Class Notes 2:

First Order Differential Equation – Linear

MAE 82 – Engineering Mathematics
Introduction – First Order Differential Equations

- First Order Differential Equation
  \[ \frac{dy}{dt} = f(t, y) \]

- Solution
  \[ y = \phi(t) \]

- Any differentiable function that satisfy this equation for all \( t \) in some interval is called a solution

- Objective
  - Determine whether such function exist
  - Develop Methods for funding them
    - Linear equations (section 2.1)
    - Separable equations (section 2.2)
    - Exact equations (section 2.6)
Classification

FIRST ORDER DIFF. EQ.

LINEAR EQ

TYPE 1 \( \frac{dy}{dt} + a(t)y = g(t) \)

SOLUTION: INTEGRATION FACTOR

TYPE 2 \( \frac{dy}{dt} + p(t)y = g(t) \)

SOLUTION: VERIFICATION OF PARAMETERS

TYPE 3 \( \frac{dy}{dt} = f(y) \)

AUTONOMOUS

NON LINEAR EQ

TYPE 1 SEPERABLE EQ.

\( M(x)dx + N(y)dy = 0 \)

SOLUTION: INTEGRATION

TYPE 2 EXACT EQ.

\( M(x,y)dx + N(x,y)dy = 0 \)

\( \frac{dm}{dy} = \frac{dn}{dx} \)

SOLUTION: INTEGRATION FACTORS

TYPE 3 CONVERGENT NON EXACT TO AN EXACT

SOLUTION: INTEGRATION FACTORS
General Form

\[ p(t) \frac{dy}{dt} + Q(t)y = G(t) \]

Given \( p, Q, G \) and \( p(t) \neq 0 \)

**Variable Coefficient**

\[ \frac{dy}{dt} + p(t)y = g(t) \]

**Constant Coefficients**

\[ \frac{dy}{dt} + ay = b \]
Example 1 – Integration

\[(4 + t^2) \frac{dy}{dt} + 2ty = 4t\]

**LEFT SIDE** ⇒ Linear combination of \(\frac{dy}{dt}\) and \(y\)

**CALCULUS** ⇒ Differentiation a product

\[\frac{d}{dt} \left[ \frac{(4 + t^2) y}{g} \right] = \frac{(4 + t^2)}{g} \frac{dy}{dt} + \frac{2t}{g} y\]

Rewrite the equation

\[\frac{d}{dt} \left[ (4 + t^2) y \right] = 4t\]

Integrate both sides with respect to \(t\)
Example 1 – Integration

\[(4+t^2)y = 2t^2 + c\]

Arbitrary constant of integration

\[y = \frac{2t^2}{4+t^2} + \frac{c}{4+t^2}\]

NOTE - Special Case - The left hand side of the eq. is a derivative of a product of \(y\) and some other function
Integration Factors Method

- For a constant $a$

$$\frac{dy}{dt} + ay = g(t)$$

- If the diff eq. is multiplied by a certain function $\mu(t)$

- The diff. eq. is converted into one that is immediately integrable by using the product rule of derivatives

$$(gh)' = gh' + g'h$$
Integration Factors Method

\[ \frac{dy}{dt} + (a) y = g(t) \]

- \( a \) - Given Constant
- \( g(t) \) - Given Function
- The integration factor must satisfy

\[
\frac{d}{dt} \left[ \mu(t)y \right] = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y = \mu \frac{dy}{dt} + (\mu a) y
\]
Integration Factors Method

\[ \frac{dy}{dt} = ay \]

- The integration factor is \( \mu(t) = e^{at} \)
- What makes \( e^{at} \) so special

\[
\begin{align*}
y &= e^{at} \\
y' &= ae^{at} \\
\frac{y}{x} &= ay
\end{align*}
\]

The slope \( \downarrow \)
Value of the function \( \uparrow \)
Integration Factors Method

- Multiply the diff eq. with the integration factor \( \mu(t) = e^{at} \)

\[
\mu(t) \left[ \frac{dy}{dt} + ay \right] = \mu(t) g(t)
\]

\[
e^{at} \frac{dy}{dt} + a e^{at} y = e^{at} g(t)
\]

\[
\frac{d}{dt} \left( \frac{e^{at} y}{g} \right) = e^{at} g(t)
\]
Integration Factors Method

Integrate both sides of the diff eq.

\[ e^{at} y = \int e^{at} g(t) + c \]

\[ y = e^{-at} \int e^{as} g(s) ds + c e^{-at} \]
Integration Factors Method – Example 3

\[ \frac{dy}{dt} - 2y = 4 - t \]

\[ \frac{dy}{dt} + ay = g(t) \]

\[ \mu(t) = e^{-2t} \]

\[ e^{-2t} \frac{dy}{dt} - 2e^{-2t} y = 4e^{-2t} - te^{-2t} \]

\[ g \quad h \quad g' \quad h' \]
Integration Factors Method – Example 3

\[ e^{-2t}y = \int 4e^{-2t} - te^{-2t} \, dt \]

\[ e^{-2t}y = -2e^{-2t} + \frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + c \]
Integration Factors Method – Example 3

\[ y = -\frac{7}{4} + \frac{1}{2}t + Ce^{2t} \]

- \( y = -\frac{7}{4} + \frac{1}{2}t \)

\( c > 0 \)

\( c = 0 \)

\( y = -\frac{7}{4} + \frac{1}{2}t \)

\( c < 0 \)

- Review Examples 4 p.37, 5 p.38
Variation of Parameters Method

\[ \frac{dy}{dt} + p(t) y = g(t) \]

\[ \text{For function } p(t) \text{ not constant} \]

Multiply the diff. eq. by \( \mu(t) \)

\[ \mu(t) \frac{dy}{dt} + \mu(t) p(t) y = \mu(t) g(t) \]

\[ \frac{d}{dt} [\mu(t) y] = \mu \frac{dy}{dt} + \frac{d\mu}{dt} y = \mu \frac{dy}{dt} + (\mu p) y \]
Variation of Parameters Method

- Condition to be met
  \[ \frac{d\mu}{dt} = \mu P \]

- Separation of variable
  \[ \frac{d\mu}{\mu} = P dt \]

- Integrating both sides
  \[ \int \frac{1}{\mu} d\mu = \int P dt \]

- Integration factor
  \[ \ln |\mu(t)| = \int P(t) dt + C \]

\[ \mu(t) = e^{\int P(t) dt} \]
Variation of Parameters Method

\[
e^{\int p(t) \, dt} \frac{dy}{dt} + p(t) e^{\int p(t) \, dt} \frac{y}{h} = e^{\int p(t) \, dt} g(t)
\]

\[
\frac{d}{dt} \left[ e^{\int p(t) \, dt} y \right] = e^{\int p(t) \, dt} g(t)
\]

\[
y = e^{-\int p(t) \, dt} \int e^{\int p(t) \, dt} g(t) \, dt + ce^{-\int p(t) \, dt}
\]
Example - Variation of Parameters Method

\[ \frac{dy}{dt} - 2ty = 2 \]

\[ y(0) = 1 \]

\[ \mu(t) = e^{\int p(t) \, dt} = e^{\int -2t \, dt} = e^{-t^2} \]

\[ e^{-t^2} \frac{dy}{dt} - 2te^{-t^2}y = 2e^{-t^2} \]

\[ \int_{h(t)}^{g(t)} \frac{dy}{dt} - \int_{g(t)}^{h(t)} 2t e^{-t^2}y = 2e^{-t^2} \]

\[ \frac{d}{dt} [e^{-t^2}y] = 2e^{-t^2} \]

\[ y = 2e^{t^2} \left[ \int_0^t e^{-s^2} \, ds \right] + ce^{t^2} \]

\[ \text{Known integral} \]
Example - Variation of Parameters Method

\[ y = 2e^{t^2} \left( \frac{1}{2} \pi \text{erf}(t) \right) + Ce^{t^2} \]

\[ y \bigg|_{t=0} = 0 \]

\[ \text{I} \text{c} \quad 1 = C e^0 \Rightarrow C = 1 \]

\[ y = e^{t^2} \left[ 1 + \sqrt{\pi} \text{erf}(t) \right] \]

- Review also example 4 p.37
Passenger Pigeon

The passenger pigeon or wild pigeon (Ectopistes migratorius) is an extinct North American bird. Named after the French word *passager* for "passing by", it was once the most abundant bird in North America, and possibly the world. It accounted for more than a quarter of all birds in North America. The species lived in enormous migratory flocks until the early 20th century, when hunting and habitat destruction led to its demise.

Some estimate 3 to 5 billion passenger pigeons were in the United States when Europeans arrived in North America. Some reduction in numbers occurred from habitat loss when European settlement led to mass deforestation. Next, pigeon meat was commercialized as a cheap food for slaves and the poor in the 19th century, resulting in hunting on a massive and mechanized scale. A slow decline between about 1800 and 1870 was followed by a catastrophic decline between 1870 and 1890. Martha, thought to be the world's last passenger pigeon, died on September 1, 1914, at the Cincinnati Zoo.
Autonomous Equation – Population Dynamics

The Tipping Point
How Little Things Can Make a Big Difference
MALCOLM GLADWELL

“A fascinating book that makes you see the world in a different way.” —FORTUNE
Autonomous Equation – Population Dynamics

Introduction

• **Autonomous** Equations - Class of first order differential equations where the independent variable $t$ does not appear explicitly.

\[
\frac{dy}{dt} = f(y)
\]

• Applications
  – Growth/Decline of population
  – Medicine
  – Ecology
  – Global Economics

• Stability / Instability of the solution
Autonomous Equation – Population Dynamics

Introduction

- Types of differential equations

  - Exponential Growth
    \[
    \frac{dy}{dt} = r y
    \]

  - Logarithmic Growth
    \[
    \frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y
    \]

  - Logarithmic Growth with Critical Threshold
    \[
    \frac{dy}{dt} = -r \left(1 - \frac{y}{K}\right) y
    \]

  - Logarithmic Growth with Threshold
    \[
    \frac{dy}{dt} = -r \left(1 - \frac{y}{L}\right) \left(1 - \frac{y}{K}\right) y
    \]
Autonomous Equation – Population Dynamics

Exponential Growth

• The population of a given species at time \( t \)

\[ y = \phi (t) \]

• The rate of change of the population \( y \) is proportional to the current population (value of the \( y \)) – Thaoms Maltus – British Economist 1766-1834

\[ \frac{dy}{dt} = ry \]

• \( r \) – The rate of growth \((r>0)\) or decline \((r<0)\)

• Solving the differential equation subject to initial condition

\[ y(0) = y_0 \]

\[ y = y_0 e^{-rt} \]
Autonomous Equation – Population Dynamics

Exponential Growth

- Under ideal condition the population will grow exponentially
- Valid during a limited period of time
- Limitation
  - Space
  - Food supplies
  - Limited resources
Annual world population growth rate (1950-2100)

The world population growth rate peaked in 1962 and 1963 with 2.2%.

The ‘Great Leap Forward’ famine in China under Mao is killing around 30 million people. It is reducing global population growth between 1958 and 1961.

Time it took for the world population to double

Historical estimates of the world population until 2015 – and UN projections until 2100

It took 697 years for the world population to double – from 0.25 billion in 837 to 0.5 billion in 1543.

1543

1803

260 years (0.5 to 1 billion)

1928

125 years (1 to 2 billion)

1975

76 years (1.5 to 3 billion)

1979

47 years (2 to 4 billion)

1983

37 years (2.5 to 5 billion) – in 1987

1987

95 years (5.5 to 11 billion) – in 2088

1990

69 years (5 to 10 billion) – in 2056

1993

49 years (4 to 8 billion) – in 2024

1997

1997

37 years (2.5 to 5 billion) – in 1987

1997

Data source: OurWorldInData annual world population series (Based on HYDE and UN until 2015. And projections from the UN after 2015 (‘Medium Variant’ 2015 Revision).
The data visualization is available at OurWorldinData.org. There you find the raw data, more visualizations, and research on this topic. Licensed under CC-BY-SA by the author Max Roser.
The World Population in 1950, 2017, and 2100

The UN projects that the number of children born in 2100 (1.31 million) will be lower than today (1.41 million).

Data source: United Nations – World Population Prospects 2015. Data in 1-year-brackets is only available up to the age of 100 years in 2017 and 2100 and only up to 80 years in 1950. The interactive data visualization is available at OurWorldinData.org. There you find the raw data and more visualizations on this topic. Licensed under CC-BY-SA by the author Max Roser.
Autonomous Equation – Population Dynamics

Logistic Growth

- The growth rate depends on the population
- Replace $+ \rightarrow h(y)$
  
  Function of the population

$$\frac{dy}{dt} = h(y) \times y$$

- Choose $h(y)$ so that
  
  \[
  h(y) = (r - ay) \begin{cases} 
    h(y) \geq r > 0 & \text{When } y \text{ is small} \\
    h(y) \downarrow & \text{Decrease as } y \text{ grow} \\
    h(y) < 0 & \text{When } y \text{ is sufficient large} 
  \end{cases}
  \]

  $a > 0$ \quad $\text{\text{const.}}$
Autonomous Equation – Population Dynamics

**Logistic Growth**

\[ \frac{dy}{dt} = (r - ay)y \]

- **Verhulst Equation or Logistic Equation**
- **Pierre F. Verhulst (1804–1849)** was a Belgian mathematician who introduced this equation as a model for human population growth in 1838.
Autonomous Equation – Population Dynamics

Logistic Growth

- Equivalent Form

\[
\frac{dy}{dt} = \Gamma \left( 1 - \frac{y}{K} \right) y
\]

\[a = \frac{\Gamma}{K}\]

- \(\Gamma\) – Intrinsic growth rate – The growth rate in the absence of any limitation

- \(K\) – Saturation level – Environmental carrying capacity
Autonomous Equation – Population Dynamics

Logistic Growth

- Investigate the solution for

\[
\frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y
\]

- Assume \( y(t) = \text{constant} \Rightarrow \frac{dy}{dt} = 0 \)

\[
- \left( 1 - \frac{y}{K} \right) y = 0
\]

\[
= 0 \text{ or } = 0
\]

- Solutions

\[
\begin{cases}
  y = \phi_1(t) = 0 \\
  y = \phi_2(t) = K
\end{cases}
\]

\( \Rightarrow \) Equilibrium Solutions
Autonomous Equation – Population Dynamics

Logistic Growth

- In $\frac{dy}{dt} = f(y)$ when $f(y) = 0$

The zeros of $f(y)$ are also called critical points.
Autonomous Equation – Population Dynamics

**Logistic Growth**

\[ f(y) = r \left(1 - \frac{y}{K}\right)y = \left(\frac{r}{K} \right) y^2 + (r)y \]

- **Slope**: \( f(y) = r \left(1 - \frac{y}{K}\right)y \)
- **Parabola**

- **Axis y is called the phase line**
- **Positive change**: \( \frac{dy}{dt} > 0 \) for \( 0 < y < K \)
- **Negative change**: \( \frac{dy}{dt} < 0 \) for \( y > K \)

Detention of the slope \( f(y) = \frac{dy}{dt} \)
Autonomous Equation – Population Dynamics

Logistic Growth

\[ \frac{dy}{dt} = r \left( 1 - \frac{y}{K} \right) y = f(y) \]

- Near 0 or k the slope is close to zero
- Inflection point - location \( \frac{d^2y}{dt^2} = 0 \)

\[
\frac{d^2y}{dt^2} = \frac{d}{dt} \frac{dy}{dt} = \frac{d}{dt} f(y) = f'(y) \frac{dy}{dt} = f'(y) f(y) = 0
\]
Autonomous Equation – Population Dynamics

Logistic Growth

- Exact solution

\[-\frac{dy}{(1 - \frac{y}{k})y} = r \, dt\]

- Partial fraction expansion

\[\left(\frac{1}{y} + \frac{1/k}{1 - \frac{y}{k}}\right) \, dy = r \, dt\]

- Integrate both sides

\[\ln |y| - \ln |1 - \frac{x}{k}| = rt + c\]
Autonomous Equation – Population Dynamics

Logistic Growth

- $c$ Arbitrary constant to be determined by $y(0) = 0$
- If $0 < y < k$, $y$ remains in this interval for all time → remove the $101$

\[ e^{\ln(y) - \ln(1 - \frac{y}{k})} = e^{rt} + c \]

\[ \frac{e^{\ln y}}{e^{\ln(1 - \frac{y}{k})}} = e^{c} e^{rt} \]

\[ \frac{y}{1 - \frac{y}{k}} = \frac{e^{c} e^{rt}}{e^{rt}} \]
Autonomous Equation – Population Dynamics

Logistic Growth

• For $y(0) = y_0 \implies \zeta = \frac{y_0}{1 - \frac{y_0}{k}}$

\[
\frac{y}{(1 - \frac{y}{k})} = \frac{y_0}{(1 - \frac{y_0}{k})} e^{-rt}
\]

\[
y = \frac{y_0 k}{y_0 + (k - y_0) e^{-rt}} \begin{cases} 
0 < y < k \\
 y > k
\end{cases}
\]

\[
\lim_{t \to \infty} y(t) = \frac{y_0 k}{y_0} = k
\]
Autonomous Equation – Population Dynamics

Logistic Growth

• Notes

• \( \phi_2(t) = k \) Asymptotically stable \( \Rightarrow \) solution equilibrium

• As \( r \) increases the solution approach the equilibrium solution more rapidly

• Problem: Even solutions that start close to zero grow as \( t \) increases and approach
Autonomous Equation – Population Dynamics
Logistic Growth with Critical Threshold

\[
\frac{dy}{dt} = -r \left( 1 - \frac{y}{T} \right) y, \quad r > 0
\]

Note - In order to obtain the exact solution for this case replace \( + \) with \( - \) in the previous solution.
Autonomous Equation – Population Dynamics
Logistic Growth with Critical Threshold

\[
\frac{dy}{dt} = -r \left(1 - \frac{y}{T}\right)y, \quad r > 0
\]

\[
(\star) \quad y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}
\]

**If** \(0 < y_0 < T\)

\[
y \to t \to \infty
\]

**If** \(y_0 > T\) - The denominator of the right hand side of the eq(\(\star\)) is zero for a certain finite value of \(t^*\) when \(y_0 + (T - y_0)e^{rt} = 0\)

\[
t^* = \frac{1}{r} \log \left(\frac{y_0}{y_0 - T}\right)
\]
Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

- Denominator of \( y(t) \) is set to 0

\[
y_0 + (T - y_0) e^{r t^*} = 0
\]
\[
y_0 - (y_0 - T) e^{r t^*} = 0
\]

\[
e^{r t^*} = \frac{y}{y_0 - T}
\]

\[
\ln \left( e^{-t^*} \right) = \frac{y_0}{y_0 - T}
\]
\[
t^* = \frac{1}{r} \cdot \frac{y_0}{y_0 - T}
\]
Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

• Note 1:
  
  – If the initial population $y_0$ is above the threshold $T$ the graph of $y(t)$ has a vertical asymptote at $t^*$

  – The population become unbounded in a finite time whose value depends on $y_0$, $T$ and $r$
Autonomous Equation – Population Dynamics

**Logistic Growth with Critical Threshold**

- **Note 2**: The population of Species exhibit the threshold phenomena if
  - (Below the threshold) Too few subjects are present, then the species can not propagate itself successfully and the population become extinct
  - (Above the threshold) further growth occurs

- **Note 3**: The population cant become unbounded
Autonomous Equation – Population Dynamics

Logarithmic Growth with Threshold

Correct the model such as unbounded growth will not occur when \( y \) is above the threshold \( T \)

\[
\frac{dy}{dt} = -r \left( 1 - \frac{y}{T} \right) \left( 1 - \frac{y}{K} \right) y, \quad r > 0 \text{ and } 0 < T < K
\]

- Three critical points
  \[
  \left\{ \begin{array}{l}
y = 0 \\
y = \frac{T}{r} \\
y = K
\end{array} \right.
\]
Autonomous Equation – Population Dynamics
Logarithmic Growth with Threshold
Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

• Notes
  – Passenger Pigeons were present in the US in vast numbers until the 19th century
  – It was heavily hunted for food and sport and reduced significantly 1880
  – Breed successfully only when present in a large concentration i.e. high number of T
Autonomous Equation – Population Dynamics

Logistic Growth with Critical Threshold

- Notes
  - Large number of individual birds remained alive in the late 1880s. However there were not enough in any one place to permit successful breeding.
  - The population decline to extinction
  - Last survivor died in 1914
  - Solution – Conservation
Linear Model – Bacterial Growth
Linear Model – Bacterial Growth
Linear Model – Bacterial Growth

• Given:
  – A culture initially has \( P_0 \) number of bacteria
  – At \( t=1 \) hr the number of bacteria is measured to be \( \frac{3}{2}P_0 \)
  – The rate of growth is propositional to the number of bacteria \( P(t) \) present at the time \( t \)

• Calculated
  – The time necessary for the population to triple
Linear Model – Bacterial Growth

- Rate of growth is proportional to the number of bacteria $p(t)$

$$\frac{dp}{dt} = kp$$

$$\frac{dp}{dt} - kp = 0 \rightarrow \text{Separable & Linear ODE}$$

- Empirical observation

$$\begin{cases} p(t_0 = 0) = P_0 \\ p(t = 1) = \frac{3}{2} P_0 \end{cases}$$
Linear Model – Bacterial Growth

\[ n = e^{-kt} \]

\[ e^{-kt} \frac{dp}{dt} - k e^{-kt} p = 0 \]

Integrating both sides \[ e^{-kt} \cdot p = c \]

\[ p(t) = c e^{kt} \]
Linear Model – Bacterial Growth

\[
\begin{align*}
A & \quad t = 0 \\
\rho(t=0) &= c e^{+k \cdot 0} = \rho_0 \quad \Rightarrow \quad c = \rho_0 \\
\rho(t) &= \rho_0 e^{+kt} \\
A & \quad t = 1h \\
\rho(t=1) &= \rho_0 e^{-k \cdot 1} = \frac{3}{2} \rho_0 \\
\end{align*}
\]

\[
e^{-k} = \frac{3}{2} \quad \Rightarrow \quad k = \ln \frac{3}{2} = 0.4055
\]
Linear Model – Bacterial Growth

\[ p(t) = P_0 e^{0.4055t} \]

- Find the time at which the number of bacteria has tripled
Linear Model – Bacterial Growth

\[ 3P_0 = P_0 e^{0.4055t} \]

\[ \ln 3 = 0.4055t \]

\[ t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ hr} \]

Note: The initial number of bacteria at \( t=0 \) \( P_0 \) played no part in determining the time it takes for a population to tripled. The initial population could be 100 or 100,000.
Linear Model – Cooling/Warming – Newton’s Law
The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body ($T$) and the temperature of the surrounding medium - ambient temperature ($T_m$).

$$
\frac{dT}{dt} \propto T - T_m
$$

$$
\frac{dt}{dt} = K \left( T - T_m \right)
$$
Linear Model – Cooling/Warming – Newton’s Law

\[ T \left( t = 0 \right) = 300^\circ \text{F} \]
\[ T \left( t = 3 \text{ min} \right) = 200^\circ \text{F} \]
\[ t = ? \quad \rightarrow \quad T = 70^\circ \text{F} \]

\[ \frac{dT}{dt} = k \left( T - 70 \right) \quad \left\{ \begin{array}{l} T(0) = 300 \\ T(3) = 200 \end{array} \right. \]

\[ \frac{dT}{T-70} = k \, dt \]

\[ \ln \left| T-70 \right| = k \, t + c \]
Linear Model – Cooling/Warming – Newton’s Law

\[ e^{\ln|T - 70|} = e^{kt + c_1} = e^{kt} \frac{e^{c_1}}{c_2} \]

\[ (T - 70) = c_2 e^{kt} \]

\[ T = 70 + c_2 e^{kt} \]

\[ T \left( t=0 \right) = 300 \rightarrow 300 = 70 + \frac{c_2 e^{k \cdot 0}}{1} \Rightarrow c_2 = 230 \]

\[ T = 70 + 230 e^{kt} \]

\[ T \left( t=3 \right) = 200 \rightarrow 200 = 70 + 230 e^{3k} \]
Linear Model – Cooling/Warming – Newton’s Law

\[ e^{3k} = \frac{13}{2^3} \]

\[ k = \frac{1}{3} \ln\left(\frac{13}{2^3}\right) = -0.19018 \]

\[ T(t) = 70 + 230 \ e^{-0.19018t} \]

\[ \lim_{t \to 00} T(t) = 70 \]
Linear Model – Series Circuits

**Kirchhoff Law**

\[ \sum V = 0 \]

Close circuit

\[ -E(t) + L \frac{di}{dt} + Ri = 0 \]

\[ L \frac{di}{dt} + Ri = E(t) \]

\[ \frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} E(t) \]
Linear Model – Series Circuits

\[ \sum V = 0 \]

\[ -E(t) + \frac{1}{C} \dot{q} + Ri = 0 \]

\[ R \frac{dq}{dt} + \frac{1}{C} q = E(t) \]
Linear Model – Series Circuits

\[ E = 12 \, \text{V} \quad \text{(Battery)} \]
\[ L = \frac{1}{2} \, \text{Henry} \]
\[ R = 10 \, \Omega \]
\[ i(t=0) = 0 \]

\[ \frac{1}{2} \frac{di}{dt} + 10i = 12 \]
\[ \frac{di}{dt} + 20i = 24 \]
Linear Model – Series Circuits

\[ \mu = e^{20t} \]

\[ e^{20t} \frac{d}{dt} \left( \frac{i}{20} \right) + e^{20t} \frac{i}{20} = 24 e^{20t} \]

\[ \frac{d}{dt} \left[ \frac{e^{20t}}{20} i \right] = 24 e^{20t} \]

\[ e^{20t} i = \frac{24}{20} e^{20t} + C \]

\[ i = \frac{6}{5} + C e^{-20t} \]
Linear Model – Series Circuits

\[ i(t = 0) = 0 \Rightarrow 0 = \frac{6}{5} + c \quad \Rightarrow \quad c = -\frac{6}{5} \]

\[ i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t} \]

If \( E = E(t) \) is a function of \( t \) and not a constant

- Based on the general solution

\[ \frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} E(t) \]

\[ \frac{dy}{dx} + p(x)y = f(x) \]
Linear Model – Series Circuits

\[ i(t) = e^{-(R/L)t} \int e^{(R/L)t} \frac{E(t)}{L} \, dt + c \cdot e^{-(R/L)t} \]

If \( E(t) = E_0 \) (constant)

\[ i(t) = \frac{E_0}{R} + c \cdot e^{-(R/L)t} \]

steady state \hspace{1cm} transient term

As \( t \to \infty \), \( e^{-(R/L)t} \to 0 \)

\[ E = iR \to \text{Ohm's Law (Steady State)} \]