22. The Tautochrone. A problem of interest in the history of mathematics is that of finding the **tautochrone**[8]—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometric methods, and later by Leibniz and Jakob Bernoulli using analytic arguments. Bernoulli's solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved. The geometric configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{1/2}. \tag{31}$$

Then it follows from the **principle of conservation of energy** that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b-y}} dy. \tag{32}$$

a. Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of equation (32) in this case, and using the convolution theorem, Theorem 6.6.1, show that

$$F(s) = \sqrt{\frac{2g}{\pi}} \frac{T_0}{\sqrt{s}}; \tag{33}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi} \frac{T_0}{\sqrt{y}}. \tag{34}$$

Hint: See Problem 24 of Section 6.1.

b. Combining equations (32) and (34), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{35}$$

where $\alpha = gT_0^2/\pi^2$.

c. Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve equation (35), and show that

$$x = \alpha(\theta + \sin\theta), \quad y = \alpha(1 - \cos\theta). \tag{36}$$

Equations (36) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

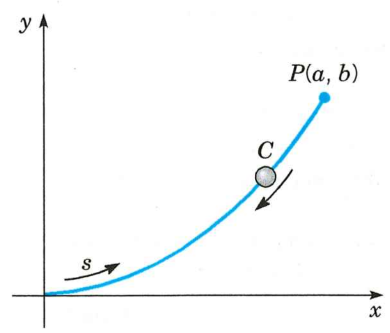

FIGURE 6.6.2 The tautochrone.

[8] The word "tautochrone" comes from the Greek words *tauto*, which means "same," and *chronos*, which means "time."

References

The books listed below contain additional information on the Laplace transform and its applications.

Churchill, R. V., *Operational Mathematics* (3rd ed.) (New York: McGraw-Hill, 1971).

Doetsch, G., *Introduction to the Theory and Application of the Laplace Transform* (trans. W. Nader) (New York: Springer, 1974).

Kaplan, W., *Operational Methods for Linear Systems* (Reading, MA: Addison-Wesley, 1962).

Kuhfittig, P. K. F., *Introduction to the Laplace Transform* (New York: Plenum, 1978).

Miles, J. W., *Integral Transforms in Applied Mathematics* (Oxford: Cambridge University Press, 2008).

Rainville, E. D., *The Laplace Transform: An Introduction* (New York: Macmillan, 1963).

Each of the books just mentioned contains a table of transforms. Extensive tables are also available. See, for example,

Erdelyi, A. (ed.), *Tables of Integral Transforms* (Vol. 1) (New York: McGraw-Hill, 1954).

Roberts, G. E., and Kaufman, H., *Table of Laplace Transforms* (Philadelphia: Saunders, 1966).

A further discussion of generalized functions can be found in

Lighthill, M. J., *An Introduction to Fourier Analysis and Generalized Functions* (Cambridge, UK: Cambridge University Press, 1958).

CHAPTER 7

Systems of First-Order Linear Equations

Many physical problems involve a number of separate but interconnected components. For example, the current and voltage in an electrical network, each mass in a mechanical system, each element (or compound) in a chemical system, or each species in a biological system have this character. In these and similar cases, the corresponding mathematical problem consists of a *system* of two or more differential equations, which can always be written as first-order differential equations. In this chapter we focus on systems of first-order *linear* differential equations and, in particular, differential equations having constant coefficients, utilizing some of the elementary aspects of linear algebra to unify the presentation. In many respects this chapter follows the same lines as the treatment of second-order linear differential equations in Chapter 3.

7.1 Introduction

Systems of simultaneous ordinary differential equations arise naturally in problems involving several dependent variables, each of which is a function of the same single independent variable. We will denote the independent variable by t and will let $x_1, x_2, x_3, \ldots$ represent dependent variables that are functions of t. Differentiation[1] with respect to t will be denoted by, for example, $\dfrac{dx_1}{dt}$ or x_1'.

Let us begin by considering the spring–mass system in Figure 7.1.1. The two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by the three springs whose constants are k_1, k_2, and k_3, respectively. We regard motion and displacement to the right as being positive.

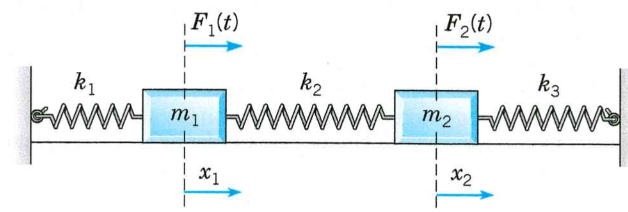

FIGURE 7.1.1 A two-mass, three-spring system.

Using arguments similar to those in Section 3.7, we find the following equations for the coordinates x_1 and x_2 of the two masses:

$$\begin{aligned} m_1 \frac{d^2x_1}{dt^2} &= k_2(x_2 - x_1) - k_1x_1 + F_1(t) = -(k_1 + k_2)x_1 + k_2x_2 + F_1(t), \\ m_2 \frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2(x_2 - x_1) + F_2(t) = k_2x_1 - (k_2 + k_3)x_2 + F_2(t). \end{aligned} \tag{1}$$

See Problem 14 for a full derivation of the system of differential equations (1).

Next, consider the parallel *LRC* circuit shown in Figure 7.1.2. Let V be the voltage drop across the capacitor and I the current through the inductor. Then, referring to Section 3.7 and

[1] In some treatments you will see differentiation with respect to time represented with a dot over the function, as in $\dot{x}_1 = \dfrac{dx_1}{dt}$ and $\ddot{x}_1 = \dfrac{d^2x_1}{dt^2}$. We reserve this notation for a specific purpose, which will be introduced in Section 9.6.

The proof of this theorem can be constructed by generalizing the argument in Section 2.8, but we do not give it here. However, note that, in the hypotheses of the theorem, nothing is said about the partial derivatives of $F_1, \ldots, F_n$ with respect to the independent variable t. Also, in the conclusion, the length $2h$ of the interval in which the solution exists is not specified exactly, and in some cases it may be very short. Finally, the same result can be established on the basis of somewhat weaker but more complicated hypotheses, so the theorem as stated is not the most general one known, and the given conditions are sufficient, but not necessary, for the conclusion to hold.

If each of the functions $F_1, \ldots, F_n$ in equations (11) is a linear function of the dependent variables $x_1, \ldots, x_n$, then the system of differential equations is said to be **linear**; otherwise, it is **nonlinear**. Thus the most general system of n first-order linear differential equations has the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t), \\ &\ \vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t). \end{aligned} \tag{14}$$

If each of the functions $g_1(t), \ldots, g_n(t)$ is zero for all t in the interval I, then the system (14) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**. Observe that the systems (1) and (2) are both linear. The system (1) is nonhomogeneous unless $F_1(t) = F_2(t) = 0$, while the system (2) is homogeneous.

For the linear system (14), the existence and uniqueness theorem is simpler and also has a stronger conclusion. It is analogous to Theorems 2.4.1 and 3.2.1.

Theorem 7.1.2

If the functions $p_{11}, p_{12}, \ldots, p_{nn}, g_1, \ldots, g_n$ are continuous on an open interval $I\colon \alpha < t < \beta$, then there exists a unique solution $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ of the system (14) that also satisfies the initial conditions (13), where t_0 is any point in I, and $x_1^0, \ldots, x_n^0$ are any prescribed numbers. Moreover, the solution exists throughout the interval I.

Note that, in contrast to the situation for a nonlinear system, the existence and uniqueness of the solution of a linear system are guaranteed throughout the interval in which the hypotheses are satisfied. Furthermore, for a linear system the initial values $x_1^0, \ldots, x_n^0$ at $t = t_0$ are completely arbitrary, whereas in the nonlinear case the initial point must lie in the region R defined in Theorem 7.1.1.

The rest of this chapter is devoted to systems of linear first-order differential equations (nonlinear systems are included in the discussions in Chapters 8 and 9). Our presentation makes use of matrix notation and assumes that you have some familiarity with the properties of matrices. The basic facts about matrices needed for this discussion are presented in Sections 7.2 and 7.3; some more advanced material is reviewed as needed in later sections.

Problems

In each of Problems 1 through 3, transform the given equation into a system of first-order equations.

1. $u'' + 0.5u' + 2u = 0$
2. $t^2u'' + tu' + (t^2 - 0.25)u = 0$
3. $u^{(4)} - u = 0$

In each of Problems 4 and 5, transform the given initial value problem into an initial value problem for two first-order equations.

4. $u'' + 0.25u' + 4u = 2\cos(3t), \quad u(0) = 1, \quad u'(0) = -2$
5. $u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u_0'$

6. Systems of first-order equations can sometimes be transformed into a single equation of higher-order. Consider the system

$$x_1' = -2x_1 + x_2, \quad x_2' = x_1 - 2x_2.$$

   a. Solve the first differential equation for x_2.
   b. Substitute the result of a into the second differential equation, thereby obtaining a second-order differential equation for x_1.
   c. Solve the differential equation found in b for x_1.
   d. Use the results of a and c to find x_2.

In each of Problems 7 through 9, proceed as in Problem 6.

   a. Transform the given system into a single equation of second-order.
   b. Find x_1 and x_2 that also satisfy the given initial conditions.
   c. Sketch the graph of the solution in the x_1x_2-plane for $t \geq 0$.

7. $x_1' = 3x_1 - 2x_2, \quad x_1(0) = 3$
   $x_2' = 2x_1 - 2x_2, \quad x_2(0) = \frac{1}{2}$
8. $x_1' = 2x_2, \quad x_1(0) = 3$
   $x_2' = -2x_1, \quad x_2(0) = 4$
9. $x_1' = -\frac{1}{2}x_1 + 2x_2, \quad x_1(0) = -2$
   $x_2' = -2x_1 - \frac{1}{2}x_2, \quad x_2(0) = 2$
10. Transform equations (2) for the parallel circuit into a single second-order equation.
11. Show that if a_{11}, a_{12}, a_{21}, and a_{22} are constants with a_{12} and a_{21} not both zero, and if the functions g_1 and g_2 are differentiable, then the initial value problem

$$x_1' = a_{11}x_1 + a_{12}x_2 + g_1(t), \quad x_1(0) = x_1^0$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + g_2(t), \quad x_2(0) = x_2^0$$

can be transformed into an initial value problem for a single second-order equation. Can the same procedure be carried out if $a_{11}, \ldots, a_{22}$ are functions of t?

12. Consider the linear homogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y,$$
$$y' = p_{21}(t)x + p_{22}(t)y.$$

Show that if $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two solutions of the given system, then $x = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$ is also a solution for any constants c_1 and c_2. This is the principle of superposition; it will be discussed in much greater detail in Section 7.4.

13. Let $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ be any two solutions of the linear nonhomogeneous system

$$x' = p_{11}(t)x + p_{12}(t)y + g_1(t),$$
$$y' = p_{21}(t)x + p_{22}(t)y + g_2(t).$$

Show that $x = x_1(t) - x_2(t)$, $y = y_1(t) - y_2(t)$ is a solution of the corresponding homogeneous system.

14. Equations (1) can be derived by drawing a free-body diagram showing the forces acting on each mass. Figure 7.1.3*a* shows the situation when the displacements x_1 and x_2 of the two masses are both positive (to the right) and $x_2 > x_1$. Then springs 1 and 2 are elongated and spring 3 is compressed, giving rise to forces as shown in Figure 7.1.3*b*. Use Newton's law ($F = ma$) to derive equations (1).

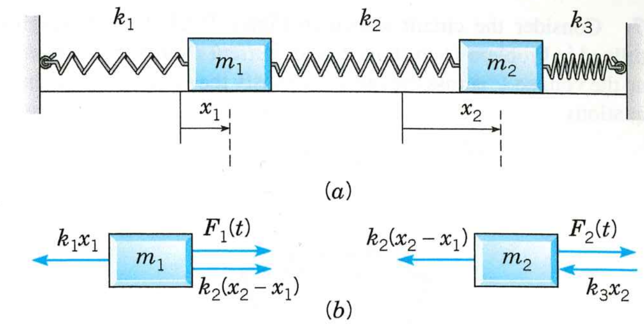

FIGURE 7.1.3 (*a*) The displacements x_1 and x_2 are both positive. (*b*) The free-body diagram for the spring–mass system.

15. Transform the system (1) into a system of first-order differential equations by letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, and $y_4 = x_2'$.

Electric Circuits. The theory of electric circuits, such as that shown in Figure 7.1.2, consisting of inductors, resistors, and capacitors, is based on Kirchhoff's laws: (1) The net flow of current into each node (or junction) is zero, and (2) the net voltage drop around each closed loop is zero. In addition to Kirchhoff's laws, we also have the relation between the current I, with units of amperes through each circuit element and the voltage drop V, measured in volts, across the element:

$$V = RI, \quad R = \text{resistance in ohms;}$$
$$C\frac{dV}{dt} = I, \quad C = \text{capacitance in farads;}^2$$
$$L\frac{dI}{dt} = V, \quad L = \text{inductance in henrys.}$$

Kirchhoff's laws and the current-voltage relation for each circuit element provide a system of algebraic and differential equations from which the voltage and current throughout the circuit can be determined. Problems 16 through 18 illustrate the procedure just described.

16. Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.

   a. Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0. \tag{15}$$

   In a similar way, show that

$$V_2 - V_3 = 0. \tag{16}$$

   b. Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0. \tag{17}$$

   c. Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3. \tag{18}$$

   d. Eliminate V_2, V_3, I_1, and I_2 among equations (15) through (18) to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1. \tag{19}$$

   Observe that if we omit the subscripts in equations (19), then we have the system (2) of this section.

[2] Actual capacitors typically have capacitances measured in microfarads. We use farad as the unit for numerical convenience.

17. Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 16 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\frac{dI}{dt} = -I - V, \quad \frac{dV}{dt} = 2I - V.$$

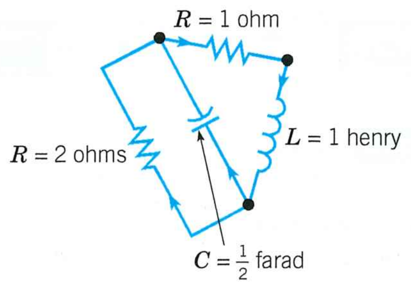

FIGURE 7.1.4 The circuit in Problem 17.

18. Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 16 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

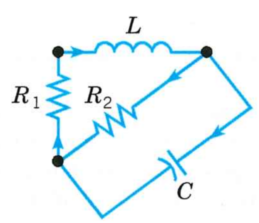

FIGURE 7.1.5 The circuit in Problem 18.

19. Consider the two interconnected tanks shown in Figure 7.1.6. Tank 1 initially contains 30 gal of water and 25 oz of salt, and Tank 2 initially contains 20 gal of water and 15 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min. The mixture flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into Tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from Tank 2 at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

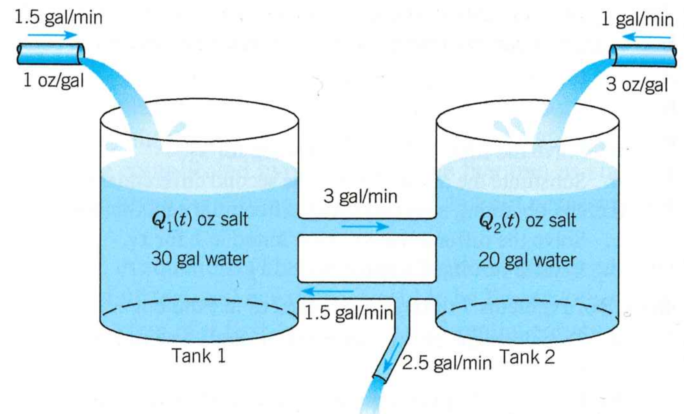

FIGURE 7.1.6 Two interconnected tanks (Problem 19).

a. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is nonhomogeneous.

b. Find the values of Q_1 and Q_2 for which the system is in equilibrium—that is, does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?

c. Let $x_1 = Q_1(t) - Q_1^E$ and $x_2 = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.

20. Consider two interconnected tanks similar to those in Figure 7.1.6. Initially, Tank 1 contains 60 gal of water and Q_1^0 oz of salt, and Tank 2 contains 100 gal of water and Q_2^0 oz of salt. Water containing q_1 oz/gal of salt flows into Tank 1 at a rate of 3 gal/min. The mixture in Tank 1 flows out at a rate of 4 gal/min, of which half flows into Tank 2, while the remainder leaves the system. Water containing q_2 oz/gal of salt also flows into Tank 2 from the outside at the rate of 1 gal/min. The mixture in Tank 2 leaves it at a rate of 3 gal/min, of which some flows back into Tank 1 at a rate of 1 gal/min, while the rest leaves the system.

a. Draw a diagram that depicts the flow process described above. Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions for Q_1 and Q_2 that model the flow process.

b. Find the equilibrium values Q_1^E and Q_2^E in terms of the concentrations q_1 and q_2.

c. Is it possible (by adjusting q_1 and q_2) to obtain $Q_1^E = 60$ and $Q_2^E = 50$ as an equilibrium state?

d. Describe which equilibrium states are possible for this system for various values of q_1 and q_2.

7.2 Matrices

For both theoretical and computational reasons, it is advisable to bring some of the results of matrix algebra[3] to bear on the initial value problem for a system of linear differential equations.

[3]The properties of matrices were first extensively explored in 1858 in a paper by the English algebraist Arthur Cayley (1821–1895), although the word "matrix" was introduced by his good friend James Sylvester (1814–1897) in 1850. Cayley did some of his best mathematical work while practicing law from 1849 to 1863; he then became professor of mathematics at Cambridge, a position he held for the rest of his life. After Cayley's groundbreaking work, the development of matrix theory proceeded rapidly, with significant contributions by Charles Hermite, Georg Frobenius, and Camille Jordan, among others.

This section and the next are devoted to a brief summary of the facts that will be needed later. More details can be found in any elementary book on linear algebra. We assume, however, that you are familiar with determinants and how to evaluate them.

We designate matrices by boldfaced capitals **A**, **B**, **C**, . . . , occasionally using boldfaced Greek capitals **Φ**, **Ψ**, ... A matrix **A** consists of a rectangular array of numbers, or elements, arranged in m rows and n columns—that is,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}. \tag{1}$$

We speak of **A** as an $\mathbf{m \times n}$ **matrix**. Although later in the chapter we will often assume that the elements of certain matrices are real numbers, in this section we allow for the possibility that the elements of matrices may be complex numbers. The element lying in the i^{th} row and j^{th} column is designated by a_{ij}, the first subscript identifying its row and the second its column. Sometimes the notation (a_{ij}) is used to denote the matrix whose generic element is a_{ij}.

Associated with each matrix **A** is the matrix $\mathbf{A}^T$, which is known as the **transpose** of **A** and is obtained from **A** by interchanging the rows and columns of **A**. Thus, if $\mathbf{A} = (a_{ij})$, then $\mathbf{A}^T = (a_{ji})$. Also, we will denote by $\overline{a_{ij}}$ the complex conjugate of a_{ij}, and by $\overline{\mathbf{A}}$ the matrix obtained from **A** by replacing each element a_{ij} by its conjugate $\overline{a_{ij}}$. The matrix $\overline{\mathbf{A}}$ is called the **conjugate** of **A**. It will also be necessary to consider the transpose of the conjugate matrix $\overline{\mathbf{A}}^T$. This matrix is called the **adjoint** of **A** and will be denoted by $\mathbf{A}^*$.

For example, let

$$\mathbf{A} = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}.$$

Then

$$\mathbf{A}^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix},$$

$$\mathbf{A}^* = \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix}.$$

We are particularly interested in two somewhat special kinds of matrices: **square matrices**, which have the same number of rows and columns—that is, $m = n$; and **vectors** (or **column vectors**), which can be thought of as $n \times 1$ matrices, or matrices having only one column. Square matrices having n rows and n columns are said to be of order n. We denote (column) vectors by boldfaced lowercase letters: $\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$. The transpose $\mathbf{x}^T$ of an $n \times 1$ column vector is a $1 \times n$ row vector—that is, the matrix consisting of one row whose elements are the same as the elements in the corresponding positions of **x**.

Properties of Matrices.

1. **Equality.** Two $m \times n$ matrices **A** and **B** are said to be equal if all corresponding elements are equal—that is, if $a_{ij} = b_{ij}$ for each i and j.

2. **Zero.** The symbol **0** will be used to denote the matrix (or vector) each of whose elements is zero.

3. **Addition.** The sum of two $m \times n$ matrices **A** and **B** is defined as the matrix obtained by adding corresponding elements:

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}). \tag{2}$$

(c) Obtain zeros in the off-diagonal positions (shaded) in the second column by adding the second row to the first row and adding (-4) times the second row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right)$$

(d) Obtain a 1 in the diagonal position (shaded) in the third column by multiplying the third row by $-\frac{1}{5}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

(e) Obtain zeros in the off-diagonal positions (shaded) in the third column by adding $\left(-\frac{3}{2}\right)$ times the third row to the first row and adding $\left(-\frac{5}{2}\right)$ times the third row to the second row.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right) = (\mathbf{I} \,|\, \mathbf{A}^{-1}).$$

Thus

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

That this matrix is, in fact, the inverse of **A** can be verified by direct multiplication with the original matrix **A**.

This example was made slightly simpler by the fact that the given matrix **A** had a 1 in the upper left corner ($a_{11} = 1$). If this is not the case, then the first step is to produce a 1 there by multiplying the first row by $1/a_{11}$, as long as $a_{11} \neq 0$. If $a_{11} = 0$, then the first row must be interchanged with some other row to bring a nonzero element into the upper left position before proceeding. If this cannot be done, because every element in the first column is zero, then the matrix has no inverse and is singular. A similar situation may occur at later stages of the process as well, and the remedy is the same: interchange the given row with a lower row so as to bring a nonzero element to the desired diagonal location. If, at any stage, this cannot be done, then the original matrix is singular.

Matrix Functions. We sometimes need to consider vectors or matrices whose elements are functions of a real variable t. We write

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \text{and} \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}, \tag{25}$$

respectively.

The matrix $\mathbf{A}(t)$ is said to be continuous at $t = t_0$ or on an interval $\alpha < t < \beta$ if each element of **A** is a continuous function at the given point or on the given interval. Similarly, $\mathbf{A}(t)$ is said to be differentiable if each of its elements is differentiable, and its derivative $d\mathbf{A}/dt$ is defined by

$$\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt}\right); \tag{26}$$

that is, each element of $d\mathbf{A}/dt$ is the derivative of the corresponding element of **A**. In the same way, the integral of a matrix function is defined as

$$\int_a^b \mathbf{A}(t)\,dt = \left(\int_a^b a_{ij}(t)\,dt\right). \tag{27}$$

For example, if

$$\mathbf{A}(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix},$$

then

$$\mathbf{A}'(t) = \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix} \quad \text{and} \quad \int_0^\pi \mathbf{A}(t)\,dt = \begin{pmatrix} 2 & \pi^2/2 \\ \pi & 0 \end{pmatrix}.$$

Many of the rules of elementary calculus extend easily to matrix functions; in particular,

$$\frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{d\mathbf{A}}{dt}, \quad \text{where } \mathbf{C} \text{ is a constant matrix;} \tag{28}$$

$$\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}; \tag{29}$$

$$\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}. \tag{30}$$

In equations (28) and (30), care must be taken in each term to avoid interchanging the order of multiplication. The definitions expressed by equations (26) and (27) also apply as special cases to vectors.

We conclude this section with an important reminder: some operations on matrices are accomplished by applying the operation separately to each element of the matrix. Examples include multiplication by a number, differentiation, and integration. However, this is not true of many other operations. For instance, the square of a matrix is not calculated by squaring each of its elements.

Problems

1. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{pmatrix}$, find

a. $2\mathbf{A} + \mathbf{B}$
b. $\mathbf{A} - 4\mathbf{B}$
c. $\mathbf{AB}$
d. $\mathbf{BA}$

2. If $\mathbf{A} = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$, find

a. $\mathbf{A} - 2\mathbf{B}$
b. $3\mathbf{A} + \mathbf{B}$
c. $\mathbf{AB}$
d. $\mathbf{BA}$

3. If $\mathbf{A} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{pmatrix}$, find

a. $\mathbf{A}^T$
b. $\mathbf{B}^T$
c. $\mathbf{A}^T + \mathbf{B}^T$
d. $(\mathbf{A}+\mathbf{B})^T$

4. If $\mathbf{A} = \begin{pmatrix} 3-2i & 1+i \\ 2-i & -2+3i \end{pmatrix}$, find

a. $\mathbf{A}^T$
b. $\overline{\mathbf{A}}$
c. $\mathbf{A}^*$

5. If $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 0 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, and

$\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, verify that

a. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
b. $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$
c. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
d. $\alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A}+\alpha\mathbf{B}$

6. Prove each of the following laws of matrix algebra:

a. $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$
b. $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$
c. $\alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A}+\alpha\mathbf{B}$
d. $(\alpha+\beta)\mathbf{A} = \alpha\mathbf{A}+\beta\mathbf{A}$
e. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
f. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

7. If $\mathbf{x} = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$, find

a. $\mathbf{x}^T\mathbf{y}$
b. $\mathbf{y}^T\mathbf{y}$
c. $(\mathbf{x}, \mathbf{y})$
d. $(\mathbf{y}, \mathbf{y})$

In each of Problems 8 through 14, if the given matrix is nonsingular, find its inverse. If the matrix is singular, verify that its determinant is zero.

8. $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 3 & -1 \\ 6 & 2 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$

12. $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 2 & 1 \\ 4 & -1 & -1 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

15. If $\mathbf{A}$ is a square matrix, and if there are two matrices $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, show that $\mathbf{B} = \mathbf{C}$. Thus, if a matrix has an inverse, it can have only one.

16. If $\mathbf{A}(t) = \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix}$ and

$\mathbf{B}(t) = \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix}$, find

a. $\mathbf{A}+3\mathbf{B}$
b. $\mathbf{AB}$
c. $\dfrac{d\mathbf{A}}{dt}$
d. $\displaystyle\int_0^1 \mathbf{A}(t)\,dt$

In each of Problems 17 and 18, verify that the given vector satisfies the given differential equation.

17. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t,$

$\mathbf{x} = \begin{pmatrix} (1+2t)e^t \\ 2te^t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^t$

18. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x},$

$\mathbf{x} = \begin{pmatrix} 6e^{-t} \\ -8e^{-t}+2e^{2t} \\ -4e^{-t}-2e^{2t} \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix}e^{-t} + 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}e^{2t}$

In each of Problems 19 and 20, verify that the given matrix satisfies the given differential equation.

19. $\mathbf{\Psi}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{\Psi}, \quad \mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$

20. $\mathbf{\Psi}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{\Psi}, \quad \mathbf{\Psi}(t) = \begin{pmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{pmatrix}$

7.3 Systems of Linear Algebraic Equations; Linear Independence, Eigenvalues, Eigenvectors

In this section we review some results from linear algebra that are particularly important for the solution of systems of linear differential equations. Some of these results are easily proved and others are not; since we are interested simply in summarizing some useful information in compact form, we give no indication of proofs in either case. All the results in this section depend on some basic facts about the solution of systems of linear algebraic equations.

Systems of Linear Algebraic Equations. A set of n simultaneous linear algebraic equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

can be written in matrix form as

$$\mathbf{Ax} = \mathbf{b}, \tag{2}$$

where the $n \times n$ matrix $\mathbf{A}$ and the n-dimensional vector $\mathbf{b}$ are given, and the components of the n-dimensional vector $\mathbf{x}$ are to be determined. If $\mathbf{b} = \mathbf{0}$, the system is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

If the coefficient matrix $\mathbf{A}$ is nonsingular—that is, if $\det \mathbf{A}$ is not zero—then there is a unique solution of the system (2) for any vector $\mathbf{b}$. Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and the solution can be found by multiplying each side of equation (2) on the left by $\mathbf{A}^{-1}$; thus

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}. \tag{3}$$

In particular, the homogeneous problem $\mathbf{Ax} = \mathbf{0}$, corresponding to $\mathbf{b} = \mathbf{0}$ in equation (2), has only the trivial solution $\mathbf{x} = \mathbf{0}$.

On the other hand, if $\mathbf{A}$ is singular—that is, if $\det \mathbf{A}$ is zero—then, depending on the specific right-hand side $\mathbf{b}$, solutions of equation (2) either do not exist, or do exist but are not unique. Since $\mathbf{A}$ is singular, $\mathbf{A}^{-1}$ does not exist, so equation (3) is no longer valid.

When $\mathbf{A}$ is singular, the homogeneous system

$$\mathbf{Ax} = \mathbf{0} \tag{4}$$

has (infinitely many) nonzero solutions in addition to the trivial solution. The situation for the nonhomogeneous system (2) is more complicated. This system has no solution unless the vector $\mathbf{b}$ satisfies a certain further condition. This condition is that

$$(\mathbf{b}, \mathbf{y}) = 0, \tag{5}$$

for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. If condition (5) is met, then the system (2) has (infinitely many) solutions. These solutions are of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}, \tag{6}$$

where $\mathbf{x}^{(0)}$ is a particular solution of equation (2), and $\boldsymbol{\xi}$ is the most general solution of the homogeneous system (4). Note the resemblance between equation (6) and the solution of a nonhomogeneous linear differential equation. The proofs of some of the preceding statements are outlined in Problems 21 through 25.

The results in the preceding paragraph are important as a means of classifying the solutions of linear systems. However, for solving particular systems, it is generally best to use row reduction to transform the system into a much simpler one from which the solution(s), if

EXAMPLE 5

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{37}$$

Solution:

The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{38}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 = 0. \tag{39}$$

The roots of equation (39) are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -1$. Thus 2 is a simple eigenvalue, and -1 is an eigenvalue of algebraic multiplicity 2, or a double eigenvalue.

To find the eigenvector $\mathbf{x}^{(1)}$ corresponding to the eigenvalue λ_1, we substitute $\lambda = 2$ in equation (38); this gives the system

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

We can use elementary row operations to reduce this to the equivalent system

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{41}$$

Solving this system yields the eigenvector

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{42}$$

For $\lambda = -1$, equations (38) reduce immediately to the single equation

$$x_1 + x_2 + x_3 = 0. \tag{43}$$

Thus values for two of the quantities x_1, x_2, x_3 can be chosen arbitrarily, and the third is determined from equation (43). For example, if $x_1 = c_1$ and $x_2 = c_2$, then $x_3 = -c_1 - c_2$. In vector notation we have

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{44}$$

For example, by choosing $c_1 = 1$ and $c_2 = 0$, we obtain the eigenvector

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{45}$$

Any nonzero multiple of $\mathbf{x}^{(2)}$ is also an eigenvector, but a second linearly independent eigenvector can be found by making another choice of c_1 and c_2—for instance, $c_1 = 0$ and $c_2 = 1$. In this case we obtain

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{46}$$

which is linearly independent of $\mathbf{x}^{(2)}$. Therefore, in this example, two linearly independent eigenvectors are associated with the double eigenvalue.

An important special class of matrices, called **self-adjoint** or **Hermitian** matrices, are those for which $\mathbf{A}^* = \mathbf{A}$; that is, $\overline{a}_{ji} = a_{ij}$. Hermitian matrices include as a subclass real symmetric matrices—that is, matrices that have real elements and for which $\mathbf{A}^T = \mathbf{A}$. The eigenvalues and eigenvectors of Hermitian matrices always have the following useful properties:

1. All eigenvalues are real.
2. There always exists a full set of n linearly independent eigenvectors, regardless of the algebraic multiplicities of the eigenvalues.
3. If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues, then $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$. Thus, if all eigenvalues are simple, then the associated eigenvectors form an orthogonal set of vectors.
4. Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal. Thus the full set of n eigenvectors can always be chosen to be orthogonal as well as linearly independent.

The proofs of statements 1 and 3 above are outlined in Problems 27 and 28. Example 5 involves a real symmetric matrix and illustrates properties 1, 2, and 3, but the choice we have made for $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ does not illustrate property 4. However, it is always possible to choose an $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$ so that $(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}) = 0$. For instance, in Example 5 we could have chosen $\mathbf{x}^{(2)}$ as before and $\mathbf{x}^{(3)}$ by using $c_1 = 1$ and $c_2 = -2$ in equation (46). In this way we obtain

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

as the eigenvectors associated with the eigenvalue $\lambda = -1$. These eigenvectors are orthogonal to each other as well as to the eigenvector $\mathbf{x}^{(1)}$ that corresponds to the eigenvalue $\lambda = 2$.

Problems

In each of Problems 1 through 5, either solve the given system of equations, or else show that there is no solution.

1. $x_1 \quad - x_3 = 0$
$3x_1 + x_2 + x_3 = 1$
$-x_1 + x_2 + 2x_3 = 2$

2. $x_1 + 2x_2 - x_3 = 1$
$2x_1 + x_2 + x_3 = 1$
$x_1 - x_2 + 2x_3 = 1$

3. $x_1 + 2x_2 - x_3 = 2$
$2x_1 + x_2 + x_3 = 1$
$x_1 - x_2 + 2x_3 = -1$

4. $x_1 + 2x_2 - x_3 = 0$
$2x_1 + x_2 + x_3 = 0$
$x_1 - x_2 + 2x_3 = 0$

5. $x_1 \quad - x_3 = 0$
$3x_1 + x_2 + x_3 = 0$
$-x_1 + x_2 + 2x_3 = 0$

In each of Problems 6 through 9, determine whether the members of the given set of vectors are linearly independent. If they are linearly dependent, find a linear relation among them. In Problems 6 to 9, 11, and 12, vectors are written as row vectors to save space but may be considered as column vectors; that is, the transposes of the given vectors may be used instead of the vectors themselves.

6. $\mathbf{x}^{(1)} = (1, 1, 0), \quad \mathbf{x}^{(2)} = (0, 1, 1), \quad \mathbf{x}^{(3)} = (1, 0, 1)$

7. $\mathbf{x}^{(1)} = (2, 1, 0), \quad \mathbf{x}^{(2)} = (0, 1, 0), \quad \mathbf{x}^{(3)} = (-1, 2, 0)$

8. $\mathbf{x}^{(1)} = (1, 2, -1, 0), \quad \mathbf{x}^{(2)} = (2, 3, 1, -1),$
$\mathbf{x}^{(3)} = (-1, 0, 2, 2), \quad \mathbf{x}^{(4)} = (3, -1, 1, 3)$

9. $\mathbf{x}^{(1)} = (1, 2, -2), \quad \mathbf{x}^{(2)} = (3, 1, 0),$
$\mathbf{x}^{(3)} = (2, -1, 1), \quad \mathbf{x}^{(4)} = (4, 3, -2)$

10. Suppose that each of the vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ has n components, where $n < m$. Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent.

In each of Problems 11 and 12, determine whether the members of the given set of vectors are linearly independent for $-\infty < t < \infty$. If they are linearly dependent, find the linear relation among them.

11. $\mathbf{x}^{(1)}(t) = (e^{-t}, 2e^{-t}), \quad \mathbf{x}^{(2)}(t) = (e^{-t}, e^{-t}),$
$\mathbf{x}^{(3)}(t) = (3e^{-t}, 0)$

12. $\mathbf{x}^{(1)}(t) = (2\sin t, \sin t), \quad \mathbf{x}^{(2)}(t) = (\sin t, 2\sin t)$

13. Let

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$. Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

In each of Problems 14 through 20, find all eigenvalues and eigenvectors of the given matrix.

14. $\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

16. $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

17. $\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$

19. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

20. $\begin{pmatrix} \frac{11}{9} & -\frac{2}{9} & \frac{8}{9} \\ -\frac{2}{9} & \frac{2}{9} & \frac{10}{9} \\ \frac{8}{9} & \frac{10}{9} & \frac{5}{9} \end{pmatrix}$

Problems 21 through 25 deal with the problem of solving $\mathbf{Ax} = \mathbf{b}$ when $\det \mathbf{A} = 0$.

21. **a.** Suppose that $\mathbf{A}$ is a real-valued $n \times n$ matrix. Show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^T\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$. *Hint:* You may find it simpler to consider first the case $n = 2$; then extend the result to an arbitrary value of n.

b. If $\mathbf{A}$ is not necessarily real, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.

c. If $\mathbf{A}$ is Hermitian, show that $(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{Ay})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$.

22. Suppose that, for a given matrix $\mathbf{A}$, there is a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. Show that there is also a nonzero vector $\mathbf{y}$ such that $\mathbf{A}^*\mathbf{y} = \mathbf{0}$.

23. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{Ax} = \mathbf{b}$ has solutions. Show that $(\mathbf{b}, \mathbf{y}) = 0$, where $\mathbf{y}$ is any solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Verify that this statement is true for the set of equations in Example 2. *Hint:* Use the result of Problem 21b.

24. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Show that if $\boldsymbol{\xi}$ is a solution of $\mathbf{A}\boldsymbol{\xi} = \mathbf{0}$ and α is any constant, then $\mathbf{x} = \mathbf{x}^{(0)} + \alpha\boldsymbol{\xi}$ is also a solution of $\mathbf{Ax} = \mathbf{b}$.

25. Suppose that $\det \mathbf{A} = 0$ and that $\mathbf{y}$ is a solution of $\mathbf{A}^*\mathbf{y} = \mathbf{0}$. Show that if $(\mathbf{b}, \mathbf{y}) = 0$ for every such $\mathbf{y}$, then $\mathbf{Ax} = \mathbf{b}$ has solutions. Note that this is the converse of Problem 23; the form of the solution is given by Problem 24. *Hint:* What does the relation $\mathbf{A}^*\mathbf{y} = \mathbf{0}$ say about the rows of $\mathbf{A}$? It may be helpful to consider the case $n = 2$ first.

26. Prove that $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

27. In this problem we show that the eigenvalues of a Hermitian matrix $\mathbf{A}$ are real. Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ.

a. Show that $(\mathbf{Ax}, \mathbf{x}) = (\mathbf{x}, \mathbf{Ax})$. *Hint:* See Problem 21c.

b. Show that $\lambda(\mathbf{x}, \mathbf{x}) = \overline{\lambda}(\mathbf{x}, \mathbf{x})$. *Hint:* Recall that $\mathbf{Ax} = \lambda\mathbf{x}$.

c. Show that $\lambda = \overline{\lambda}$; that is, the eigenvalue λ is real.

28. Show that if λ_1 and λ_2 are eigenvalues of a Hermitian matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal. *Hint:* Use the results of Problems 21c and 27 to show that $(\lambda_1 - \lambda_2)(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$.

29. Show that if λ_1 and λ_2 are eigenvalues of any matrix $\mathbf{A}$, and if $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. *Hint:* Start from $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = \mathbf{0}$; multiply by $\mathbf{A}$ to obtain $c_1\lambda_1\mathbf{x}^{(1)} + c_2\lambda_2\mathbf{x}^{(2)} = \mathbf{0}$. Then show that $c_1 = c_2 = 0$.

7.4 Basic Theory of Systems of First-Order Linear Equations

The general theory of a system of n first-order linear equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1}$$

closely parallels that of a single linear equation of nth order. The discussion in this section therefore follows the same general lines as that in Sections 3.2 and 4.1. To discuss the system (1) most effectively, we write it in matrix notation. That is, we consider $x_1 = x_1(t), \ldots, x_n = x_n(t)$ to be components of a vector $\mathbf{x} = \mathbf{x}(t)$; similarly,

$g_1(t), \ldots, g_n(t)$ are components of a vector $\mathbf{g}(t)$, and $p_{11}(t), \ldots, p_{nn}(t)$ are elements of an $n \times n$ matrix $\mathbf{P}(t)$. Equation (1) then takes the form

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t). \tag{2}$$

The use of vectors and matrices not only saves a great deal of space and facilitates calculations but also emphasizes the similarity between systems of differential equations and single (scalar) differential equations.

A vector $\mathbf{x} = \mathbf{x}(t)$ is said to be a solution of equation (2) if its components satisfy the system of equations (1). Throughout this section we assume that $\mathbf{P}$ and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$; that is, each of the scalar functions $p_{11}, \ldots, p_{nn}, g_1, \ldots, g_n$ is continuous there. According to Theorem 7.1.2, this is sufficient to guarantee the existence of solutions of equation (2) on the interval $\alpha < t < \beta$.

It is convenient to consider first the homogeneous equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{3}$$

obtained from equation (2) by setting $\mathbf{g}(t) = \mathbf{0}$. Just as we have seen for a single linear differential equation (of any order), once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous equation (2); this is taken up in Section 7.9.

We use the notation

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}, \ldots \tag{4}$$

to designate specific solutions of the system (3). Note that $x_{ij}(t) = x_i^{(j)}(t)$ refers to the i^{th} component of the j^{th} solution $\mathbf{x}^{(j)}(t)$. The main facts about the structure of solutions of the system (3) are stated in Theorems 7.4.1 to 7.4.5. They closely resemble the corresponding theorems in Sections 3.2 and 4.1; some of the proofs are left to you as exercises.

Theorem 7.4.1 | Principle of Superposition

If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system (3), then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

The **principle of superposition** is proved simply by differentiating $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ and using the facts that $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ satisfy equation (3). As an example, it can be verified that

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \tag{5}$$

satisfy the equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}. \tag{6}$$

Then, according to Theorem 7.4.1,

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = \begin{pmatrix} c_1e^{3t} + c_2e^{-t} \\ 2c_1e^{3t} - 2c_2e^{-t} \end{pmatrix} \tag{7}$$

also satisfies equation (6).

Finally, it may happen (just as for second-order linear equations) that a system whose coefficients are all real-valued may give rise to solutions that are complex-valued. In this case, the following theorem is analogous to Theorem 3.2.6 and enables us to obtain real-valued solutions instead.

Theorem 7.4.5

Consider the system (3)

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$

where each element of $\mathbf{P}$ is a real-valued continuous function. If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex-valued solution of equation (3), then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions of this equation.

To prove this result, we substitute $\mathbf{u}(t) + i\mathbf{v}(t)$ for $\mathbf{x}$ in equation (3), thereby obtaining

$$\mathbf{x}' - \mathbf{P}(t)\mathbf{x} = \mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) + i(\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t)) = \mathbf{0}. \tag{17}$$

We have used the assumption that $\mathbf{P}(t)$ is real-valued to separate equation (17) into its real and imaginary parts. Since a complex number is zero if and only if its real and imaginary parts are both zero, we conclude that $\mathbf{u}'(t) - \mathbf{P}(t)\mathbf{u}(t) = \mathbf{0}$ and $\mathbf{v}'(t) - \mathbf{P}(t)\mathbf{v}(t) = \mathbf{0}$. Therefore, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions of equation (3).

To summarize the results of this section:

1. Any set of n linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ constitutes a fundamental set of solutions.
2. Under the conditions given in this section, such fundamental sets always exist.
3. Every solution of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ can be represented as a linear combination of any fundamental set of solutions.

Problems

In Problems 1 through 6 you are given a homogeneous system of first-order linear differential equations and two vector-valued functions, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

a. Show that the given functions are solutions of the given system of differential equations.
b. Show that $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution of the given system for any values of c_1 and c_2.
c. Show that the given functions form a fundamental set of solutions of the given system.
d. Find the solution of the given system that satisfies the initial condition $\mathbf{x}(0) = (1, 2)^T$.
e. Find $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right](t)$.
f. Show that the Wronskian, $W = W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right]$, found in e is a solution of Abel's equation: $W' = (p_{11}(t) + p_{22}(t))W$.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x};\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x};\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t},\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x};\ \mathbf{x}^{(1)} = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix},$
$\mathbf{x}^{(2)} = \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x};\ \mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 4 \end{pmatrix},\ \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

5. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}\ (t > 0);\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}t,\ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}t^{-1}$

6. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}\ (t > 0);\ \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1},\ \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2$

7. Prove the generalization of Theorem 7.4.1, as expressed in the sentence that includes equation (8), for an arbitrary value of the integer k.

8. In this problem we outline a proof of Theorem 7.4.3 in the case $n = 2$. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be solutions of equation (3) for $\alpha < t < \beta$, and let W be the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
a. Show that
$$\frac{dW}{dt} = \begin{vmatrix} \dfrac{dx_1^{(1)}}{dt} & \dfrac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \dfrac{dx_2^{(1)}}{dt} & \dfrac{dx_2^{(2)}}{dt} \end{vmatrix}.$$
b. Using equation (3), show that
$$\frac{dW}{dt} = (p_{11} + p_{22})W.$$
c. Find $W(t)$ by solving the differential equation obtained in part b. Use this expression to obtain the conclusion stated in Theorem 7.4.3.
d. Prove Theorem 7.4.3 for an arbitrary value of n by generalizing the procedure of parts a, b, and c.

9. Show that the Wronskians of two fundamental sets of solutions of the system (3) can differ at most by a multiplicative constant.
Hint: Use equation (15).

10. If $x_1 = y$ and $x_2 = y'$, then the second-order equation
$$y'' + p(t)y' + q(t)y = 0 \tag{18}$$
corresponds to the system
$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2. \tag{19}$$
Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of equations (19), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of equation (18), then $W[y^{(1)}, y^{(2)}] = cW[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$, where c is a nonzero constant. *Hint:* $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

11. Show that the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is the sum of any particular solution $\mathbf{x}^{(p)}$ of this equation and the general solution $\mathbf{x}^{(c)}$ of the corresponding homogeneous equation.

12. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.
a. Compute the Wronskian of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
b. In what intervals are $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ linearly independent?
c. What conclusion can be drawn about the coefficients in the system of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$?
d. Find this system of equations and verify the conclusions of part c.

13. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, and answer the same questions as in Problem 12.

Problems 14 and 15 indicate an alternative derivation of Theorem 7.4.2.

14. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$. Assume that $\mathbf{P}$ is continuous, and let t_0 be an arbitrary point in the given interval. Show that $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent for $\alpha < t < \beta$ if (and only if) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(m)}(t_0)$ are linearly dependent. In other words, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}$ are linearly dependent on the interval (α, β) if they are linearly dependent at any point in it. *Hint:* There are constants $c_1, \ldots, c_m$ that satisfy
$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_m\mathbf{x}^{(m)}(t_0) = \mathbf{0}.$$
Let $\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_m\mathbf{x}^{(m)}(t)$, and use the uniqueness theorem to show that $\mathbf{z}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$.

15. Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be linearly independent solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}$ is continuous on $\alpha < t < \beta$.
a. Show that any solution $\mathbf{x} = \mathbf{z}(t)$ can be written in the form
$$\mathbf{z}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$
for suitable constants $c_1, \ldots, c_n$. *Hint:* Use the result of Problem 10 of Section 7.3, and also Problem 14 above.
b. Show that the expression for the solution $\mathbf{z}(t)$ in part a is unique; that is, if $\mathbf{z}(t) = k_1\mathbf{x}^{(1)}(t) + \cdots + k_n\mathbf{x}^{(n)}(t)$, then $k_1 = c_1, \ldots, k_n = c_n$.
Hint: Show that $(k_1 - c_1)\mathbf{x}^{(1)}(t) + \cdots + (k_n - c_n)\mathbf{x}^{(n)}(t) = \mathbf{0}$ for each t in $\alpha < t < \beta$, and use the linear independence of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$.

7.5 Homogeneous Linear Systems with Constant Coefficients

We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients—that is, systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Unless stated otherwise, we will assume further that all the elements of $\mathbf{A}$ are real (rather than complex) numbers.

If $n = 1$, then the system reduces to a single first-order equation

$$\frac{dx}{dt} = ax, \tag{2}$$

whose solution is $x(t) = ce^{at}$. Note that $x = 0$ is the only critical point when $a \neq 0$. If $a < 0$, then all nontrivial solutions approach $x(t) = 0$ as t increases, and in this case we say that $x(t) = 0$ is an asymptotically stable equilibrium solution. On the other hand, if $a > 0$, then every solution (except the equilibrium solution $x(t) = 0$ itself) moves further from the equilibrium solution as t increases. Thus, in this case, $x(t) = 0$ is unstable.

For systems of n equations, the situation is similar but more complicated. Equilibrium solutions are found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$. We usually assume that $\det\mathbf{A} \neq 0$, so $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution. An important question is whether other solutions approach this equilibrium solution or depart from it as t increases; in other words, is $\mathbf{x} = \mathbf{0}$ asymptotically stable or unstable? Or are there still other possibilities?

Problems

In each of Problems 1 through 4:

G a. Draw a direction field.

b. Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

G c. Plot a few trajectories of the system.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x}$

In each of Problems 5 and 6 the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text. For each system:

G a. Draw a direction field.

b. Find the general solution of the given system of equations.

G c. Draw a few of the trajectories.

5. $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 7 through 9, find the general solution of the given system of equations.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}\mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\mathbf{x}$

9. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$

In each of Problems 10 through 12, solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

10. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

11. $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

12. $\mathbf{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$

13. The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the second-order Euler equation (Section 5.4). Assuming that $\mathbf{x} = \boldsymbol{\xi} t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ in order to obtain nontrivial solutions of the given differential equation.

Referring to Problem 13, solve the given system of equations in each of Problems 14 through 16. Assume that $t > 0$.

14. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x}$

15. $t\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

16. $t\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\mathbf{x}$

In each of Problems 17 through 19, the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ are given. Consider the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

G a. Sketch a phase portrait of the system.

G b. Sketch the trajectory passing through the initial point $(2, 3)$.

G c. For the trajectory in part b, sketch the graphs of x_1 versus t and of x_2 versus t.

17. $r_1 = -1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

18. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \quad r_2 = -2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

19. $r_1 = 1, \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad r_2 = 2, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

20. Consider a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$, provided that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent. In this problem we establish the linear independence of $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.

a. Explain how we know that $\boldsymbol{\xi}^{(1)}$ satisfies the matrix equation $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; similarly, explain why $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(2)} = \mathbf{0}$.

b. Show that $(\mathbf{A} - r_2\mathbf{I})\boldsymbol{\xi}^{(1)} = (r_1 - r_2)\boldsymbol{\xi}^{(1)}$.

c. Suppose that $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly dependent. Then $c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 (say, c_1) is not zero. Show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = \mathbf{0}$, and also show that $(\mathbf{A} - r_2\mathbf{I})(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}) = c_1(r_1 - r_2)\boldsymbol{\xi}^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are linearly independent.

d. Modify the argument of part c if we assume that $c_2 \neq 0$.

e. Carry out a similar argument for the case $\mathbf{A}$ is 3×3; note that the procedure can be extended to an arbitrary value of n.

21. Consider the equation

$$ay'' + by' + cy = 0, \tag{35}$$

where a, b, and c are constants with $a \neq 0$. In Chapter 3 it was shown that the general solution depended on the roots of the characteristic equation

$$ar^2 + br + c = 0. \tag{36}$$

a. Transform equation (35) into a system of first-order equations by letting $x_1 = y$, $x_2 = y'$. Find the system of equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ satisfied by $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

b. Find the equation that determines the eigenvalues of the coefficient matrix $\mathbf{A}$ in part a. Note that this equation is just the characteristic equation (36) of equation (35).

22. The two-tank system of Problem 19 in Section 7.1 leads to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -17 \\ -21 \end{pmatrix},$$

where x_1 and x_2 are the deviations of the salt levels Q_1 and Q_2 from their respective equilibria.

a. Find the solution of the given initial value problem.

G b. Plot x_1 versus t and x_2 versus t on the same set of axes.

N c. Find the smallest time T such that $|x_1(t)| \le 0.5$ and $|x_2(t)| \le 0.5$ for all $t \ge T$.

23. Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}\mathbf{x}.$$

a. Solve the system for $\alpha = \frac{1}{2}$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

b. Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

c. In parts a and b, solutions of the system exhibit two quite different types of behavior. Find the eigenvalues of the coefficient matrix in terms of α, and determine the value of α between $\frac{1}{2}$ and 2 where the transition from one type of behavior to the other occurs. This value of α is called a **bifurcation value** for this problem.

Electric Circuits. Problems 24 and 25 are concerned with the electric circuit described by the system of differential equations in Problem 18 of Section 7.1:

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}, \quad I(0) = I_0, \ V(0) = V_0. \tag{37}$$

24. a. Find the general solution of equation (37) if $R_1 = 1\ \Omega$, $R_2 = \frac{3}{5}\ \Omega$, $L = 2$ H, and $C = \frac{2}{3}$ F.

b. Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial values I_0 and V_0.

25. Consider the preceding system of differential equations (37).

a. Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different.

b. If the condition found in part a is satisfied, show that both eigenvalues are negative. Then show that both $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$, regardless of the initial conditions.

c. If the condition found in part a is not satisfied, then the eigenvalues are either complex or repeated. Do you think that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ in these cases as well?

Hint: In part c, one approach is to change the system (37) into a single second-order equation. We also discuss complex and repeated eigenvalues in Sections 7.6 and 7.8.

7.6 Complex-Valued Eigenvalues

In this section we consider again a system of n linear homogeneous equations with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where the coefficient matrix $\mathbf{A}$ is real-valued. If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$, then it follows, as in Section 7.5, that r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$. Recall that the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are the roots of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0 \tag{2}$$

and that the corresponding eigenvectors are nonzero vectors that satisfy

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}. \tag{3}$$

If $\mathbf{A}$ is real-valued, then the coefficients in the polynomial equation (2) for r are real-valued, and any complex-valued eigenvalues must occur in conjugate pairs. For example, if $r_1 = \lambda + i\mu$, where λ and μ are real, is an eigenvalue of $\mathbf{A}$, then so is $r_2 = \lambda - i\mu$. To explore the effect of complex-valued eigenvalues, we begin with an example.

Figure 7.6.4(b) shows a superposition of the trajectories for the two masses in their respective phase planes. Both graphs are ellipses, the inner one corresponding to the first mass and the outer one to the second. The trajectory on the inner ellipse starts at $(3, 0)$, and the trajectory on the outer ellipse starts at $(-4, 0)$. Both are traversed clockwise, and a circuit is completed in time π. The origin is a center in the respective y_1y_3- and y_2y_4-planes. Once again, similar graphs are obtained from $\mathbf{v}^{(2)}$ or from a linear combination of $\mathbf{u}^{(2)}$ and $\mathbf{v}^{(2)}$.

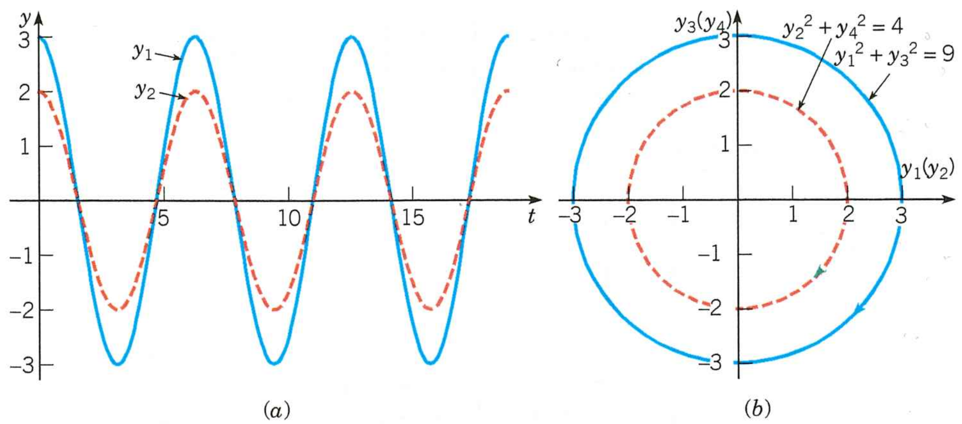

FIGURE 7.6.3 (a) A plot of y_1 versus t (solid blue) and y_2 versus t (dashed red) for the solution $\mathbf{u}^{(1)}(t)$. (b) Superposition of projections of trajectories in the y_1y_3- and y_2y_4-planes for the solution $\mathbf{u}^{(1)}(t)$.

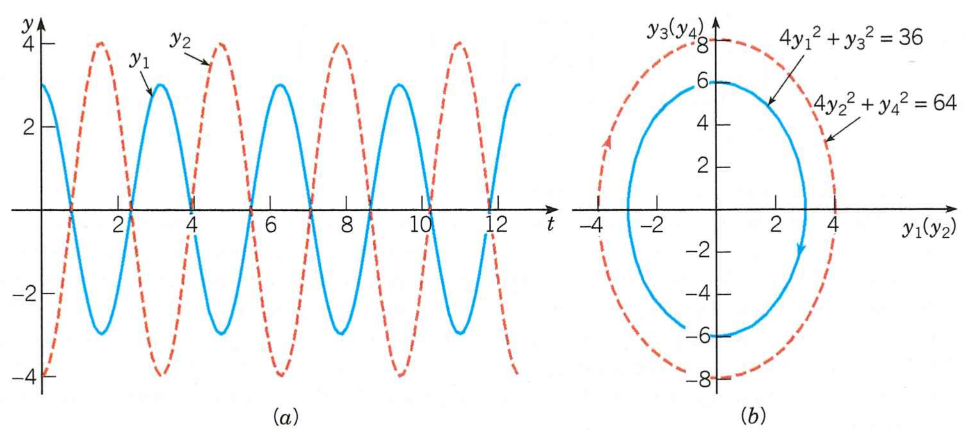

FIGURE 7.6.4 (a) A plot of y_1 versus t (solid blue) and y_2 versus t (dashed red) for the solution $\mathbf{u}^{(2)}(t)$. (b) Superposition of projections of trajectories in the y_1y_3- and y_2y_4-planes for the solution $\mathbf{u}^{(2)}(t)$.

The types of motion described in the two preceding paragraphs are called **fundamental modes** of vibration for the two-mass system. Each of them results from fairly special initial conditions. For example, to obtain the fundamental mode of frequency 1, both of the constants c_3 and c_4 in equation (31) must be zero. This occurs only for initial conditions in which $3y_2(0) = 2y_1(0)$ and $3y_4(0) = 2y_3(0)$. Similarly, the mode of frequency 2 is obtained only when both of the constants c_1 and c_2 in equation (31) are zero—that is, when the initial conditions are such that $3y_2(0) = -4y_1(0)$ and $3y_4(0) = -4y_3(0)$.

For more general initial conditions the solution is a combination of the two fundamental modes. A plot of y_1 versus t for a typical case is shown in Figure 7.6.5(a), and the projection of the corresponding trajectory in the y_1y_3-plane is shown in Figure 7.6.5(b). Observe that this latter figure may be a bit misleading in that it shows the projection of the trajectory crossing itself. This cannot be the case for the actual trajectory in four dimensions, because it would violate the general uniqueness theorem: there cannot be two different solutions issuing from the same initial point.

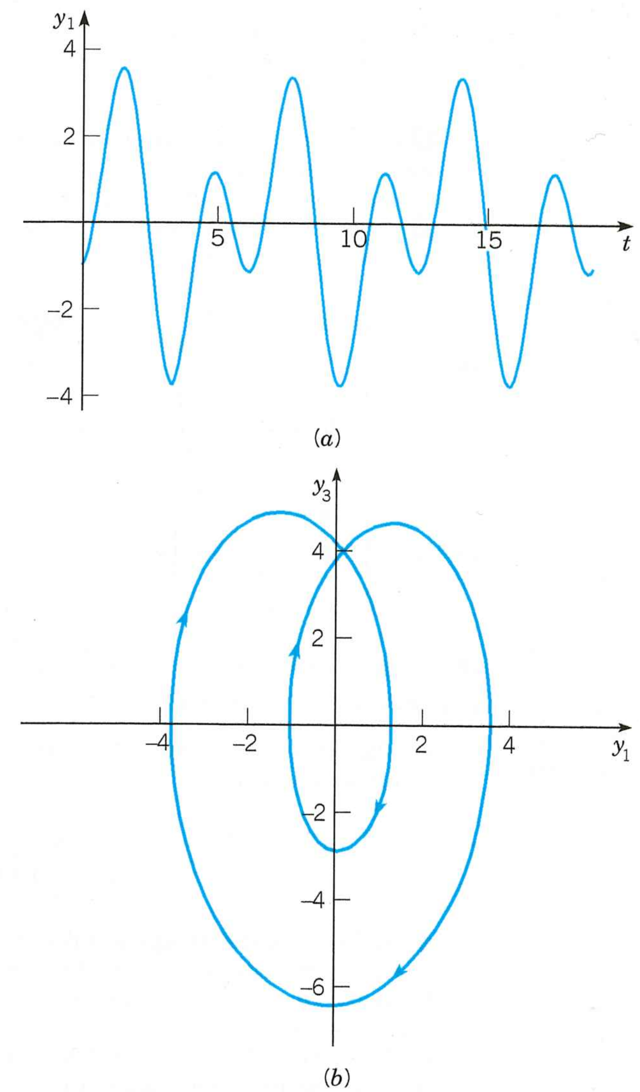

FIGURE 7.6.5 A solution of the system (25) satisfying the initial condition $\mathbf{y}(0) = (-1, 4, 1, 1)^T$. ($a$) A plot of y_1 versus t. (b) The projection of the trajectory in the y_1y_3-plane. As stated in the text, the actual trajectory in four dimensions does not intersect itself.

Problems

In each of Problems 1 through 4:

G a. Draw a direction field and sketch a few trajectories.

b. Express the general solution of the given system of equations in terms of real-valued functions.

c. Describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}\mathbf{x}$

In each of Problems 5 and 6, express the general solution of the given system of equations in terms of real-valued functions.

5. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\mathbf{x}$

In each of Problems 7 and 8, find the solution of the given initial-value problem. Describe the behavior of the solution as $t \to \infty$.

7. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 9 and 10:

- **a.** Find the eigenvalues of the given system.
- **G b.** Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane.
- **G c.** For your trajectory in part b, draw the graphs of x_1 versus t and of x_2 versus t.
- **G d.** For your trajectory in part b, draw the corresponding graph in three-dimensional tx_1x_2-space. Note that the projections of this plot onto each of the coordinate planes should produce the three plots produced in parts b and c.

9. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix}\mathbf{x}$

10. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix}\mathbf{x}$

In each of Problems 11 through 15, the coefficient matrix contains a parameter α. In each of these problems:

- **a.** Determine the eigenvalues in terms of α.
- **b.** Find the bifurcation value or values of α where the qualitative nature of the phase portrait for the system changes.
- **G c.** Draw a phase portrait for a value of α slightly below, and for another value slightly above, each bifurcation value.

11. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}\mathbf{x}$

12. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix}\mathbf{x}$

13. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix}\mathbf{x}$

14. $\mathbf{x}' = \begin{pmatrix} -1 & \alpha \\ -1 & -1 \end{pmatrix}\mathbf{x}$

15. $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix}\mathbf{x}$

In each of Problems 16 and 17, solve the given system of equations by the method of Problem 13 of Section 7.5. Assume that $t > 0$.

16. $t\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{x}$

17. $t\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

In each of Problems 18 and 19:

- **a.** Find the eigenvalues of the given system.
- **G b.** Choose an initial point (other than the origin) and draw the corresponding trajectory in the x_1x_2-plane. Also draw the trajectories in the x_1x_3- and x_2x_3-planes.
- **G c.** For the initial point in part b, draw the corresponding trajectory in $x_1x_2x_3$-space.

18. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}\mathbf{x}$

19. $\mathbf{x}' = \begin{pmatrix} -\frac{1}{4} & 1 & 0 \\ -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix}\mathbf{x}$

20. Consider the electric circuit shown in Figure 7.6.6. Suppose that $R_1 = R_2 = 4\ \Omega$, $C = \frac{1}{2}$ F, and $L = 8$ H.

- **a.** Show that this circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}, \tag{32}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. *Hint:* See Problem 18 of Section 7.1.

- **b.** Find the general solution of equations (32) in terms of real-valued functions.
- **c.** Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 3$ V.
- **d.** Determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

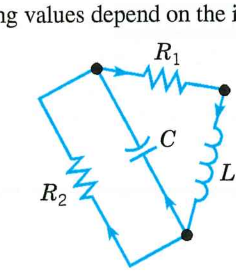

FIGURE 7.6.6 The circuit in Problem 20.

21. The electric circuit shown in Figure 7.6.7 is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}, \tag{33}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. These differential equations were derived in Problem 16 of Section 7.1.

- **a.** Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.
- **b.** Suppose that $R = 1\ \Omega$, $C = \frac{1}{2}$ F, and $L = 1$ H. Find the general solution of the system (33) in this case.

- **c.** Find $I(t)$ and $V(t)$ if $I(0) = 2$ A and $V(0) = 1$ V.
- **d.** For the circuit of part b, determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

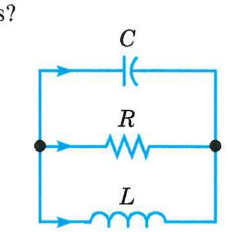

FIGURE 7.6.7 The circuit in Problem 21.

22. In this problem we indicate how to show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$, as given by equations (17), are linearly independent. Let $r_1 = \lambda + i\mu$ and $\overline{r_1} = \lambda - i\mu$ be a pair of conjugate eigenvalues of the coefficient matrix $\mathbf{A}$ of equation (1); let $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$ and $\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{a} - i\mathbf{b}$ be the corresponding eigenvectors. Recall that it was stated in Section 7.3 that two different eigenvalues have linearly independent eigenvectors, so if $r_1 \neq \overline{r_1}$, then $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$ are linearly independent.

- **a.** First we show that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Consider the equation $c_1\mathbf{a} + c_2\mathbf{b} = \mathbf{0}$. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of $\boldsymbol{\xi}^{(1)}$ and $\overline{\boldsymbol{\xi}^{(1)}}$, and then show that $(c_1 - ic_2)\boldsymbol{\xi}^{(1)} + (c_1 + ic_2)\overline{\boldsymbol{\xi}^{(1)}} = \mathbf{0}$.
- **b.** Show that $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$ and then that $c_1 = 0$ and $c_2 = 0$. Consequently, $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.
- **c.** To show that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, consider the equation $c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{0}$, where t_0 is an arbitrary point. Rewrite this equation in terms of $\mathbf{a}$ and $\mathbf{b}$, and then proceed as in part b to show that $c_1 = 0$ and $c_2 = 0$. Hence $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent at the arbitrary point t_0. Therefore, they are linearly independent at every point and on every interval.

23. A mass m on a spring with constant k satisfies the differential equation (see Section 3.7)

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

- **a.** Let $x_1 = u$, $x_2 = u'$, and show that the resulting system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}\mathbf{x}.$$

- **b.** Find the eigenvalues of the matrix for the system in part a.
- **c.** Sketch several trajectories of the system. Choose one of your trajectories, and sketch the corresponding graphs of x_1 versus t and x_2 versus t. Sketch both graphs on one set of axes.
- **d.** What is the relation between the eigenvalues of the coefficient matrix and the natural frequency of the spring-mass system?

24. Consider the two-mass, three-spring system of Example 3 in the text. Instead of converting the problem into a system of four first-order equations, we indicate here how to proceed directly from equations (22).

- **a.** Show that equations (22) can be written in the form

$$\mathbf{x}'' = \begin{pmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{pmatrix}\mathbf{x} = \mathbf{A}\mathbf{x}. \tag{34}$$

- **b.** Assume that $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ and show that

$$(\mathbf{A} - r^2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}.$$

Note that r^2 (rather than r) is an eigenvalue of $\mathbf{A}$ corresponding to an eigenvector $\boldsymbol{\xi}$.

- **c.** Find the eigenvalues and eigenvectors of $\mathbf{A}$.
- **d.** Write down expressions for x_1 and x_2. There should be four arbitrary constants in these expressions.
- **e.** By differentiating the results from part d, write down expressions for x_1' and x_2'. Your results from parts d and e should agree with equation (31) in the text.

25. Consider the two-mass, three-spring system whose equations of motion are equations (22). Let $m_1 = 1$, $m_2 = 4/3$, $k_1 = 1$, $k_2 = 3$, and $k_3 = 4/3$.

- **a.** As in Example 3, convert the system to four first-order equations of the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Determine the coefficient matrix $\mathbf{A}$.
- **b.** Find the eigenvalues and eigenvectors of $\mathbf{A}$.
- **c.** Write down the general solution of the system.
- **G d.** Describe the fundamental modes of vibration. For each fundamental mode, draw graphs of y_1 versus t and y_2 versus t. Also draw the corresponding trajectories in the y_1y_3- and y_2y_4-planes.
- **G e.** Consider the initial conditions $\mathbf{y}(0) = (2, 1, 0, 0)^T$. Evaluate the arbitrary constants in the general solution in part c. What is the period of the motion in this case? Plot graphs of y_1 versus t and y_2 versus t. Also plot the corresponding trajectories in the y_1y_3- and y_2y_4-planes. Be sure you understand how the trajectories are traversed for a full period.
- **G f.** Consider other initial conditions of your own choice, and plot graphs similar to those requested in part e.

7.7 Fundamental Matrices

The structure of the solutions of systems of linear differential equations can be further illuminated by introducing the idea of a fundamental matrix. Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the equation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{1}$$

on some interval $\alpha < t < \beta$. Then the matrix

$$\boldsymbol{\Psi}(t) = \left(\mathbf{x}^{(1)}(t)\,\middle|\,\mathbf{x}^{(2)}(t)\,\middle|\,\cdots\,\middle|\,\mathbf{x}^{(n)}(t)\right) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix}, \tag{2}$$

Problems

In each of Problems 1 through 8:

a. Find a fundamental matrix for the given system of equations.
b. Find the fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(0) = \mathbf{I}$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{x}$

7. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}\mathbf{x}$

8. $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$

9. Use the fundamental matrix $\mathbf{\Phi}(t)$ found in Problem 4 to solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

10. Show that $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)$, where $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are as defined in this section.

11. The fundamental matrix $\mathbf{\Phi}(t)$ for the system (3) was found in Example 2. Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$ by multiplying $\mathbf{\Phi}(t)$ and $\mathbf{\Phi}(s)$.

12. Let $\mathbf{\Phi}(t)$ denote the fundamental matrix satisfying $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$, $\mathbf{\Phi}(0) = \mathbf{I}$. In the text we also denoted this matrix by $\exp(\mathbf{A}t)$. In this problem we show that $\mathbf{\Phi}$ does indeed have the principal algebraic properties associated with the exponential function.

a. Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$; that is, show that $\exp(\mathbf{A}t)\exp(\mathbf{A}s) = \exp(\mathbf{A}(t+s))$. *Hint:* Show that if s is fixed and t is variable, then both $\mathbf{\Phi}(t)\mathbf{\Phi}(s)$ and $\mathbf{\Phi}(t+s)$ satisfy the initial value problem $\mathbf{Z}' = \mathbf{A}\mathbf{Z}$, $\mathbf{Z}(0) = \mathbf{\Phi}(s)$.

b. Show that $\mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$; that is, $\exp(\mathbf{A}t)\exp(\mathbf{A}(-t)) = \mathbf{I}$. Then show that $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$.

c. Show that $\mathbf{\Phi}(t-s) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(s)$.

13. Show that if $\mathbf{A}$ is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$, then $\exp(\mathbf{A}t)$ is also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \ldots, \exp(a_n t)$.

14. Consider an oscillator satisfying the initial value problem

$$u'' + \omega^2 u = 0, \quad u(0) = u_0, \quad u'(0) = v_0. \tag{50}$$

a. Let $x_1 = u$, $x_2 = u'$, and transform equations (53) into the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^0. \tag{51}$$

b. Use the series (23) to show that

$$\exp(\mathbf{A}t) = \mathbf{I}\cos(\omega t) + \mathbf{A}\frac{\sin(\omega t)}{\omega}. \tag{52}$$

c. Find the solution of the initial value problem (51).

15. The method of successive approximations (see Section 2.8) can also be applied to systems of equations. For example, consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^0, \tag{53}$$

where $\mathbf{A}$ is a constant matrix and $\mathbf{x}^0$ is a prescribed vector.

a. Assuming that a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ exists, show that it must satisfy the integral equation

$$\boldsymbol{\phi}(t) = \mathbf{x}^0 + \int_0^t \mathbf{A}\boldsymbol{\phi}(s)\,ds. \tag{54}$$

b. Start with the initial approximation $\boldsymbol{\phi}^{(0)}(t) = \mathbf{x}^0$. Substitute this expression for $\boldsymbol{\phi}(s)$ on the right-hand side of equation (51) and obtain a new approximation $\boldsymbol{\phi}^{(1)}(t)$. Show that

$$\boldsymbol{\phi}^{(1)}(t) = (\mathbf{I} + \mathbf{A}t)\mathbf{x}^0. \tag{55}$$

c. Repeat this process and thereby obtain a sequence of approximations $\boldsymbol{\phi}^{(0)}, \boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \ldots, \boldsymbol{\phi}^{(n)}, \ldots$. Use an inductive argument to show that

$$\boldsymbol{\phi}^{(n)}(t) = \left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!}\right)\mathbf{x}^0. \tag{56}$$

d. Let $n \to \infty$ and show that the solution of the initial value problem (53) is

$$\boldsymbol{\phi}(t) = \exp(\mathbf{A}t)\mathbf{x}^0. \tag{57}$$

7.8 Repeated Eigenvalues

We conclude our consideration of the linear homogeneous system of differential equations with constant coefficients

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

with a discussion of the case in which the matrix $\mathbf{A}$ has a repeated eigenvalue. Recall that in Section 7.3 we stated that a repeated eigenvalue with algebraic multiplicity $m \geq 2$ may have a geometric multiplicity less than m. In other words, there may be fewer than m linearly independent eigenvectors associated with this eigenvalue. The following example illustrates this possibility.

EXAMPLE 1

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}. \tag{2}$$

Solution:

The eigenvalues r and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or

$$\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3}$$

The eigenvalues are the roots of the equation

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0. \tag{4}$$

Thus the two eigenvalues are $r_1 = r_2 = 2$; that is, the eigenvalue 2 has algebraic multiplicity 2.

To determine the eigenvectors, we must return to equation (3) and use $r = 2$. This gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

Hence we obtain the single condition $\xi_1 + \xi_2 = 0$, which determines ξ_2 in terms of ξ_1, or vice versa. Thus the eigenvector corresponding to the eigenvalue $r = 2$ is

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{6}$$

or any nonzero multiple of this vector. Observe that there is only one linearly independent eigenvector associated with the double eigenvalue.

Returning to the system (1), suppose that $r = \rho$ is an m-fold root of the characteristic equation

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{7}$$

Then ρ is an eigenvalue of algebraic multiplicity m of the matrix $\mathbf{A}$. In this event, there are two possibilities: either there are m linearly independent eigenvectors corresponding to the eigenvalue ρ, or else, as in Example 1, there are fewer than m linearly independent eigenvectors.

In the first case, let $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(m)}$ be m linearly independent eigenvectors associated with the eigenvalue ρ of algebraic multiplicity m. Then there are m linearly independent solutions $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{\rho t}, \ldots, \mathbf{x}^{(m)}(t) = \boldsymbol{\xi}^{(m)}e^{\rho t}$ of equation (1). Thus in this case, it makes no difference that the eigenvalue $r = \rho$ is repeated; there is still a fundamental set of

Consider again the matrix $\mathbf{A}$ given by equation (2). To transform $\mathbf{A}$ into its Jordan form, we construct the transformation matrix $\mathbf{T}$ with the single eigenvector $\boldsymbol{\xi}$ from equation (6) in its first column and the generalized eigenvector $\boldsymbol{\eta}$ from equation (17) with $k = 0$ in the second column. Then $\mathbf{T}$ and its inverse are given by

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{28}$$

As you can verify, it follows that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \mathbf{J}. \tag{29}$$

The matrix $\mathbf{J}$ in equation (29) is the **Jordan form** of $\mathbf{A}$. It is typical of all Jordan forms in that it has a 1 above the main diagonal in the column corresponding to the eigenvector that is lacking (and is replaced in $\mathbf{T}$ by the generalized eigenvector).

If we start again from equation (1)

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T}$ is given by equation (28), produces the system

$$\mathbf{y}' = \mathbf{J}\mathbf{y}, \tag{30}$$

where $\mathbf{J}$ is given by equation (29). In scalar form the system (30) is

$$y_1' = 2y_1 + y_2, \quad y_2' = 2y_2. \tag{31}$$

These equations can be solved readily in reverse order—that is, by starting with the equation for y_2. In this way we obtain

$$y_2(t) = c_1e^{2t} \quad \text{and} \quad y_1(t) = c_1te^{2t} + c_2e^{2t}. \tag{32}$$

Thus two independent solutions of the system (30) are

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{y}^{(2)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{2t}, \tag{33}$$

and the corresponding fundamental matrix is

$$\hat{\boldsymbol{\Psi}}(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}. \tag{34}$$

Since $\hat{\boldsymbol{\Psi}}(0) = \mathbf{I}$, we can also identify the matrix in equation (34) as $\exp(\mathbf{J}t)$. The same result can be reached by calculating powers of $\mathbf{J}$ and substituting them into the exponential series (see Problems 19 through 21). To obtain a fundamental matrix for the original system, we now form the product

$$\boldsymbol{\Psi}(t) = \mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -e^{2t} - te^{2t} \end{pmatrix}, \tag{35}$$

which is the same as the fundamental matrix given in equation (25).

We will not discuss $n \times n$ systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in more detail here. For large n it is possible that there may be eigenvalues of high algebraic multiplicity m, perhaps with much lower geometric multiplicity q, thus giving rise to $m - q$ generalized eigenvectors. Problems 17 and 18 explore the use of Jordan forms for systems of three differential equations. For $n \geq 4$ there may also be repeated complex eigenvalues. A full discussion[9] of the Jordan form of a general $n \times n$ matrix requires a greater background in linear algebra than we assume for most readers of this book.

[9]For example, see the books listed in the References at the end of this chapter.

The amount of arithmetic required in the analysis of a general $n \times n$ system may be prohibitive to do by hand even if n is no greater than 3 or 4. Consequently, suitable computer software should be used routinely in most cases. This does not overcome all difficulties by any means, but it does make many problems much more tractable. Finally, for a set of equations arising from modeling a physical system, it is likely that some of the elements in the coefficient matrix $\mathbf{A}$ result from measurements of some physical quantity. The inevitable uncertainties in such measurements lead to uncertainties in the values of the eigenvalues of $\mathbf{A}$. For example, in such a case it may not be clear whether two eigenvalues are actually equal or are merely close together.

Problems

In each of Problems 1 through 3:

a. Draw a direction field and sketch a few trajectories.
G b. Describe how the solutions behave as $t \to \infty$.
c. Find the general solution of the system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\mathbf{x}$

In each of Problems 4 and 5, find the general solution of the given system of equations.

4. $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\mathbf{x}$

5. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x}$

In each of Problems 6 through 8:

a. Find the solution of the given initial value problem.
b. Sketch the trajectory of the solution in the x_1x_2-plane, and also sketch the graph of x_1 versus t.

6. $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

7. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

8. $\mathbf{x}' = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

In each of Problems 9 and 10:

a. Find the solution of the given initial value problem.
G b. Draw the corresponding trajectory in $x_1x_2x_3$-space.
c. Sketch the graph of x_1 versus t.

9. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

10. $\mathbf{x}' = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

In each of Problems 11 and 12, solve the given system of equations by the method of Problem 13 of Section 7.5. Assume that $t > 0$.

11. $t\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

12. $t\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}$

13. Show that all solutions of the system

$$\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 28 in Section 3.4.

14. Consider again the electric circuit in Problem 21 of Section 7.6. This circuit is described by the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}.$$

a. Show that the eigenvalues are real and equal if $L = 4R^2C$.
b. Suppose that $R = 1\,\Omega$, $C = 1$ F, and $L = 4$ H. Suppose also that $I(0) = 1$ A and $V(0) = 2$ V. Find $I(t)$ and $V(t)$.

15. Consider again the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x} \tag{36}$$

that we discussed in Example 2. We found there that $\mathbf{A}$ has a double eigenvalue $r_1 = r_2 = 2$ with a single independent eigenvector $\boldsymbol{\xi}^{(1)} = (1, -1)^T$, or any nonzero multiple thereof. Thus one solution of the system (36) is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{2t}$ and a second independent solution has the form

$$\mathbf{x}^{(2)}(t) = \boldsymbol{\xi}te^{2t} + \boldsymbol{\eta}e^{2t},$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{37}$$

In the text we solved the first equation for $\boldsymbol{\xi}$ and then the second equation for $\boldsymbol{\eta}$. Here, we ask you to proceed in the reverse order.

a. Show that η satisfies $(\mathbf{A}-2\mathbf{I})^2\eta = \mathbf{0}$.

b. Show that $(\mathbf{A}-2\mathbf{I})^2 = \mathbf{0}$. Thus the generalized eigenvector η can be chosen arbitrarily, except that it must be independent of $\xi^{(1)}$.

c. Let $\eta = (0, -1)^T$. Then determine ξ from the second of equations (37) and observe that $\xi = (1, -1)^T = \xi^{(1)}$. This choice of η reproduces the solution found in Example 2.

d. Let $\eta = (1, 0)^T$ and determine the corresponding eigenvector ξ.

e. Let $\eta = (k_1, k_2)^T$, where k_1 and k_2 are arbitrary numbers. What condition on k_1 and k_2 ensures that η and $\xi^{(1)}$ are linearly independent? Then determine ξ. How is it related to the eigenvector $\xi^{(1)}$?

16. In Example 2, with $\mathbf{A}$ given in equation (36) above, it was claimed that equation (16) is solvable even though the matrix $\mathbf{A}-2\mathbf{I}$ is singular. This problem justifies that claim.

a. Find all eigenvalues and eigenvectors for $\mathbf{A}^*$, the adjoint of $\mathbf{A}$.

b. Show that the eigenvectors of $\mathbf{A}$ and the corresponding eigenvectors of $\mathbf{A}^*$ are orthogonal.

c. Explain why this proves that equation (16) is solvable.

Eigenvalues of Multiplicity 3. If the matrix $\mathbf{A}$ has an eigenvalue of algebraic multiplicity 3, then there may be either one, two, or three corresponding linearly independent eigenvectors. The general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is different, depending on the number of eigenvectors associated with the triple eigenvalue. As noted in the text, there is no difficulty if there are three eigenvectors, since then there are three independent solutions of the form $\mathbf{x} = \xi e^{rt}$. The following two problems illustrate the solution procedure for a triple eigenvalue with one or two eigenvectors, respectively.

17. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\mathbf{x}. \qquad (38)$$

a. Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

b. Using the information in part a, write down one solution $\mathbf{x}^{(1)}(t)$ of the system (38). There is no other solution of the purely exponential form $\mathbf{x} = \xi e^{rt}$.

c. To find a second solution, assume that $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$. Show that ξ and η satisfy the equations

$$(\mathbf{A}-2\mathbf{I})\xi = \mathbf{0}, \quad (\mathbf{A}-2\mathbf{I})\eta = \xi.$$

Since ξ has already been found in part a, solve the second equation for η. Neglect the multiple of $\xi^{(1)}$ that appears in η, since it leads only to a multiple of the first solution $\mathbf{x}^{(1)}$. Then write down a second solution $\mathbf{x}^{(2)}(t)$ of the system (38).

d. To find a third solution, assume that

$$\mathbf{x} = \xi \frac{t^2}{2} e^{2t} + \eta t e^{2t} + \zeta e^{2t}.$$

Show that ξ, η, and ζ satisfy the equations

$$(\mathbf{A}-2\mathbf{I})\xi = \mathbf{0}, \quad (\mathbf{A}-2\mathbf{I})\eta = \xi, \quad (\mathbf{A}-2\mathbf{I})\zeta = \eta.$$

The first two equations are the same as in part c, so solve the third equation for ζ, again neglecting the multiple of $\xi^{(1)}$ that appears. Then write down a third solution $\mathbf{x}^{(3)}(t)$ of the system (38).

e. Write down a fundamental matrix $\Psi(t)$ for the system (38).

f. Form a matrix $\mathbf{T}$ with the eigenvector $\xi^{(1)}$ in the first column and the generalized eigenvectors η and ζ in the second and third columns. Then find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

18. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}\mathbf{x}. \qquad (39)$$

a. Show that $r = 1$ is a triple eigenvalue of the coefficient matrix $\mathbf{A}$ and that there are only two linearly independent eigenvectors, which we may take as

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}. \qquad (40)$$

Write down two linearly independent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ of equation (39).

b. To find a third solution, assume that $\mathbf{x} = \xi t e^t + \eta e^t$; then show that ξ and η must satisfy

$$(\mathbf{A}-\mathbf{I})\xi = \mathbf{0}, \qquad (41)$$

$$(\mathbf{A}-\mathbf{I})\eta = \xi. \qquad (42)$$

c. Equation (41) is satisfied if ξ is an eigenvector, so one way to proceed is to choose ξ to be a suitable linear combination of $\xi^{(1)}$ and $\xi^{(2)}$ so that equation (42) is solvable, and then to solve that equation for η. However, let us proceed in a different way and follow the pattern of Problem 15. First, show that η satisfies

$$(\mathbf{A}-\mathbf{I})^2\eta = \mathbf{0}.$$

Further, show that $(\mathbf{A}-\mathbf{I})^2 = \mathbf{0}$. Thus η can be chosen arbitrarily, except that it must be independent of $\xi^{(1)}$ and $\xi^{(2)}$.

d. A convenient choice for η is $\eta = (0, 0, 1)^T$. Find the corresponding ξ from equation (42). Verify that ξ is an eigenvector of $\mathbf{A}$.

e. Write down a fundamental matrix $\Psi(t)$ for the system (39).

f. Form a matrix $\mathbf{T}$ with the eigenvector $\xi^{(1)}$ in the first column and with the eigenvector ξ from part d and the generalized eigenvector η in the other two columns. Find $\mathbf{T}^{-1}$ and form the product $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. The matrix $\mathbf{J}$ is the Jordan form of $\mathbf{A}$.

19. Let $\mathbf{J} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is an arbitrary real number.

a. Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

b. Use an inductive argument to show that $\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

c. Determine $\exp(\mathbf{J}t)$.

d. Use $\exp(\mathbf{J}t)$ to solve the initial value problem $\mathbf{x}' = \mathbf{J}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$.

20. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

a. Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

b. Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

c. Determine $\exp(\mathbf{J}t)$.

d. Observe that if you choose $\lambda = 1$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 18f. Using the matrix $\mathbf{T}$ from Problem 18f, form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 1$.

e. Is the resulting matrix the same as the fundamental matrix $\Psi(t)$ in Problem 18e? If not, explain the discrepancy.

21. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

a. Find $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$.

b. Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

c. Determine $\exp(\mathbf{J}t)$.

d. Note that if you choose $\lambda = 2$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 17f. Using the matrix $\mathbf{T}$ from Problem 17f, form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 2$. The resulting matrix is the same as the fundamental matrix $\Psi(t)$ in Problem 17e. If not, explain the discrepancy.

7.9 Nonhomogeneous Linear Systems

In this section we turn to the nonhomogeneous system of linear first-order differential equations

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad (1)$$

where the $n \times n$ matrix $\mathbf{P}(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$. By the same argument as in Section 3.5 (see also Problem 12 in this section), the general solution of equation (1) can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t), \qquad (2)$$

where $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the corresponding homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system (1). We will briefly describe several methods for determining $\mathbf{v}(t)$.

Diagonalization. We begin with systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \qquad (3)$$

where $\mathbf{A}$ is an $n \times n$ diagonalizable constant matrix. By diagonalizing the coefficient matrix $\mathbf{A}$, as indicated in Section 7.7, we can transform equation (3) into a system of equations that is readily solved.

Let $\mathbf{T}$ be the matrix whose columns are the eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$ of $\mathbf{A}$, and define a new dependent variable $\mathbf{y}$ by

$$\mathbf{x} = \mathbf{T}\mathbf{y}. \qquad (4)$$

Then, substituting for $\mathbf{x}$ in equation (3), we obtain

$$\mathbf{T}\mathbf{y}' = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t).$$

When we multiply this equation (on the left) by $\mathbf{T}^{-1}$, it follows that

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \qquad (5)$$

where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$ and where $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \dots, r_n$ of $\mathbf{A}$, arranged in the same order as the corresponding eigenvectors $\xi^{(1)}, \dots, \xi^{(n)}$ that appear as columns of $\mathbf{T}$. Equation (5) is a system of n *uncoupled* first-order linear differential equations for $y_1(t), \dots, y_n(t)$; as a consequence, the differential equations can be solved separately. In scalar form, equation (5) has the form

$$y_j'(t) = r_j y_j(t) + h_j(t), \quad j = 1, \dots, n, \qquad (6)$$

is a fundamental matrix. Then the solution $\mathbf{x}$ of equation (35) is given by $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or

$$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{37}$$

Solving equation (37) by row reduction, we obtain

$$u_1' = e^{2t} - \frac{3}{2}te^{3t},$$

$$u_2' = 1 + \frac{3}{2}te^{t}.$$

Hence

$$u_1(t) = \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} + c_1,$$

$$u_2(t) = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} + c_2,$$

and

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$$

$$= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \tag{38}$$

which is the same as the solution obtained in Example 1 (compare with equation (15)) and is equivalent to the solution obtained in Example 2 (compare with equation (21)).

Laplace Transforms. We used the Laplace transform in Chapter 6 to solve linear equations of arbitrary order. It can also be used in very much the same way to solve systems of equations. Since the transform is an integral, the transform of a vector is computed component by component. Thus $\mathcal{L}\{\mathbf{x}(t)\}$ is the vector whose components are the transforms of the respective components of $\mathbf{x}(t)$, and similarly for $\mathcal{L}\{\mathbf{x}'(t)\}$. We will denote $\mathcal{L}\{\mathbf{x}(t)\}$ by $\mathbf{X}(s)$. Then, by an extension of Theorem 6.2.1 to vectors, we also have

$$\mathcal{L}\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0). \tag{39}$$

EXAMPLE 4

Use the method of Laplace transforms to solve the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t). \tag{40}$$

This is the same system of equations as in Examples 1, 2, and 3.

Solution:

We take the Laplace transform of each term in equation (40), obtaining

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s), \tag{41}$$

where $\mathbf{G}(s)$ is the transform of $\mathbf{g}(t)$. The transform $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{pmatrix} \dfrac{2}{s+1} \\ \dfrac{3}{s^2} \end{pmatrix}. \tag{42}$$

To proceed further we need to choose the initial vector $\mathbf{x}(0)$. For simplicity let us choose $\mathbf{x}(0) = \mathbf{0}$. Then equation (41) becomes

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s), \tag{43}$$

where, as usual, $\mathbf{I}$ is the 2×2 identity matrix. Consequently, $\mathbf{X}(s)$ is given by

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s). \tag{44}$$

The matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ is called the **transfer matrix** because multiplying it by the transform of the input vector $\mathbf{g}(t)$ yields the transform of the output vector $\mathbf{x}(t)$. In this example we have

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}, \tag{45}$$

and by a straightforward calculation, we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}. \tag{46}$$

Then, substituting from equations (42) and (46) in equation (44) and carrying out the indicated multiplication, we find that

$$\mathbf{X}(s) = \begin{pmatrix} \dfrac{2(s+2)}{(s+1)^2(s+3)} + \dfrac{3}{s^2(s+1)(s+3)} \\ \dfrac{2}{(s+1)^2(s+3)} + \dfrac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}. \tag{47}$$

Finally, we need to obtain the solution $\mathbf{x}(t)$ from its transform $\mathbf{X}(s)$. This can be done by expanding the expressions in equation (47) in partial fractions and using Table 6.2.1, or (more efficiently) by using appropriate computational tools. In any case, after some simplification the result is

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-t} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{48}$$

Equation (48) gives the particular solution of system (40) that satisfies the initial condition $\mathbf{x}(0) = \mathbf{0}$. As a result, it differs slightly from the particular solutions obtained in the preceding three examples. To obtain the general solution of equation (40), you must add to the expression in equation (48) the general solution (10) of the homogeneous system corresponding to equation (40).

Each of the methods for solving nonhomogeneous equations has its own advantages and disadvantages. The method of undetermined coefficients requires no integration, but it is limited in scope and may entail the solution of several sets of algebraic equations. The method of diagonalization requires finding the inverse of the transformation matrix and the solution of a set of uncoupled first-order linear differential equations, followed by a matrix multiplication. Its main advantage is that for Hermitian coefficient matrices, the inverse of the transformation matrix can be written down without calculation—a feature that is more important for large systems. The method of Laplace transforms involves a matrix inversion to find the transfer matrix, followed by a multiplication, and finally by the determination of the inverse transform of each term in the resulting expression. It is particularly useful in problems with forcing functions that involve discontinuous or impulsive terms. Variation of parameters is the most general method. On the other hand, it involves the solution of a set of linear algebraic equations with variable coefficients, followed by an integration and a matrix multiplication, so it may also be the most complicated from a computational viewpoint. For many small systems with constant coefficients, such as the one in the examples in this section, all of these methods work well, and there may be little reason to select one over another.

Problems

In each of Problems 1 through 8 find the general solution of the given system of equations.

1. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

2. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$

3. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$

4. $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0$

5. $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}e^t$

6. $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t$

7. $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$

8. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \quad 0 < t < \pi$

9. The electric circuit shown in Figure 7.9.1 is described by the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} I(t), \qquad (49)$$

where x_1 is the current through the inductor, x_2 is the voltage drop across the capacitor, and $I(t)$ is the current supplied by the external source.

a. Determine a fundamental matrix $\mathbf{\Psi}(t)$ for the homogeneous system corresponding to equation (49). Refer to Problem 20 of Section 7.6.

b. If $I(t) = e^{-t/2}$, determine the solution of the system (49) that also satisfies the initial conditions $\mathbf{x}(0) = \mathbf{0}$.

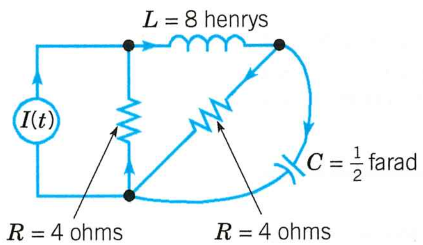

FIGURE 7.9.1 The circuit in Problem 9.

In each of Problems 10 and 11, verify that the given vector is the general solution of the corresponding homogeneous system, and then solve the nonhomogeneous system. Assume that $t > 0$.

10. $t\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1-t^2 \\ 2t \end{pmatrix},$

$\mathbf{x}^{(c)} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}t^{-1}$

11. $t\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2t \\ t^4-1 \end{pmatrix},$

$\mathbf{x}^{(c)} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}t^{-1} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}t^2$

12. Let $\mathbf{x} = \boldsymbol{\phi}(t)$ be the general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, and let $\mathbf{x} = \mathbf{v}(t)$ be some particular solution of the same system. By considering the difference $\boldsymbol{\phi}(t) - \mathbf{v}(t)$, show that $\boldsymbol{\phi}(t) = \mathbf{u}(t) + \mathbf{v}(t)$, where $\mathbf{u}(t)$ is the general solution of the corresponding homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Alternate Derivation of Variation of Parameters. When we first encountered variation of parameters for a second-order linear differential equation in Section 3.6 and again for higher-order linear differential equations in Section 4.4, some of the equations used to determine the unknown variable coefficients appeared to have been chosen primarily to prevent higher-order derivatives of the unknown variable coefficients from entering into the process. In fact, as we show in Problems 13 through 15, the variation of parameter equations are completely explained when viewed from the perspective of the equivalent system of first-order linear differential equations. Problems 13 and 14 reconsider two problems from Section 3.6; Problem 15 shows that this connection is true for any second-order linear differential equation. The same ideas can be used to explain variation of parameters for higher-order linear differential equations.[10]

In Problems 13 and 14, you are given a nonhomogeneous second-order linear differential equation and two linearly independent solutions, y_1 and y_2, to the corresponding homogeneous differential equation. Use this information to complete the following steps:

a. Find the equivalent nonhomogeneous system of first-order linear differential equations for $x_1 = y$ and $x_2 = y'$.

b. Show that $\mathbf{x}^{(1)} = (y_1, y_1')^T$ and $\mathbf{x}^{(2)} = (y_2, y_2')^T$ are solutions to the homogeneous system of differential equations corresponding to the system found in a. (As a consequence, $\mathbf{\Psi} = \left(\mathbf{x}^{(1)} | \mathbf{x}^{(2)}\right)$ is a fundamental matrix for the same homogeneous system.)

c. Find the variation of parameters equations that have to be satisfied for $y = y_1(t)u_1(t) + y_2(t)u_2(t)$ to be a particular solution of the given nonhomogeneous second-order differential equation.

d. Find the variation of parameters equations that have to be satisfied for $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$ to be a particular solution of the nonhomogeneous system of first-order linear differential equations found in a.

e. Use the definition of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ in b to show that the systems of equations found in c and the equations found in d are equivalent.

13. $y'' - 5y' + 6y = 2e^t$, $y_1 = e^{2t}$, $y_2 = e^{3t}$ (Problem 1, Section 3.6)

14. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$ $(t > 0)$, $y_1 = t$, $y_2 = te^t$ (Problem 11, Section 3.6)

15. Carry out steps a through e for the general nonhomogeneous second-order linear differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $y_1 = y_1(t)$ and $y_2 = y_2(t)$ form a fundamental set of solutions to the corresponding homogeneous differential equation.

16. Consider the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{x}^0.$$

a. By referring to Problem 12c in Section 7.7, show that

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \int_0^t \mathbf{\Phi}(t-s)\mathbf{g}(s)\,ds.$$

b. Show also that

$$\mathbf{x} = \exp(\mathbf{A}t)\mathbf{x}^0 + \int_0^t \exp(\mathbf{A}(t-s))\mathbf{g}(s)\,ds.$$

Compare these results with those of Problem 22 in Section 3.6.

[10] These problems were motivated by correspondence with Weishi Liu, University of Kansas.

17. Use the Laplace transform to solve the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t) \qquad (50)$$

used in the examples in this section. Instead of using zero initial conditions, as in Example 4, let

$$\mathbf{x}(0) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \qquad (51)$$

where α_1 and α_2 are arbitrary. How must α_1 and α_2 be chosen so that the solution is identical to equation (38)?

References

Further information on matrices and linear algebra is available in any introductory book on the subject. Here is a representative sample:

Anton, H. and Rorres, C., *Elementary Linear Algebra* (10th ed.) (Hoboken, NJ: Wiley, 2010).

Johnson, L. W., Riess, R. D., and Arnold, J. T., *Introduction to Linear Algebra* (6th ed.) (Boston: Addison-Wesley, 2008).

Kolman, B. and Hill, D. R., *Elementary Linear Algebra* (8th ed.) (Upper Saddle River, NJ: Pearson, 2004).

Lay, D. C., *Linear Algebra and Its Applications* (4th ed.)(Boston: Addison-Wesley, 2012).

Leon, S. J., *Linear Algebra with Applications* (8th ed.) (Upper Saddle River, NJ: Pearson/Prentice-Hall, 2010).

Strang, G., *Linear Algebra and Its Applications* (4th ed.)(Belmont, CA: Thomson, Brooks/Cole, 2006).

A more extended treatment of systems of first-order linear equations may be found in many books, including the following:

Coddington, E. A. and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mahematics, 1997).

Hirsch, M. W., Smale, S., and Devaney, R. L., *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (2nd ed.)(San Diego, CA: Academic Press, 2004).

The following book treats elementary differential equations with a particular emphasis on systems of first-order equations:

Brannan, J. R. and Boyce, W. E., *Differential Equations: An Introduction to Modern Methods and Applications* (3rd ed.) (New York: Wiley, 2015).