Class Notes 1:

Introduction

MAE 82 – Engineering Mathematics

CHANGE

Rate of Change

Basic Mathematical Models

- Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving **rates at which things happen.**

- Principles / Laws
  - Relations -> Equations
  - Rates -> Derivative

- A differential equation that describes a physical process is often called a **mathematical model**.

Air Motion - Air Flow - Wing


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Air Motion – Hurrian Katrina

Air Motion – Hurrian Katrina

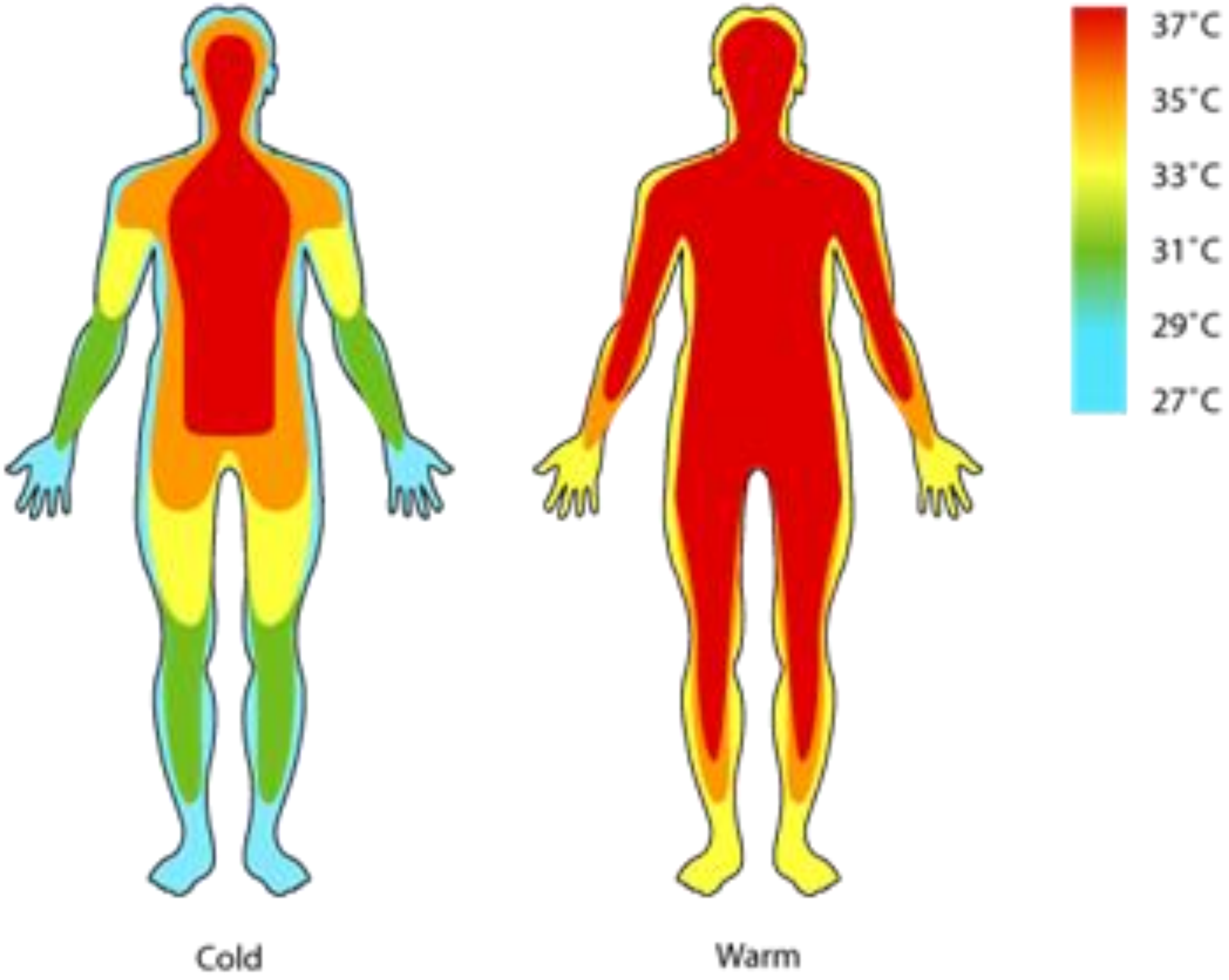

Water Motion - Fluid Flow - Waves

Water Motion - Fluid Flow - Waves

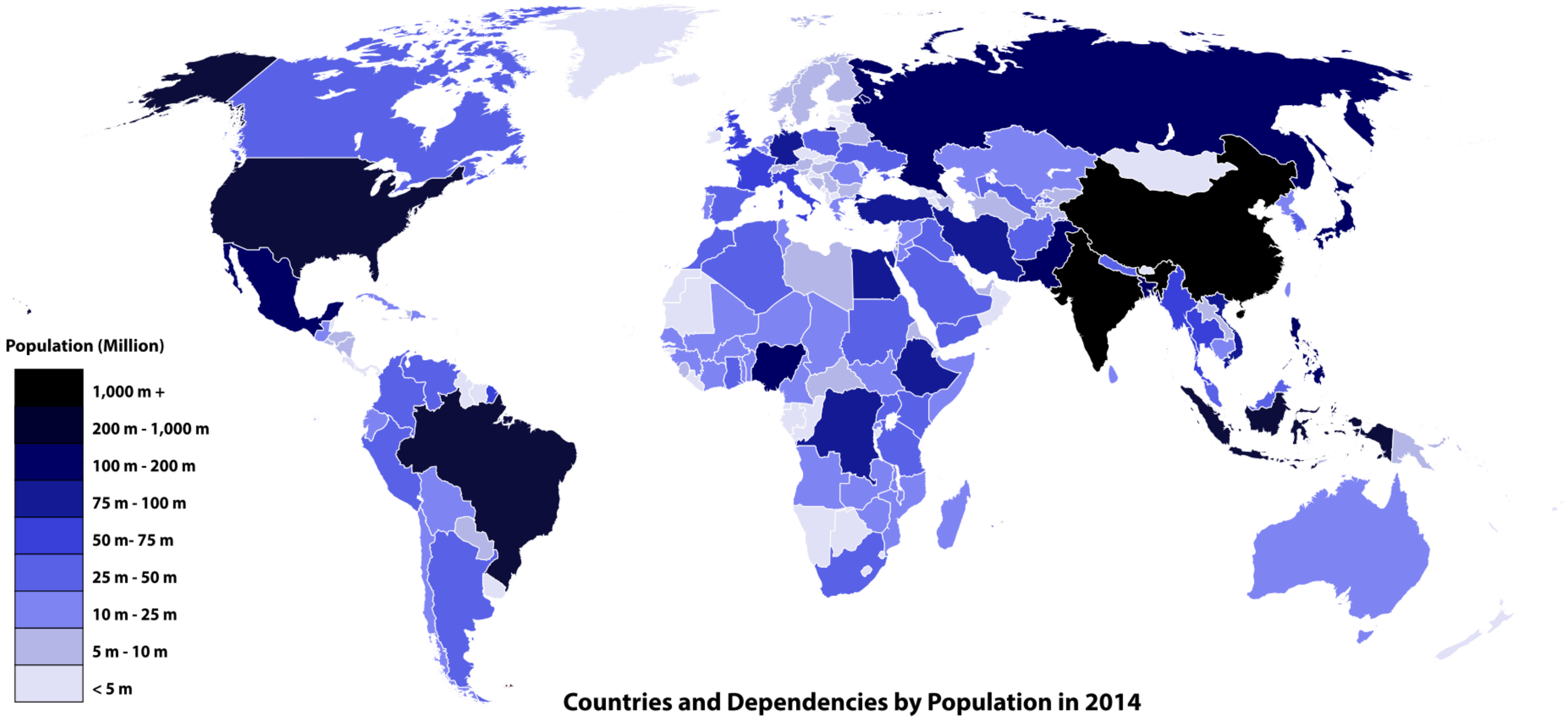
2004 Indian Ocean earthquake and tsunami

Solid Flow

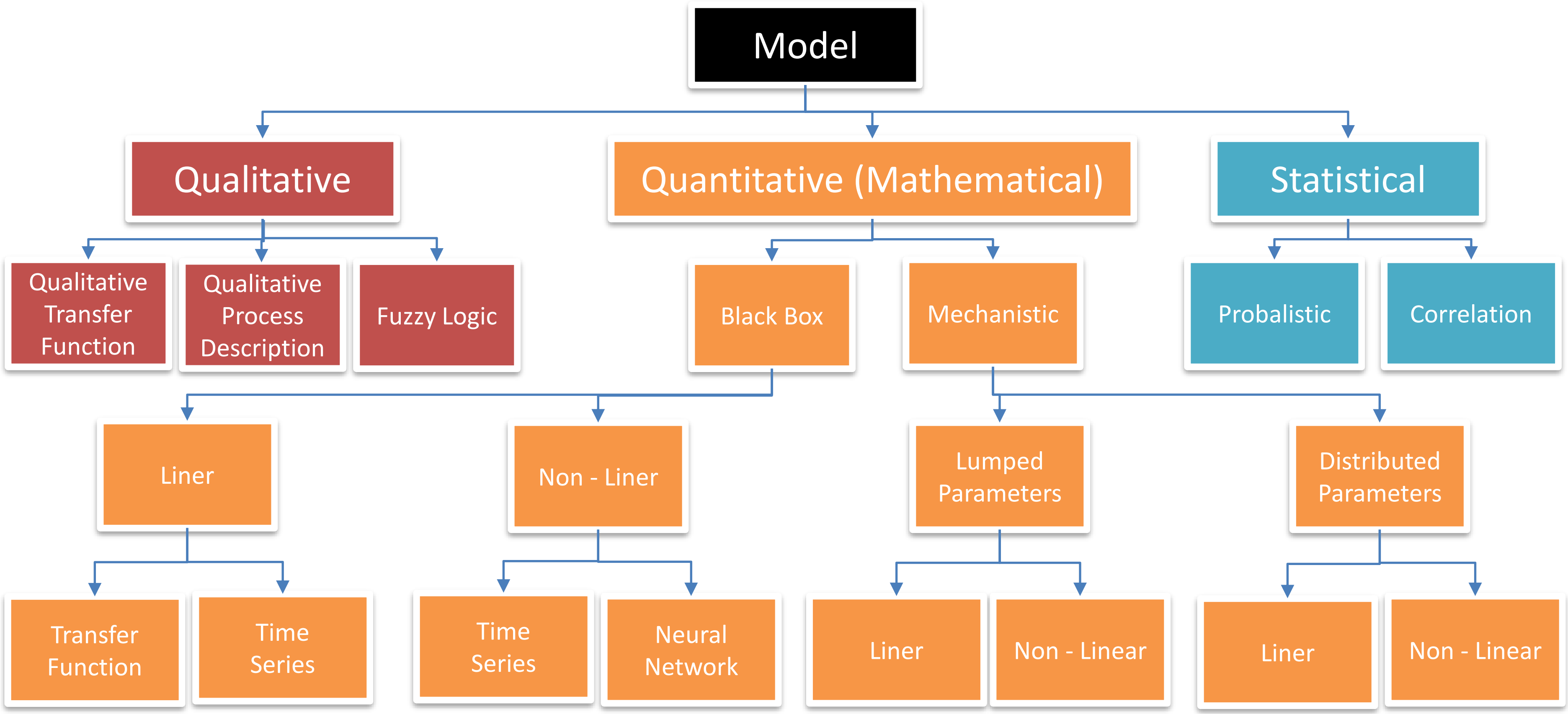
Heat Dissipation – Human Body



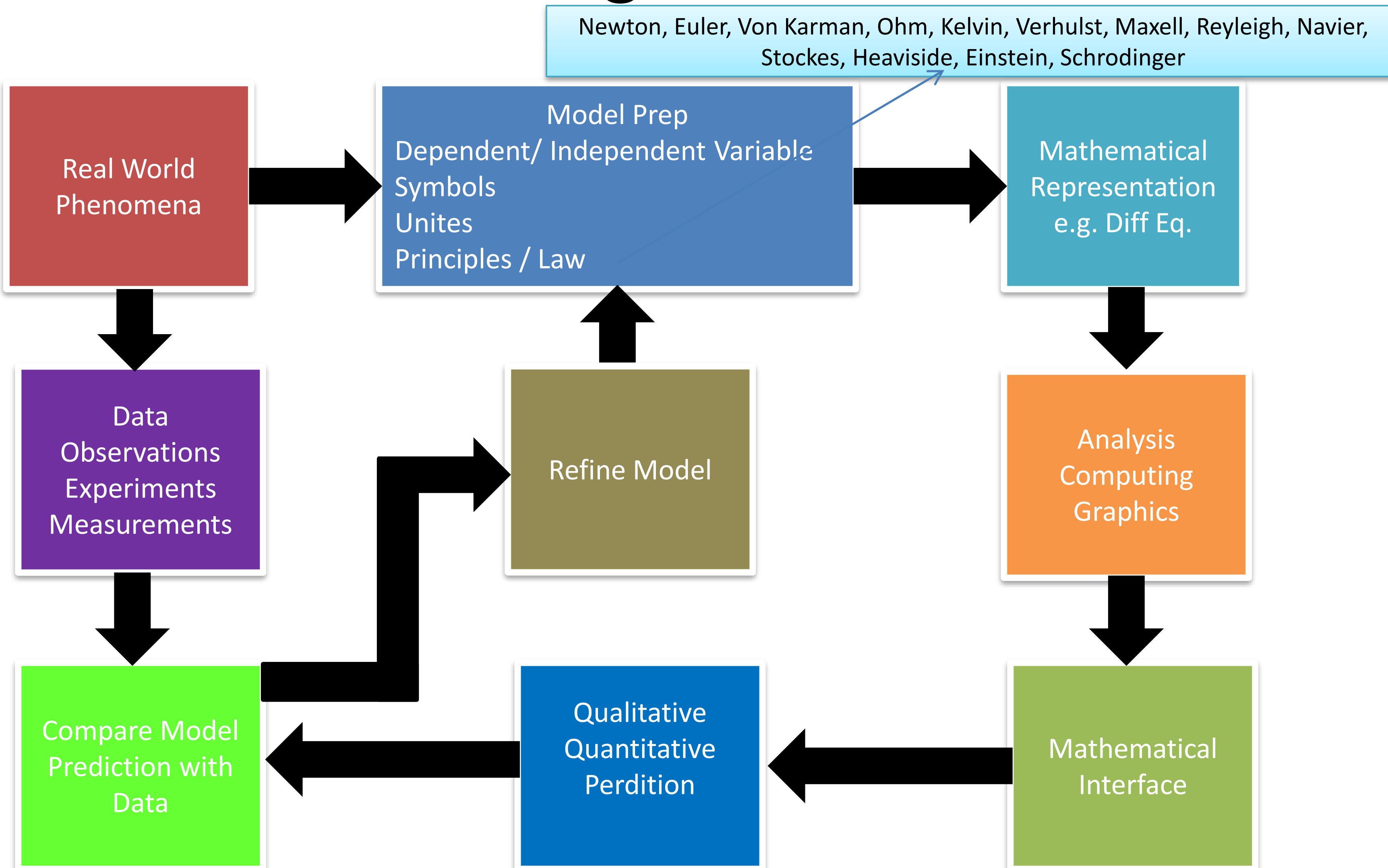
Population Dynamics



Modeling – Types



Modeling Process



Constructing Mathematical Models

1. **Independent and dependent variables** - Identify independent and dependent variables and assign letters to represent them.
2. **Unites** - Choose the units of measure for each variable.
3. **Basic Principle** - Articulate the basic principle that underlies or governs the problem you are investigating. This requires your being familiar with the field in which the problem originates.
4. **Mathematical Expression** - Express the principle or law in the previous step in terms of the variables identified at the start. This may involve the use of intermediate variables related to the primary variables.
5. **Unites Unification** - Make sure each term of your equation has the same physical units.
6. **Equations / Set of Equations** - The result may involve one or more differential equations.

Example 1 – You Have Changed

Example 2 – Fosbury Flop

Example 3 – Free Fall Particle – Modeling

- Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.
- Variables / Units:

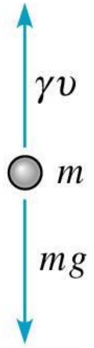
- Basic Principle
- Law

Example 3 – Free Fall Particle – Modeling

Example 3 – Free Fall Particle

- Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.
- Variables: time t, velocity v
- Newton's 2nd Law: $F = ma = m(\mathrm{d}v/\mathrm{d}t)$ ←net force
- Force of gravity: $F = mg$ ←downward force
- Force of air resistance: $F = \gamma v$ ←upward force
- Then

$$m\frac{dv}{dt} = mg - \gamma v$$



- Taking $g = 9.8$ m/sec^2, $m = 10$ kg, $\gamma = 2$ kg/sec, we obtain

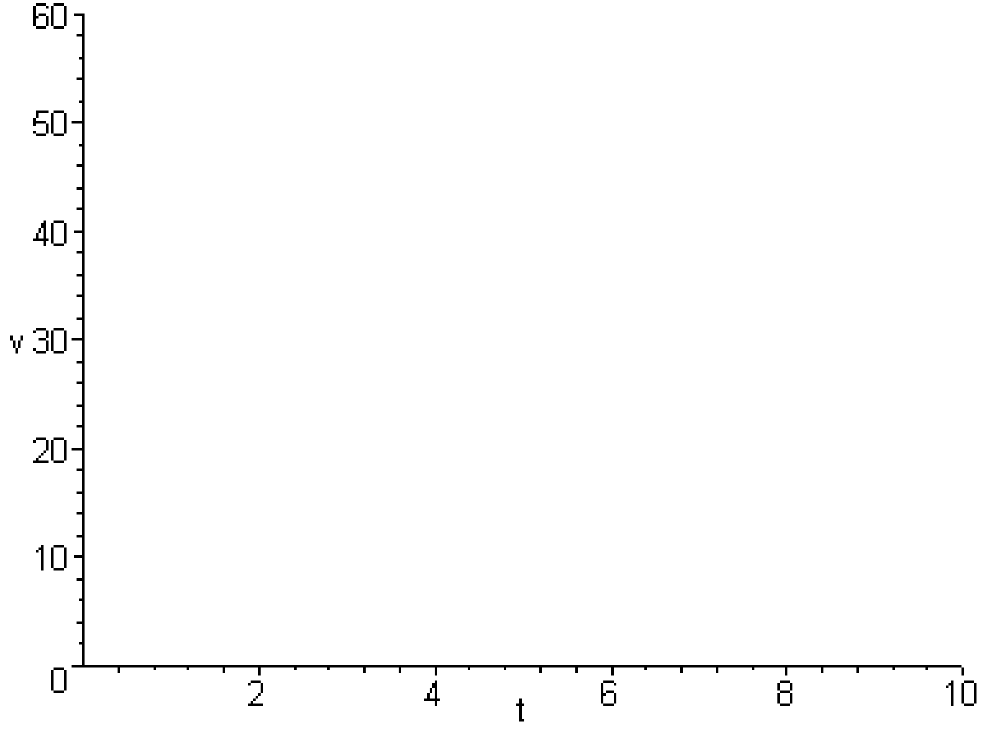
$$\frac{dv}{dt} = 9.8 - 0.2v$$

Example 3 – Free Fall Particle – Direction Field

$$v' = 9.8 - 0.2v$$

- Using differential equation and table, plot slopes (estimates) on axes below. The resulting graph is called a **direction field**. (Note that values of *v* do not depend on *t*.)

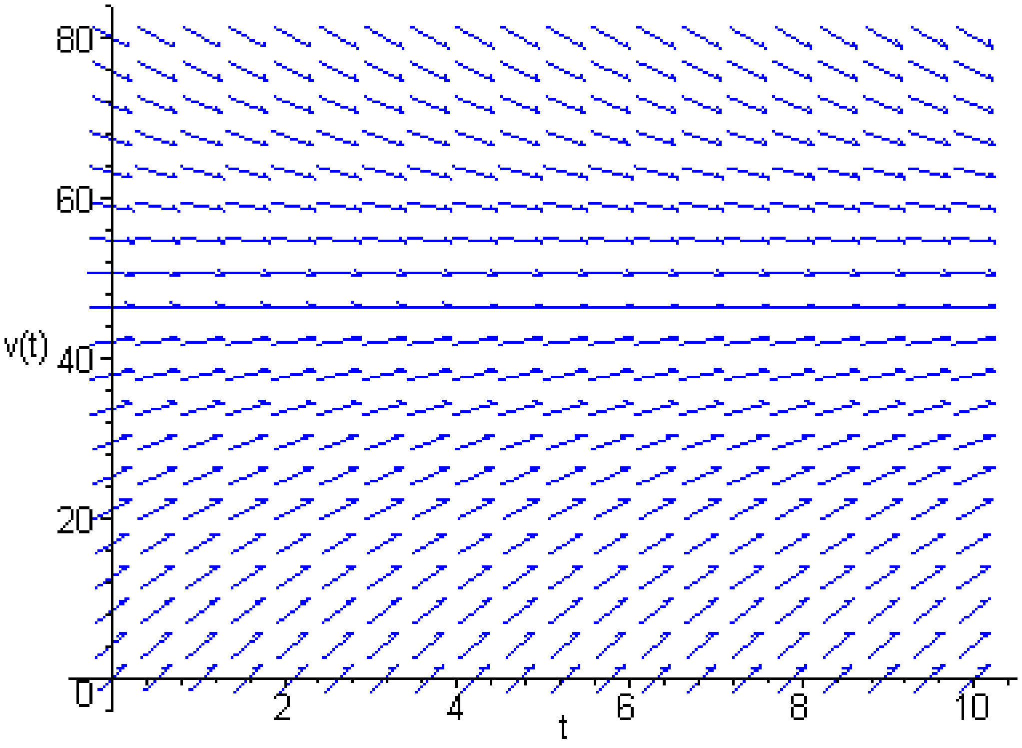
v	v'
0	9.8
5	8.8
10	7.8
15	6.8
20	5.8
25	4.8
30	3.8
35	2.8
40	1.8
45	0.8
50	-0.2
55	-1.2
60	-2.2



Example 3 – Free Fall Particle – Direction Field

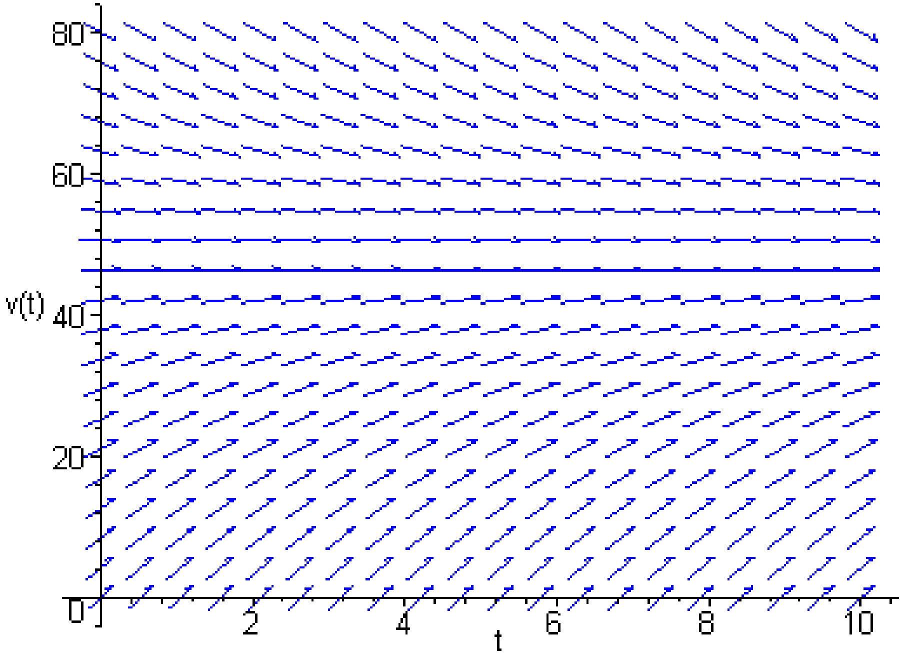
$$v' = 9.8 - 0.2v$$

- When graphing direction fields, be sure to use an appropriate window, in order to display all equilibrium solutions and relevant solution behavior.



Example 3 – Free Fall Particle – Equilibrium Solution

- Arrows give tangent lines to solution curves, and indicate where soln is increasing & decreasing (and by how much).
- Horizontal solution curves are called **equilibrium solutions**.
- Use the graph below to solve for equilibrium solution, and then determine analytically by setting $v' = 0$.



$$v' = 9.8 - 0.2v$$

$$\text{Set } v' = 0:$$

$$\Leftrightarrow 9.8 - 0.2v = 0$$

$$\Leftrightarrow v = \frac{9.8}{0.2}$$

$$\Leftrightarrow v = 49$$

Example 4 – Mice & Owls

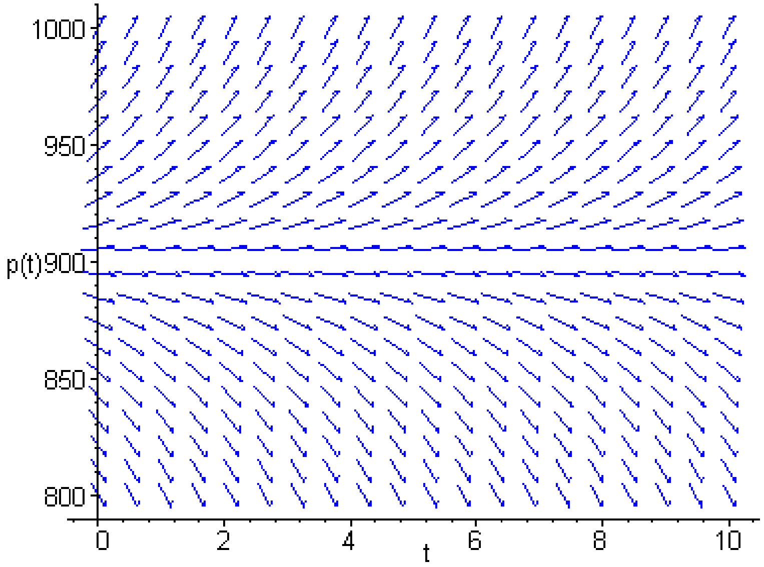
- **Populations:** Mice (pray) / Owls (Predators)

- **Mice (pray)** - A mouse population reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month
- Write a differential equation describing mouse population assuming no owls present.

- **Owls (Predator)** - When owls are present, they eat the mice. Suppose that the owls eat 15 per day (average).
- Write a differential equation describing mouse population in the presence of owls. (Assume that there are 30 days in a month.)

Example 4 – Mice & Owls – Directional Field

- Solution curve behavior, and equilibrium solution

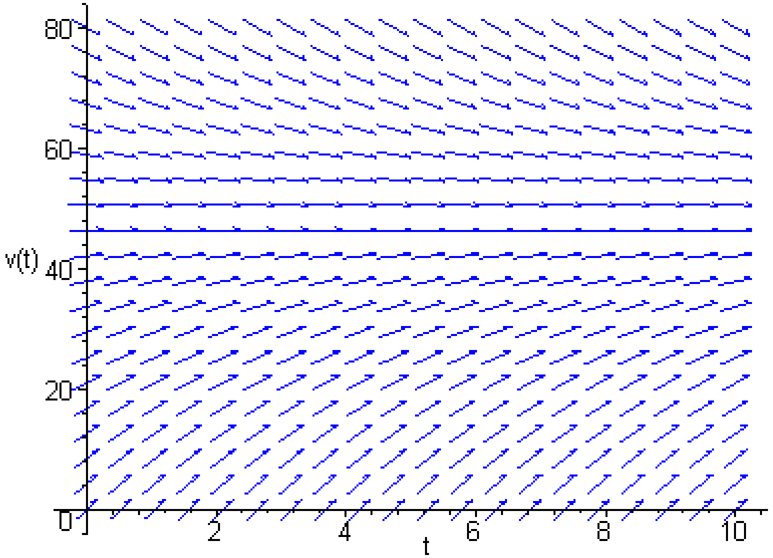


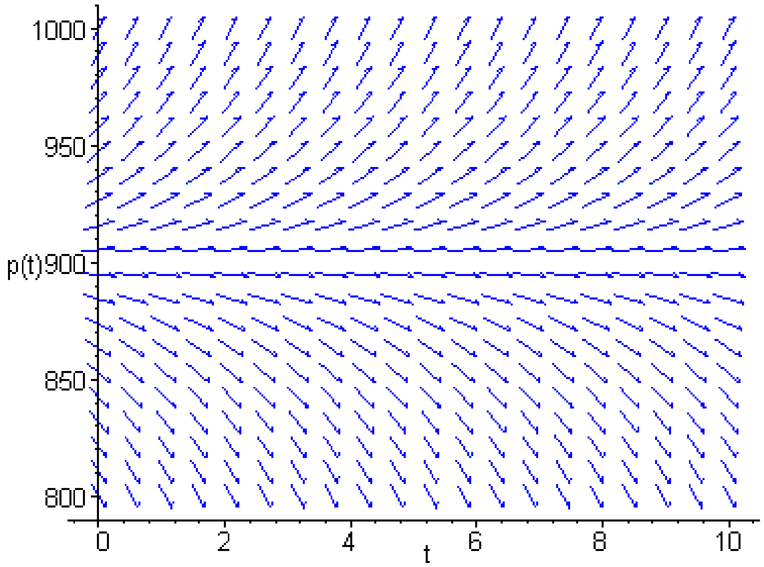
$$p' = 0.5p - 450$$

Example 3 & 4 – Directional Field – Solution Behavior

$$v' = -0.2v + 9.8$$

$$p' = 0.5p - 450$$





Solution of the General Equation

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Solution of the General Equation

Linearization of a Non Linear ODE – Example

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Linearization of a Non Linear ODE – Example

Classification of Differential Equations

- **Ordinary differential equations (ODE).**

- When the unknown function depends on a **single independent variable**, only ordinary derivatives appear in the equation.

- The equations discussed in the preceding two sections are ordinary differential equations. For example,

$$\frac{dv}{dt} = 9.8 - 0.2v, \quad \frac{dp}{dt} = 0.5p - 450$$

Classification of Differential Equations

- **Partial Differential Equations (PDE)**
- When the unknown function depends on **<u>several independent variables</u>**, partial derivatives appear in the equation.
- For example,

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t} \quad \text{(heat equation)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave equation)}$$

Classification of Differential Equations

- **System of Differential Equations**

- One Unknown – One equation
- Two or more unknown – System of equations

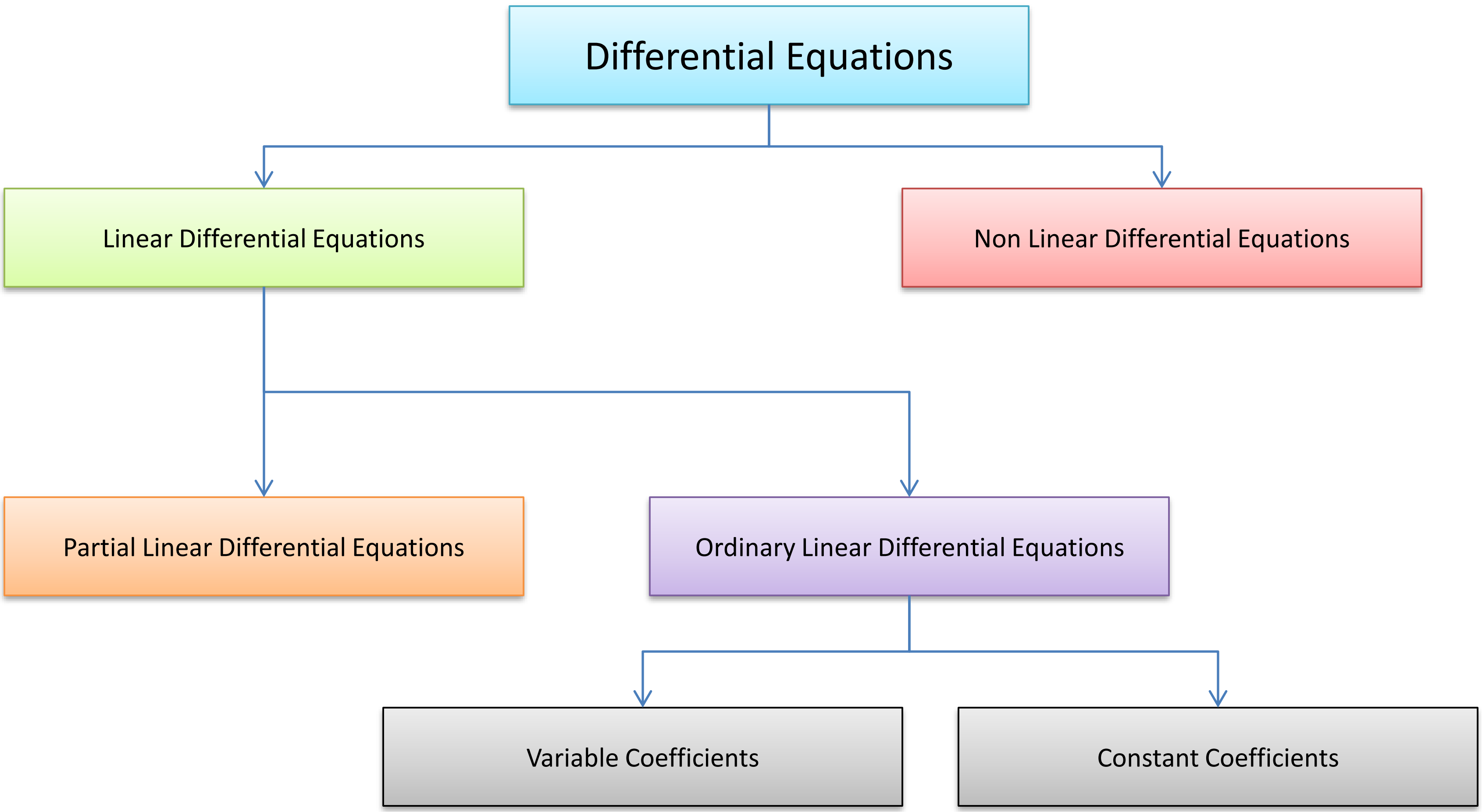
- For example, predator-prey equations have the form

$$du/dt = \ ax - \alpha xy$$
$$dv/dt = -cy + \gamma uy$$

- where x(t) and y(t) are the respective populations of prey and predator species. The constants a, c, α, γ depend on the particular species being studied.

- Systems of equations are discussed in Chapter 7.

Classification of Differential Equations Block Diagram



Order of Classification

- The order of a differential equation is the order of the highest derivative that appears in the equation.
- Examples:

$$y' + 3y = 0$$

$$y'' + 3y' - 2t = 0$$

$$\frac{d^4 y}{dt^4} - \frac{d^2 y}{dt^2} + 1 = e^{2t}$$

$$u_{xx} + u_{yy} = \sin t$$

- We will be studying differential equations for which the highest derivative can be isolated:

$$y^{(n)}(t) = f\left(t, y, y', y'', y''', \ldots, y^{(n-1)}\right)$$

Linear & Non Linear ODE

- An ordinary differential equation

$$F\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

is **linear** if *F* is linear in the variables

$$y, y', y'', y''', \ldots, y^{(n)}$$

- Thus the general linear ODE has the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

Linear & Non Linear ODE – Examples

$$y' + 3y = 0$$

$$y'' + 3e^{y} y' - 2t = 0$$

$$y'' + 3y' - 2t^{2} = 0$$

$$\frac{d^{4}y}{dt^{4}} - t\frac{d^{2}y}{dt^{2}} + 1 = t^{2}$$

$$u_{xx} + uu_{yy} = \sin t$$

$$u_{xx} + \sin(u)u_{yy} = \cos t$$

Solution to Differential Equations

- A solution $\phi(t)$ to an ordinary differential equation

$$y^{(n)}(t) = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

- satisfies the equation:

$$\phi^{(n)}(t) = f\left(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)}\right)$$

- Example: Verify the following solutions of the ODE

$$y'' + y = 0; \;\; y_1(t) = \sin t, \;\; y_2(t) = -\cos t, \;\; y_3(t) = 2\sin t$$

Solution to Differential Equations

- Three important questions in the study of differential equations:
  - Is there a solution? (Existence)
  - If there is a solution, is it unique? (Uniqueness)
  - If there is a solution, how do we find it?

    (Analytical Solution, Numerical Approximation, etc)

Solution to Differential Equations

- A solution of the ordinary differential equation

$$y^{(n)} = f(t, y, y', y'', \cdots y^{(n-1)})$$

on the interval

$$\alpha < t < \beta$$

is a function ϕ such that $\phi', \phi'', \cdots \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f(t, \phi(t), \phi'(t) \cdots \phi^{(n-1)}(t))$$

for every t in $\alpha < t < \beta$

- Unless stated otherwise, we assume that the function f is a real value function, and we are interested in obtaining real valued solutions of $y = \phi(t)$

Solution to Differential Equations

① IS THERE A SOLUTION? (EXISTENCE)

- It is answered by theorems stating that under certain restrictions on the function f in

$$y^{(n)} = f(t, y, y', y'' \dots y^{(n-1)})$$

allways has a solution

Reasons

(A) If a problem has no solution, we would prefer to know that fact before investing time and effort in a rain attemp to solve to problem

(B) Diff eq of physical model – somthing is wrong with the formulation – check the validity of the mathematical model

Solution to Differential Equations

(2) If there is a solution, is it unique? (Uniqueness)

(A) One solution → can be sure that we have completely solve the problem

(B) If there is more than one solution continue to search for more

(3) If there is a solution, how do we find it? (Analytical solution, Numerical Approximation etc.)

- Even though we may know that a solution exist, it may be that the solution is not expressible in therm of the usual elementary functions - polynomial, trigonometric exponential, logarithmic, and hyperbolic functions
- Common situation for most of the dif. eq.