method

$$y = 2e^{t^2}\Big(\tfrac{1}{2}\pi \,\mathrm{erf}(t)\Big) + ce^{t^2}$$

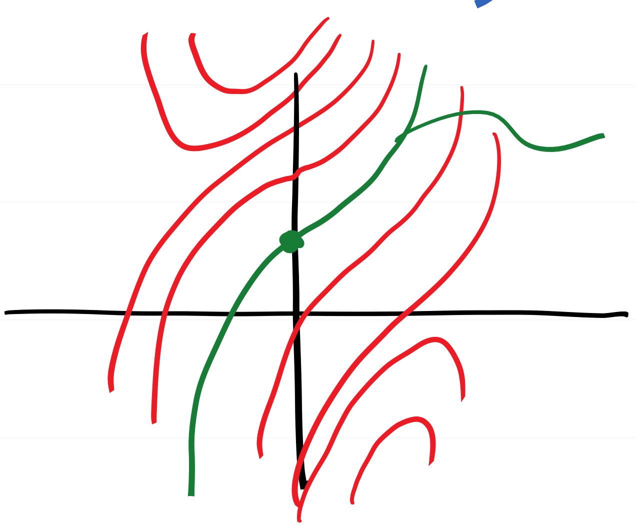
ERROR FUNCTION

$t=0$

$= 0$

IC $\quad 1 = c\overbrace{e^{0}}^{1} \Rightarrow c = 1$

$$y = e^{t^2}\left[1 + \sqrt{\pi}\,\mathrm{erf}(x)\right]$$



SELECTED SOLUTION GIVEN THE IC

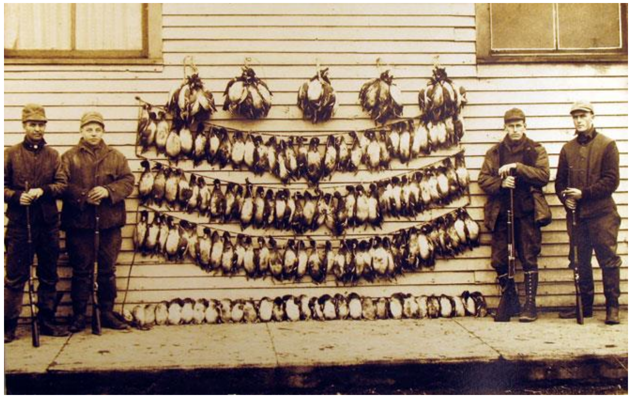
- Review also example 4 p.37

Autonomous Equation – Population Dynamics

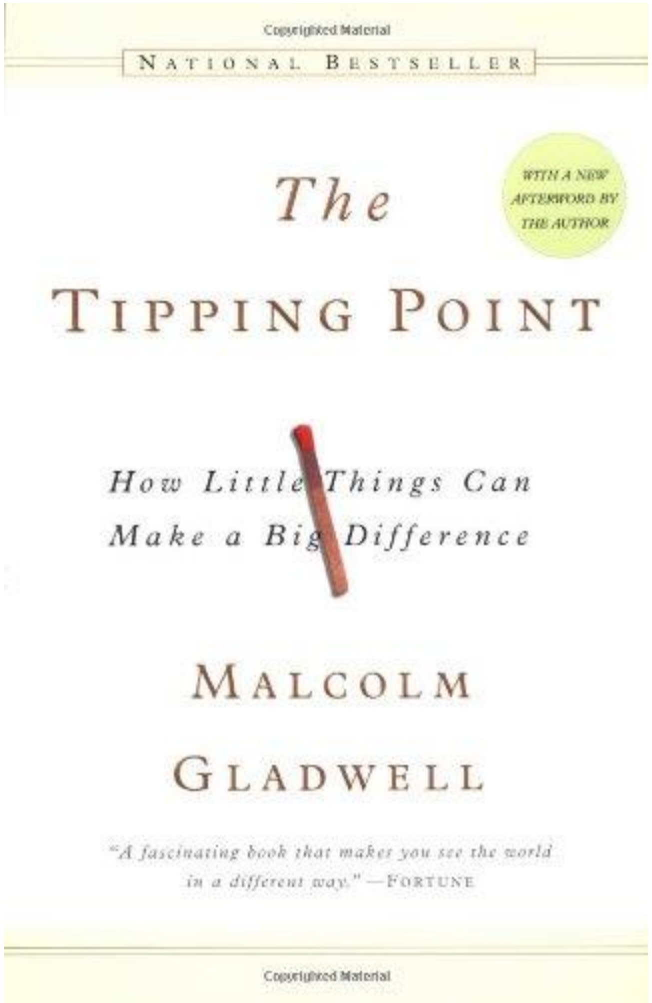
Passenger Pigeon

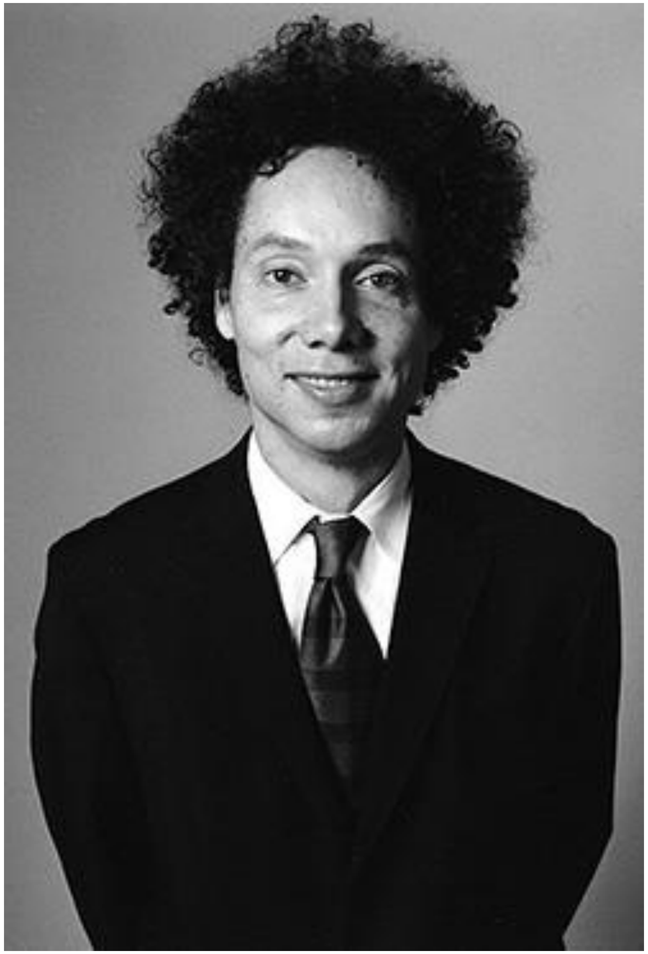
The passenger pigeon or wild pigeon (Ectopistes migratorius) is an extinct North American bird. Named after the French word *passager* for "passing by", it was once the most abundant bird in North America, and possibly the world. It accounted for more than a quarter of all birds in North America. The species lived in enormous migratory flocks until the early 20th century, when hunting and habitat destruction led to its demise.

Some estimate 3 to 5 billion passenger pigeons were in the United States when Europeans arrived in North America. Some reduction in numbers occurred from habitat loss when European settlement led to mass deforestation. Next, pigeon meat was commercialized as a cheap food for slaves and the poor in the 19th century, resulting in hunting on a massive and mechanized scale. A slow decline between about 1800 and 1870 was followed by a catastrophic decline between 1870 and 1890. Martha, thought to be the world's last passenger pigeon, died on September 1, 1914, at the Cincinnati Zoo.



Autonomous Equation – Population Dynamics





Autonomous Equation – Population Dynamics Introduction

- **Autonomous** Equations - Class of first order differential equations where the independent variable *t* does not appear explicitly.

$$\frac{dy}{dt} = f(y)$$

- Applications
  - Growth/Decline of population
  - Medicine
  - Ecology
  - Global Economics

- Stability / Instability of the solution

Autonomous Equation – Population Dynamics Introduction

- Types of differential equations
  - Exponential Growth $$\frac{dy}{dt} = ry$$
  - Logarithmic Growth $$\frac{dy}{dt} = +r\left(1-\frac{y}{K}\right)y$$
  - Logarithmic Growth with Critical Threshold $$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y$$
  - Logarithmic Growth with Threshold $$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$$

Autonomous Equation – Population Dynamics
Exponential Growth

- The population of a given species at time t

$$y = \phi(t)$$

- The rate of change of the population y is proportional to the current population (value of the y) – Thaoms Maltus – British Economist 1766-1834

$$\boxed{\frac{dy}{dt}} = \boxed{r}\,\boxed{y}$$

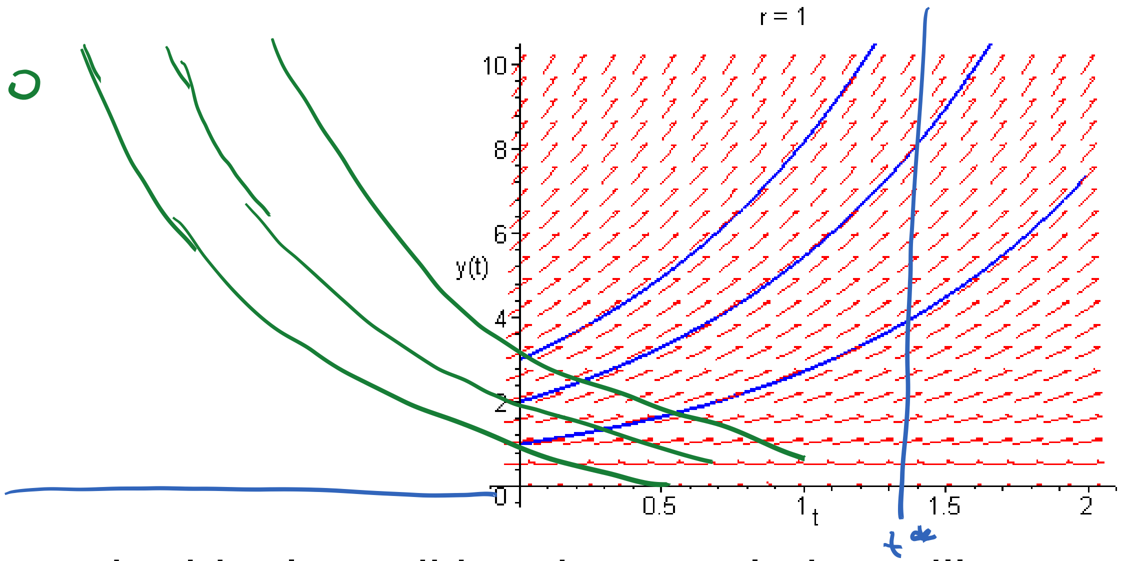
- r – The rate of growth (r>0) or decline (r<0)
- Solving the differential equation subject to initial condition

$$y(0) = y_0$$

$$y = y_0 e^{rt}$$

Autonomous Equation – Population Dynamics
Exponential Growth

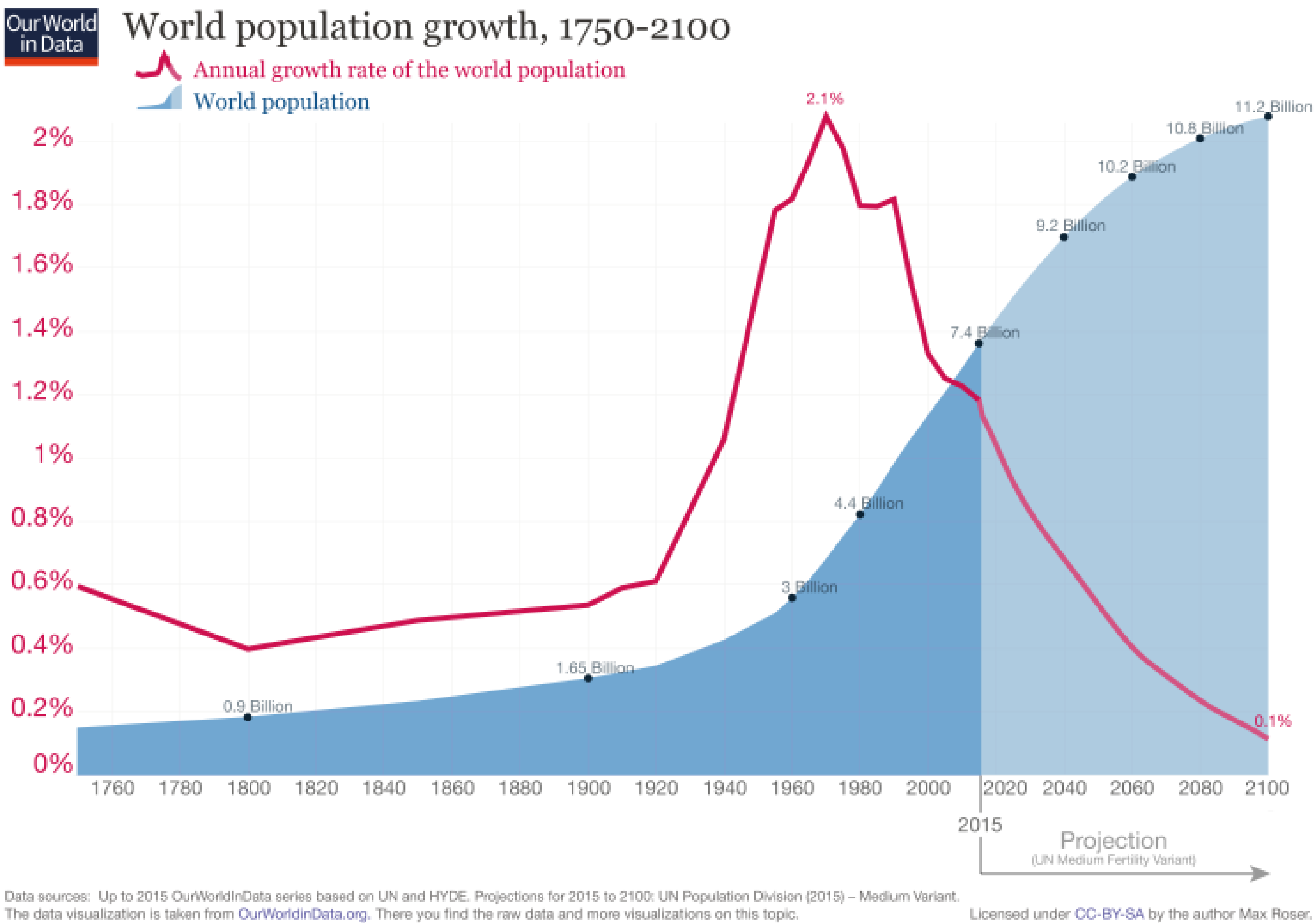
$r<0$ $r>0$



- Under ideal condition the population will grow exponanatially
- Valid during a limited period of time
- Limitation
  - Space
  - Food supplies
  - Limited resources

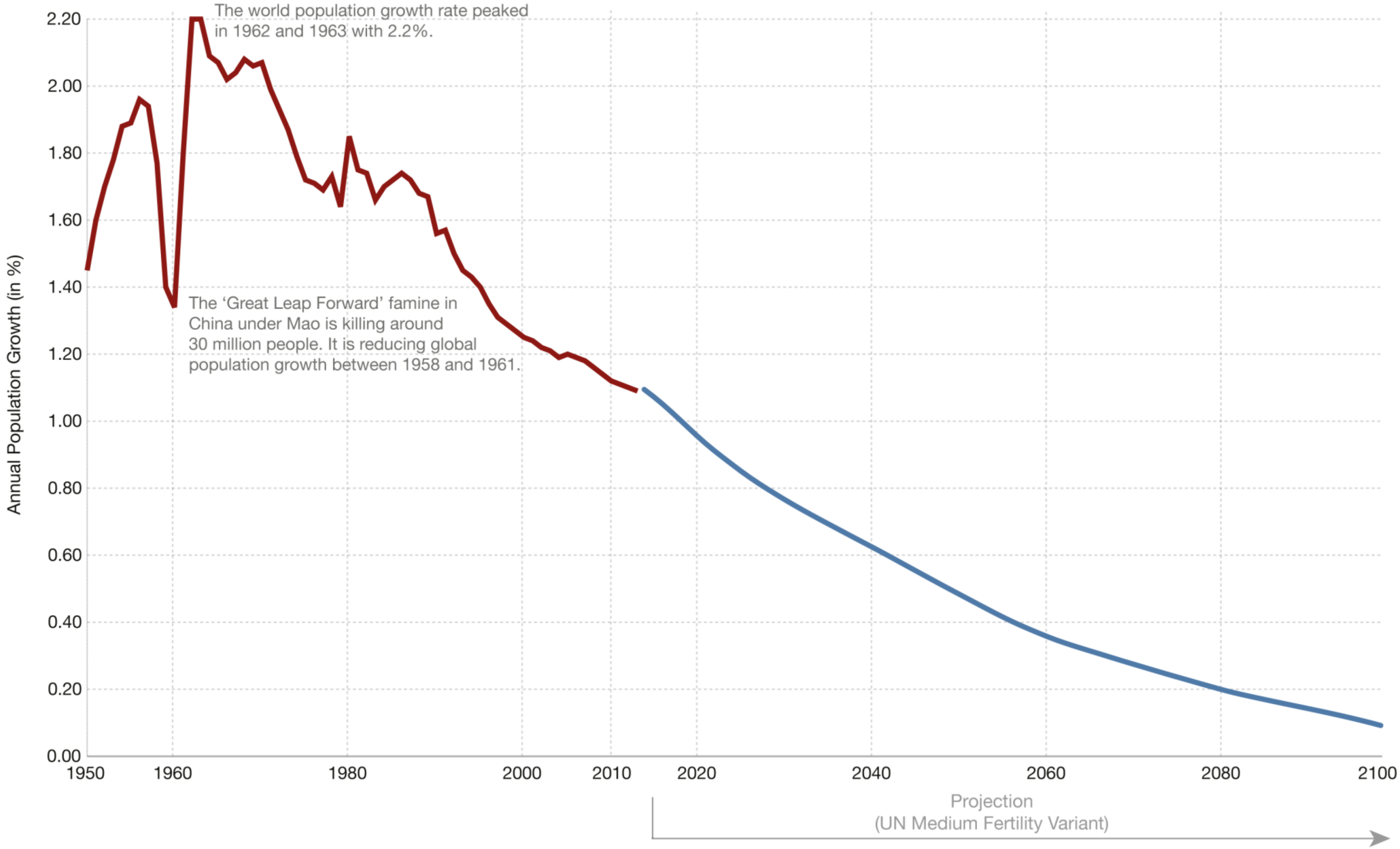
$t: 0 \to t^*$

$t \to \infty \quad y(t) \to \infty$



https://ourworldindata.org/world-population-growth

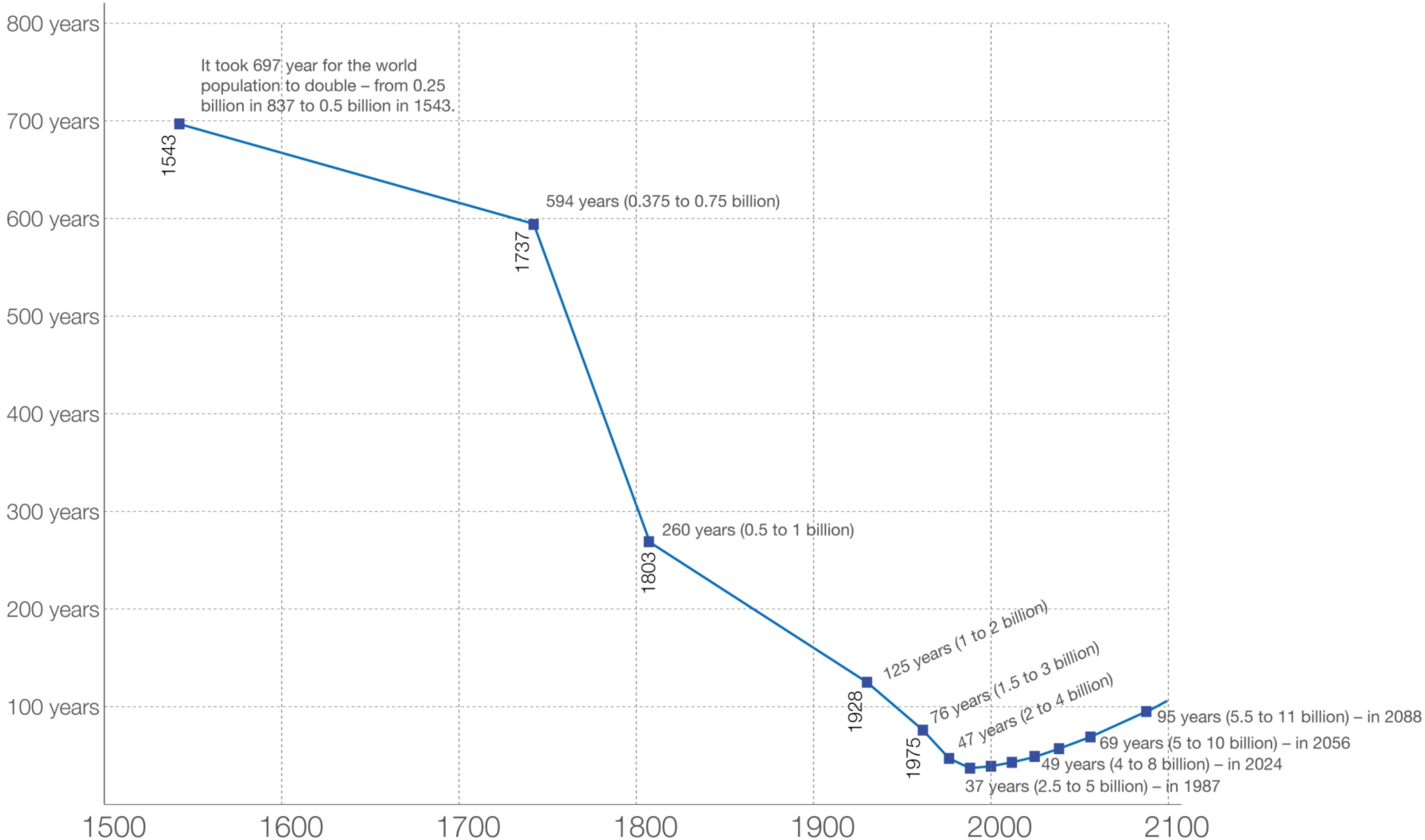
Annual world population growth rate (1950-2100)



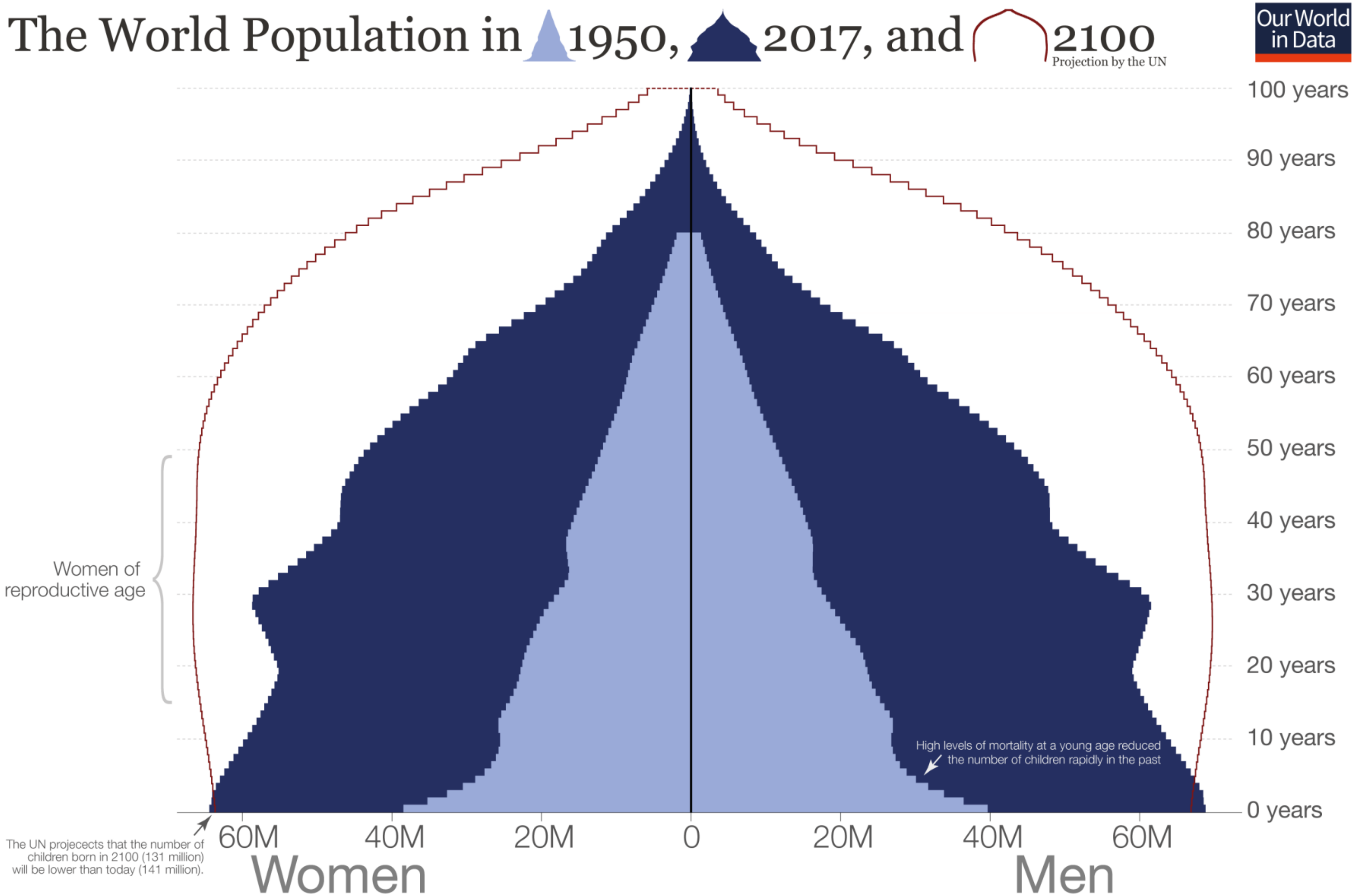
Data sources: Observations: US Census Bureau & Projections: United Nations Population Division (Medium Variant (2015 revision).
The interactive data visualization is available at OurWorldinData.org. There you find the raw data and more visualizations on this topic.

Time it took for the world population to double

Historical estimates of the world population until 2015 – and UN projections until 2100



Data source: OurWorldInData annual world population series (Based on HYDE and UN until 2015. And projections from the UN after 2015 ('Medium Variant' 2015 Revision).
The data visualization is available at OurWorldinData.org. There you find the raw data, more visualizations, and research on this topic.



Data source: United Nations – World Population Prospects 2015. Data in 1-year-brackets is only available up to the age of 100 years in 2017 and 2100 and only up to 80 years in 1950.
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Autonomous Equation – Population Dynamics

Logistic Growth

- The growth rate depends on the population
- Replace $r \to h(y)$

$\leftarrow$ Function of the population

$$\frac{dy}{dt} = h(y)\, y$$

- choose $h(y)$ so that

$$h(y) = (r - ay) \quad \left\{ \begin{array}{ll} h(y) \simeq r > 0 & \text{when } y \text{ is small} \\ h(y) \downarrow & \text{Decrease as } y \text{ grow} \\ h(y) < 0 & \text{when } y \text{ is sufficient large} \end{array} \right.$$

$a > 0$

$\hookrightarrow$ const.

Autonomous Equation – Population Dynamics

Logistic Growth

$$\frac{dy}{dt} = (r - ay)y$$

- Verhulst Equation or Logistic Equation
- Pierre F. Verhulst (1804–1849) was a Belgaim mathematician who introduce this equation as a model for human population growth is 1838
-

Autonomous Equation – Population Dynamics
Logistic Growth

- Equivalent Form

$$\frac{dy}{dt} = \underbrace{r\left(1 - \frac{y}{K}\right)}\,y$$

$$\hookrightarrow \left(r - \boxed{\frac{r}{K}}\,y\right)y$$

$a = \frac{r}{K}$

- r – Intrinsic growth rate – The growth rate in the absents of any limitation
- K – Saturation level – Enviromental carrying capasity

Autonomous Equation – Population Dynamics

Logistic Growth

- Investigate the solution for

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

- Assume $y(t) = \text{CONSTANT} \Rightarrow \frac{dy}{dt} = 0$

$$\underbrace{r\left(1 - \frac{y}{K}\right)}_{=0} \;\text{OR}\; \underbrace{y}_{=0} = 0$$

- solutions $\left\{\begin{array}{l} y = \phi_1(t) = 0 \\ y = \phi_2(t) = K \end{array}\right] \Rightarrow$ Equilibrium Solutions

Autonomous Equation – Population Dynamics

Logistic Growth

- In $\frac{dy}{dt} = f(y)$ when $f(y) = 0$

The zeros of $f(y)$ are also called critical points

Autonomous Equation – Population Dynamics

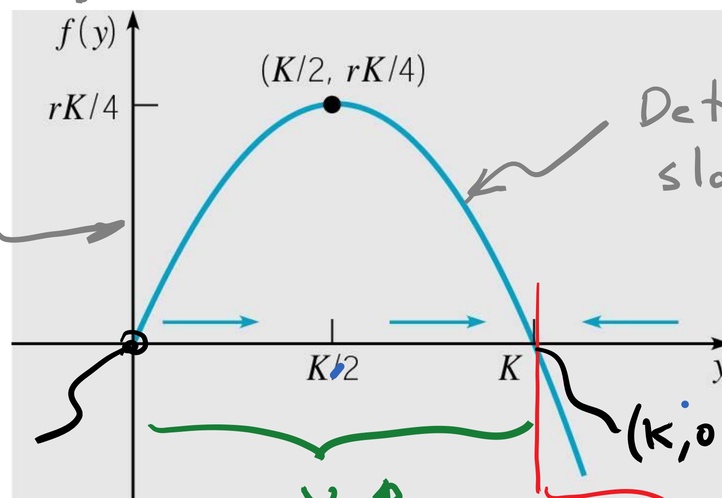
Logistic Growth

parabola

slope → $f(y) = r\left(1-\dfrac{y}{K}\right)y = \left(-\dfrac{r}{k}\right)y^2 + (r)y$

Axis y is called the phase line

Definition of the slope $f(y) = \dfrac{dy}{dt}$



(0,0) (K,0)

$y \uparrow$ $y \downarrow$

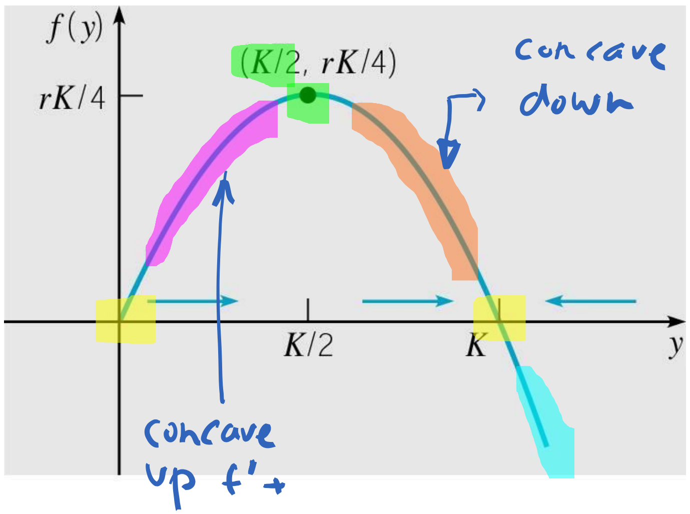
positive chang $dy/dt > 0$ for $0 < y < k$

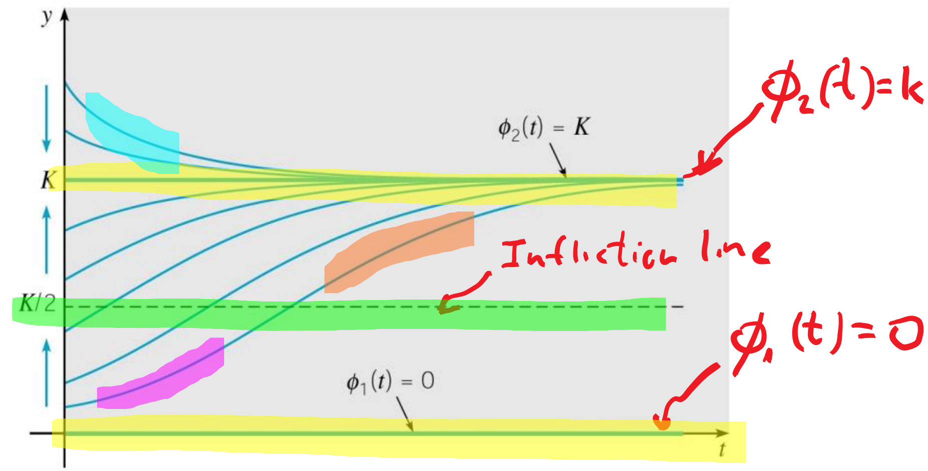
Negative chang $dy/dt < 0$ for $y > K$

Autonomous Equation – Population Dynamics

Logistic Growth

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y = f(y)$$





- Near 0 or k the slope is close to zero
- Inflection point - Location $d^2y/dt^2 = 0$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y) = 0$$

Autonomous Equation – Population Dynamics

Logistic Growth

- Exact solution

$$\frac{dy}{(1 - y/k)y} = r\,dt$$

- Partial fraction expension

$$\left(\frac{1}{y} + \frac{1/k}{1 - y/k}\right) dy = r\,dt$$

- Integrate both sides

$$\ln|y| - \ln\left|1 - \frac{y}{k}\right| = rt + c$$

Autonomous Equation – Population Dynamics

Logistic Growth

- C: Arbitrary constant to be determined by $y(0) = 0$
- If $0 < y_0 < k$, y remains in this interval for all time $\rightarrow$ remove the $|\,\,|$

$$e^{\ln(y) - \ln\left(1 - \frac{y}{k}\right)} = e^{rt + c}$$

$$\frac{e^{\ln y}}{e^{\ln\left(1-\frac{y}{k}\right)}} = \underbrace{e^{c}}_{\tilde{c}} e^{rt}$$

$$\frac{y}{1 - \frac{y}{k}} = \tilde{c} e^{rt}$$

Autonomous Equation – Population Dynamics
Logistic Growth

- For $y(0) = y_0 \Rightarrow \quad \tilde{c} = \dfrac{y_0}{1 - \frac{y_0}{k}}$

$$\frac{y}{\left(1 - \frac{y}{k}\right)} = \frac{y_0}{\left(1 - \frac{y_0}{k}\right)} e^{rt}$$

$$y = \frac{y_0 k}{y_0 + (k - y_0) e^{-rt}} \quad \begin{cases} 0 < y < k \\ y > k \end{cases}$$

$$\lim_{t \to \infty} y(t) = \frac{y_0 k}{y_0} = k$$

Autonomous Equation – Population Dynamics

Logistic Growth

- Notes

- $\phi_2(t) = K$ Asymptotically stable →
  Solution equlibrium
  Critical point

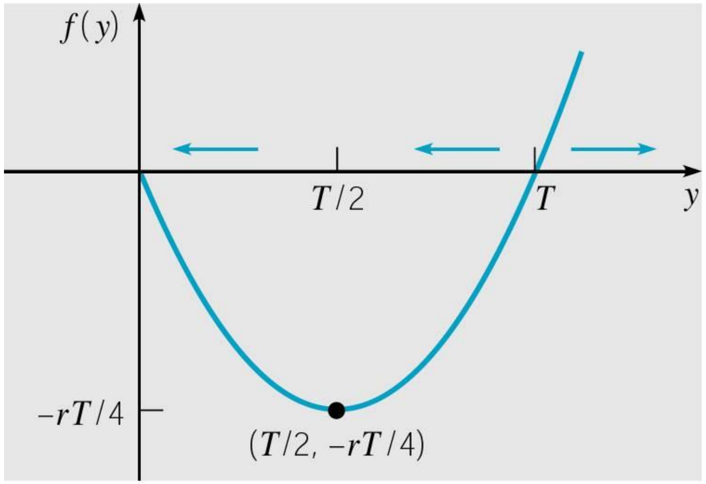
- As t increases the solution approuch the equlibrium solution more rapidly

- Problem: Even solutions that start close to zero grow as t increases and approuch

Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

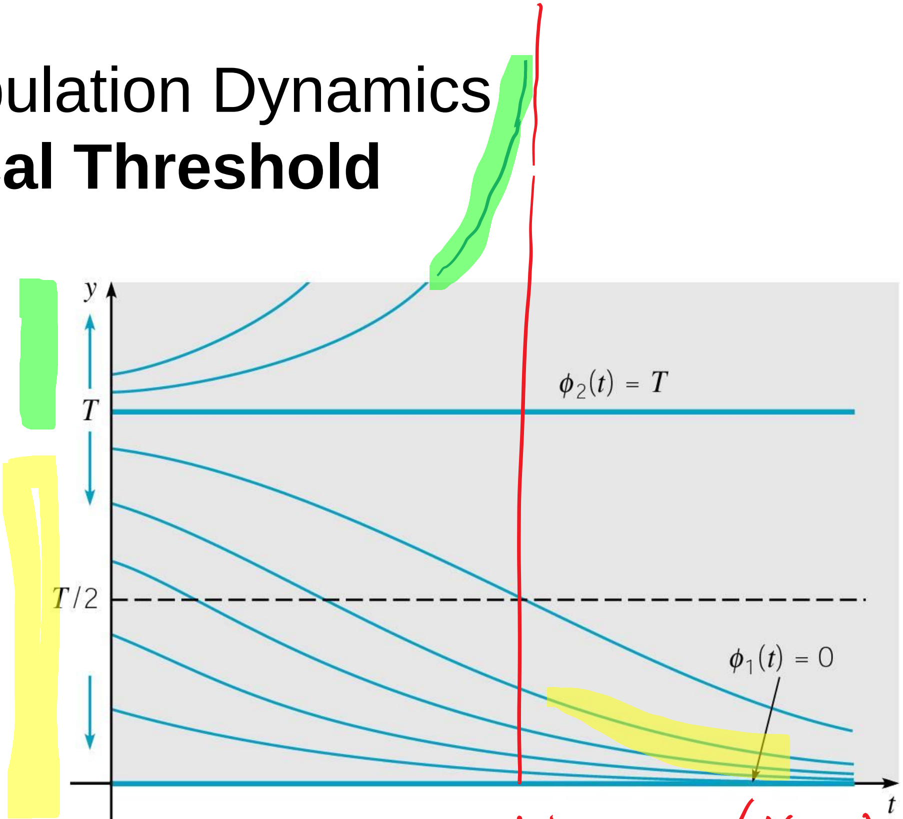


- Note - In order to obtain the exact solution for this case replace $+r$ with $-r$ in the previous solution

Autonomous Equation – Population Dynamics
Logistic Growth with Critical Threshold

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y, \quad r>0$$

$$(*) \quad y = \frac{y_0 T}{y_0 + (T-y_0)e^{rt}}$$



$t^* = \frac{1}{r}\ln\left(\frac{y_0}{y_0-T}\right)$

- If $0 < y_0 < T$ $\quad y \to 0$ $\quad t \to \infty$
- If $y_0 > T$ - The denominator of the right hand side of the eq $(*)$ is zero for a certain finite value of t^* when $y_0 + (T-y_0)e^{rt^*} = 0$

Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

- Denominator of $y(t)$ is set to 0

$$y_0 + (T - y_0)e^{rt^*} = 0$$

$$y_0 - (y_0 - T)e^{rt^*} = 0$$

$$e^{rt^*} = \frac{y}{y_0 - T}$$

$$\ln\left(e^{rt^*}\right) = \frac{y_0}{y_0 - T}$$

$$t^* = \frac{1}{r}\,\frac{y_0}{y_0 - T}$$

Autonomous Equation – Population Dynamics
Logistic Growth with Critical Threshold

- Note 1:

  - If the initial population $\mathbf{y_0}$ is above the threshold **T** the graph of y(t) has a vertical asymptote at t*

  - The population become unbounded in a finite time whose value depends on $\mathbf{y_0}$, **T** and **r**

Autonomous Equation – Population Dynamics
Logistic Growth with Critical Threshold

- Note 2: The population of Species exhibit the threshold phenomena if

  - (Below the threshold) Too few subjects are present, then the species can not propagate itself successfully and the population become extinct
  - (Above the threshold) further growth occurs

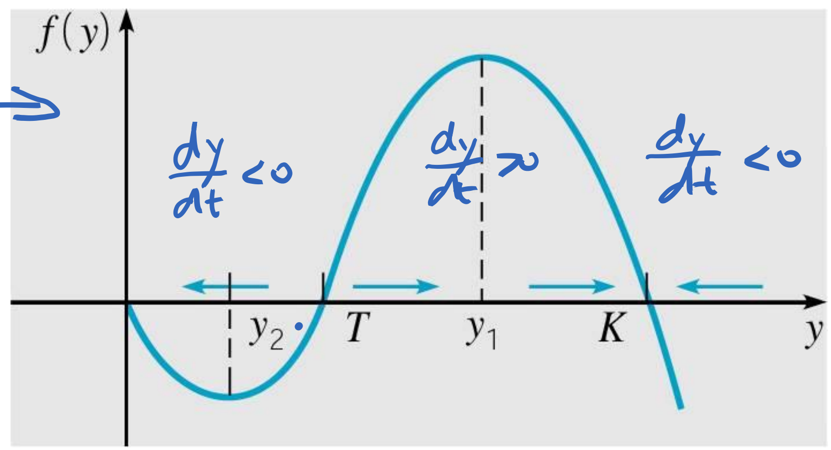
- Note 3: The population cant become unbounded

Autonomous Equation – Population Dynamics

Logarithmic Growth with Threshold

- Correct the model such as unbounded growth will not occure when y is above the thershold T

$$\frac{dy}{dt} = \overbrace{-r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y}^{f(t)}, \quad r>0 \text{ and } 0<T<K$$

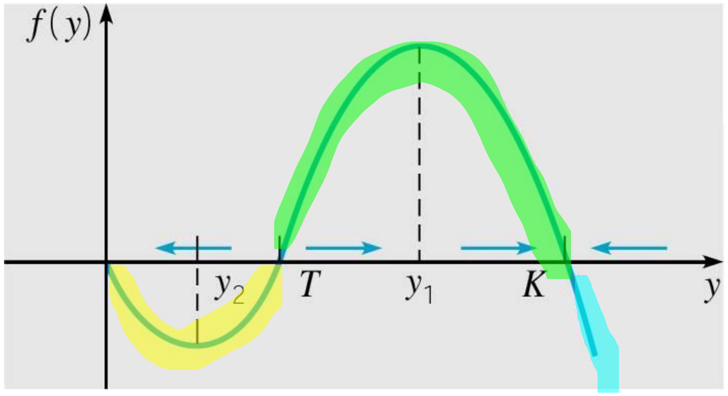


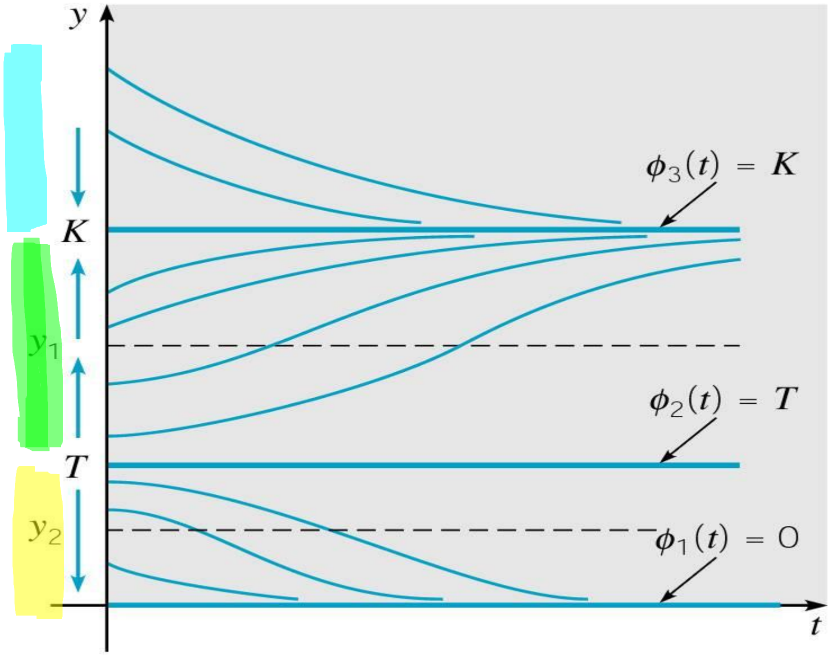
- Three critical points

$$\begin{cases} y = 0 \\ y = T \\ y = K \end{cases}$$

Autonomous Equation – Population Dynamics

Logarithmic Growth with Threshold





Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

- Notes
  - Passenger Pigeons were present in the US in vast numbers until the 19th century
  - It was heavily hunted for food and sport and reduced significantly 1880
  - Breed successfully only when present in a large concentration i.e. high number of **T**

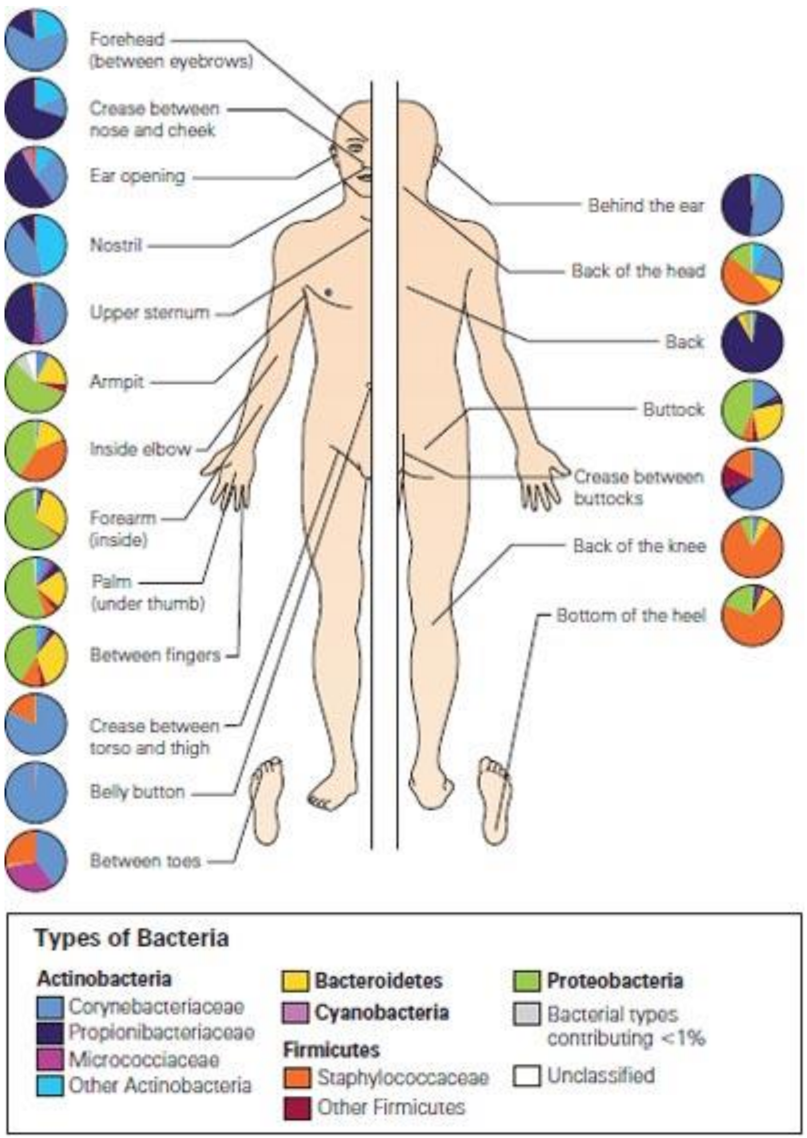
Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

- Notes
  - Large number of individual birds remained alive in the late 1880s. However there were not enough in any one place to permit successful breeding.
  - The population decline to extinction
  - Last survivor died in 1914
  - Solution – Conservation

Linear Model – Bacterial Growth

Linear Model – Bacterial Growth



Linear Model – Bacterial Growth

- Given:
  - A culture initially has $\mathbf{P_0}$ number of bacteria
  - At t=1hr the number of bacteria is measured to be $\mathbf{3/2P_0}$
  - The rate of growth is propositional to the number of bacteria **P(t)** present at the time **t**
- Calculated
  - The time necessary for the population to triple

Linear Model – Bacterial Growth

- Rate of growth is proportional to the number of bacteria $p(t)$

$$\frac{dp}{dt} = kp$$

$$\frac{dp}{dt} - kp = 0 \longrightarrow$$ seperable & linear ODE

- Emperical observation

$$\begin{cases} p(t_0 = 0) = P_0 \\ p(t = 1) = \frac{3}{2} P_0 \end{cases}$$

Linear Model – Bacterial Growth

$$\mu = e^{-kt}$$

$$\underbrace{e^{-kt}}_{g} \underbrace{\frac{dp}{dt}}_{h} - \underbrace{k e^{-kt}}_{g'} \underbrace{p}_{h} = 0$$

$$\frac{d}{dt}\left[e^{-kt} \cdot p \right] = 0$$

Integrating both sides $\quad e^{-kt} \cdot p = c$

$$p(t) = c e^{+kt}$$

Linear Model – Bacterial Growth

At $t = 0$

$$p(t=0) = c\, e^{+k0} = P_0 \Rightarrow c = P_0$$

$$p(t) = P_0\, e^{+kt}$$

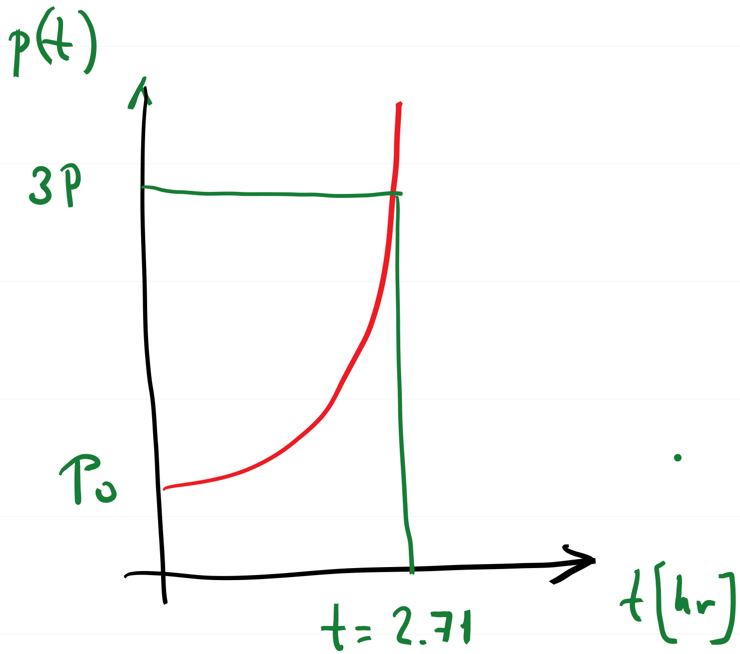
At $t = 1h$

$$p(t=1) = \cancel{P_0}\, e^{+k1} = \frac{3}{2}\cancel{P_0}$$

$$e^{+k} = \frac{3}{2} \Rightarrow k = \ln\frac{3}{2} = 0.4055$$

Linear Model – Bacterial Growth

$$p(t) = P_0 e^{0.4055t}$$



- Find the time at which the number of bacteria has tripled

Linear Model – Bacterial Growth

$$3\cancel{P_0} = \cancel{P_0} e^{0.4055t}$$

$$\ln 3 = 0.4055t$$

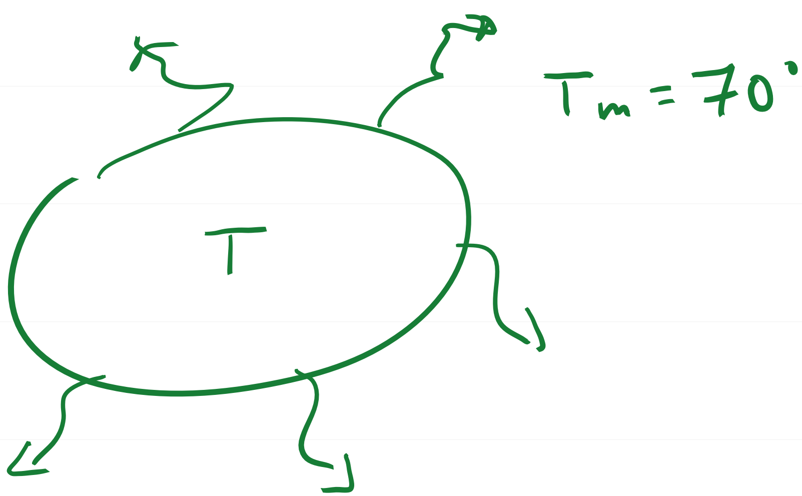
$$t = \frac{\ln 3}{0.4055} = 2.71 \text{ hr}$$

Note: The intial number of bacteria at $t=0$ P_0 played no part in determening the time it takes for a population to tripled. The intial population could be 100 or 100,000

Linear Model – Cooling/Warming – Newton's Law

Linear Model – Cooling/Warming – Newton's Law

The rate at which the temperature of a body changes is proportional to the differance between the temperature of the body (T) and the temperature of the surrounding medium - ambient temperature (T_m)



$$\frac{dT}{dt} \propto T - T_m$$

$$\frac{dt}{dt} = K(T - T_m)$$

Linear Model – Cooling/Warming – Newton's Law

$T(t=0) = 300°F$

$T(t=3\text{ min}) = 200°F$

$t = ? \longrightarrow T = 70°F$

$$\frac{dT}{dt} = k(T-70) \qquad \begin{cases} T(0) = 300 \\ T(3) = 200 \end{cases}$$

$$\frac{dT}{T-70} = k\,dt$$

$$\ln|T-70| = kt + c_1$$

Linear Model – Cooling/Warming – Newton's Law

$$e^{\ln|T-70|} = e^{kt+c_1} = e^{kt}\underbrace{e^{c_1}}_{c_2}$$

$$(T-70) = c_2 e^{kt}$$

$$T = 70 + c_2 e^{kt}$$

$$T(t=0) = 300 \longrightarrow 300 = 70 + c_2 \underbrace{e^{k\cdot 0}}_{1} \Rightarrow c_2 = 230$$

$$T = 70 + 230 e^{kt}$$

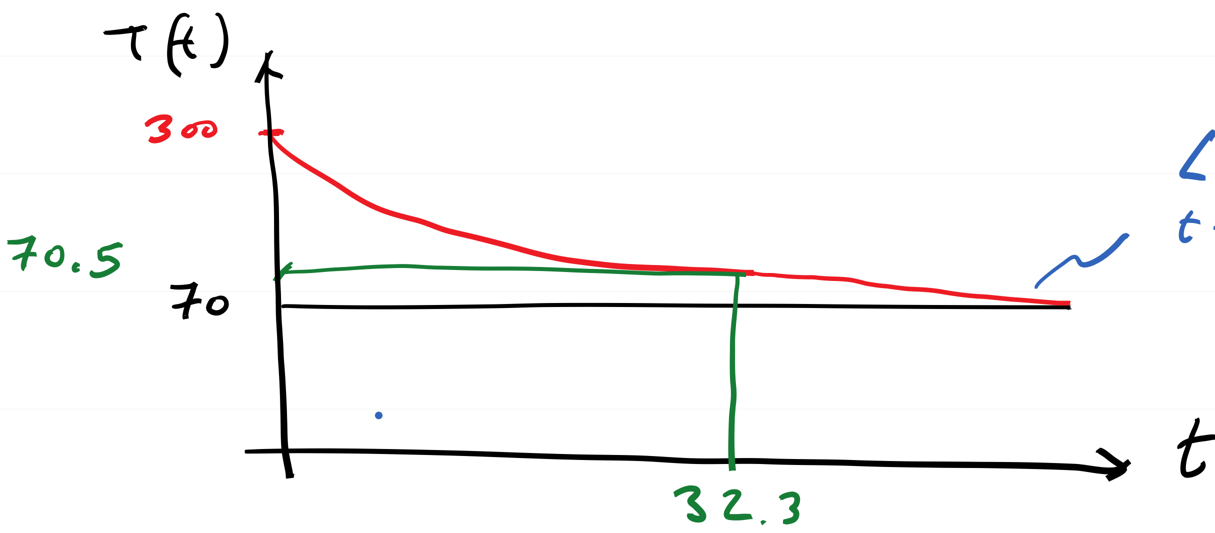
$$T(t=3) = 200 \longrightarrow 200 = 70 + 230 e^{3k}$$

Linear Model – Cooling/Warming – Newton's Law

$$e^{3k} = \frac{13}{23}$$

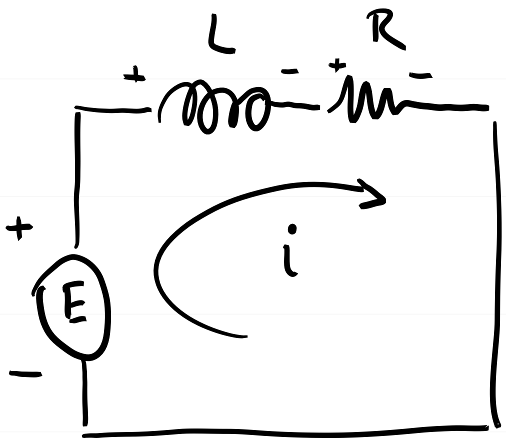
$$k = \frac{1}{3} \ln\left(\frac{13}{23}\right) = -0.19018$$

$$T(t) = 70 + 230\, e^{-0.19018t}$$



$$\lim_{t\to\infty} T(t) = 70$$

Linear Model – Series Circuits



KIRCHOFF LAW

$$\sum V = 0$$

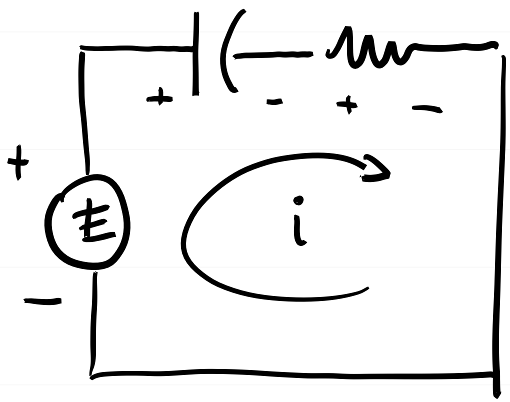
close circuite

$$-E(t) + L\frac{di}{dt} + Ri = 0$$

$$L\frac{di}{dt} + Ri = E(t)$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t)$$

Linear Model – Series Circuits



$$\sum V = 0$$

$$-E(t) + \frac{1}{C} q + R i = 0$$

$$\frac{dq}{dt}$$

$$R \frac{dq}{dt} + \frac{1}{C} q = E(t)$$

Linear Model – Series Circuits

$E = 12\,V$ (Battery)

$L = 1/2$ Henry

$R = 10\ \Omega$

$i(t=0) = 0$

$$\frac{1}{2}\frac{di}{dt} + 10i = 12$$

$$\frac{di}{dt} + 20i = 24$$

Linear Model – Series Circuits

$$\mu = e^{20t}$$

$$\underbrace{e^{20t}}_{g}\,\underbrace{\frac{di}{dt}}_{h'} + \underbrace{e^{20t}\,20}_{g'}\,\underbrace{i}_{h} = 24e^{20t}$$

$$=$$

$$\frac{d}{dt}\left[\underbrace{e^{20t}}_{g}\,\underbrace{i}_{h}\right] = 24\,e^{20t}$$

$$e^{20t}\,i = \frac{24}{20}e^{20t} + c$$

$$i = \frac{6}{5} + c\,e^{-20t}$$

Linear Model – Series Circuits

$$i(t=0)=0 \to \quad 0=\frac{6}{5}+C \implies C=-\frac{6}{5}$$

$$i(t)=\frac{6}{5}-\frac{6}{5}e^{-20t}$$

If $E=E(t)$ Function of t and not a constant

- Based on the general solution

$$\frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}E(t)$$

$$\frac{dy}{dx}+P(x)y=f(x)$$

Linear Model – Series Circuits

$$i(t) = e^{-(R/L)t} \int e^{(R/L)t} \frac{E(t)}{L}\, dt + c\, e^{-(R/L)t}$$

If $E(t) = E_0$ (constant)

$$i(t) = \underbrace{\frac{E_0}{R}}_{\text{steady state}} + \underbrace{c\, e^{-(R/L)T}}_{\text{transient term}}$$

As $t \to \infty \quad e^{-(R/L)t} \longrightarrow 0$

$E = iR \rightarrow$ Ohm's Law (Steady State)