11. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of

$$x^2y'' + \left(\alpha^2\beta^2x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right)y = 0, \quad x > 0$$

is given by $y = x^{1/2}f(\alpha x^{\beta})$, where $f(\xi)$ is a solution of the Bessel equation of order ν.

12. Using the result of Problem 11, show that the general solution of the Airy equation

$$y'' - xy = 0, \quad x > 0$$

is $y = x^{1/2}\left(c_1 f_1\left(\frac{2}{3}ix^{3/2}\right) + c_2 f_2\left(\frac{2}{3}ix^{3/2}\right)\right)$, where $f_1(\xi)$ and $f_2(\xi)$ are a fundamental set of solutions of the Bessel equation of order one-third.

13. It can be shown that J_0 has infinitely many zeros for $x > 0$. In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.7.1). Let λ_j, $j = 1, 2, 3, \ldots$, denote the zeros of J_0; it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that $y = J_0(\lambda_j x)$ satisfies the differential equation

$$y'' + \frac{1}{x}y' + \lambda_j^2 y = 0, \quad x > 0.$$

Hence show that

$$\int_0^1 xJ_0(\lambda_i x)J_0(\lambda_j x)\,dx = 0 \quad \text{if } \lambda_i \neq \lambda_j.$$

This important property of $J_0(\lambda_i x)$, which is known as the **orthogonality property**, is useful in solving boundary value problems.

Hint: Write the differential equation for $J_0(\lambda_i x)$. Multiply it by $xJ_0(\lambda_j x)$ and subtract that result from $xJ_0(\lambda_i x)$ times the differential equation for $J_0(\lambda_j x)$. Then integrate from 0 to 1.

References

Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice-Hall, 1961; New York: Dover, 1989).

Coddington, E. A., and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mathematics, 1997).

Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable* (Oxford: Oxford University Press, 1935).

K. Knopp, *Theory and Applications of Infinite Series* (NewYork: Hafner, 1951).

Proofs of Theorems 5.3.1 and 5.6.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, Chapters 5 and 6 of Coddington and Carlson, or Chapters 3 and 4 of

Rainville, E. D., *Intermediate Differential Equations* (2nd ed.) (New York: Macmillan, 1964).

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 32 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of

Coddington, E. A., and Levinson, N., *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955; Malabar, FL: Krieger, 1984).

Fuller discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. One text dealing with special functions such as the Legendre polynomials and the Bessel functions is

Hochstadt, H., *Special Functions of Mathematical Physics* (New York: Holt, 1961).

An excellent compilation of formulas, graphs, and tables of Bessel functions, Legendre functions, and other special functions of mathematical physics may be found in

Abramowitz, M., and Stegun, I. A. (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover, 1965); originally published by the National Bureau of Standards, Washington, DC, 1964.

The digital successor to Abramowitz and Stegun is

Digital Library of Mathematical Functions. Released August 29, 2011. National Institute of Standards and Technology from http://dlmf.nist.gov/.

CHAPTER 6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those that arise in engineering applications.

6.1 Definition of the Laplace Transform

Improper Integrals. Since the Laplace transform involves an integral from zero to infinity, a knowledge of improper integrals of this type is necessary to appreciate the subsequent development of the properties of the transform. We provide a brief review of such improper integrals here. If you are already familiar with improper integrals, you may wish to skip over this review. On the other hand, if improper integrals are new to you, then you should probably consult a calculus book, where you will find many more details and examples.

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt, \tag{1}$$

where A is a positive real number. If the definite integral from a to A exists for each $A > a$, and if the limit of these values as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Does the improper integral $\int_1^\infty \frac{dt}{t}$ diverge or converge?

Solution:

From equation (1) we have

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty}\int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

EXAMPLE 7

Find $\mathcal{L}\{\sin(at)\}$. For what values of s is this transform defined?

Solution:

Let $f(t) = \sin(at),\ t \geq 0$. Then

$$\mathcal{L}\{\sin(at)\} = F(s) = \int_0^{\infty} e^{-st}\sin(at)\,dt, \quad s > 0.$$

Since

$$F(s) = \lim_{A\to\infty}\int_0^{A} e^{-st}\sin(at)\,dt,$$

upon integrating by parts, we obtain

$$F(s) = \lim_{A\to\infty}\left[-\frac{e^{-st}\cos(at)}{a}\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos(at)\,dt\right]$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^{\infty} e^{-st}\cos(at)\,dt.$$

A second integration by parts then yields

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^{\infty} e^{-st}\sin(at)\,dt$$

$$= \frac{1}{a} - \frac{s^2}{a^2}F(s).$$

Now, solving for $F(s)$, we have

$$F(s) = \frac{a}{s^2+a^2}, \quad s > 0.$$

In Problem 5 you will use a similar process to find $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ for $s > 0$. Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = \int_0^{\infty} e^{-st}\big(c_1f_1(t) + c_2f_2(t)\big)dt$$

$$= c_1\int_0^{\infty} e^{-st}f_1(t)\,dt + c_2\int_0^{\infty} e^{-st}f_2(t)\,dt;$$

hence

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \tag{6}$$

Equation (6) states that the Laplace transform is a **linear operator**, and we make frequent use of this property later. The sum in equation (6) can be readily extended to an arbitrary number of terms.

EXAMPLE 8

Find the Laplace transform of $f(t) = 5e^{-2t} - 3\sin(4t),\ t \geq 0$.

Solution:

Using equation (6), we write

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin(4t)\}.$$

Then, from Examples 5 and 7, we obtain

$$\mathcal{L}\{f(t)\} = \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0.$$

Problems

In each of Problems 1 through 3, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \leq t \leq 3$.

1. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 2+t, & 1 < t \leq 2 \\ 6-t, & 2 < t \leq 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ (t-1)^{-1}, & 1 < t \leq 2 \\ 1, & 2 < t \leq 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 1, & 1 < t \leq 2 \\ 3-t, & 2 < t \leq 3 \end{cases}$

4. Find the Laplace transform of each of the following functions:
  a. $f(t) = t$
  b. $f(t) = t^2$
  c. $f(t) = t^n$, where n is a positive integer

5. Find the Laplace transform of $f(t) = \cos(at)$, where a is a real constant.

Recall that

$$\cosh(bt) = \frac{1}{2}(e^{bt} + e^{-bt}) \text{ and } \sinh(bt) = \frac{1}{2}(e^{bt} - e^{-bt}).$$

In each of Problems 6 through 7, use the linearity of the Laplace transform to find the Laplace transform of the given function; a and b are real constants.

6. $f(t) = \cosh(bt)$

7. $f(t) = \sinh(bt)$

Recall that

$$\cos(bt) = \frac{1}{2}(e^{ibt} + e^{-ibt}) \text{ and } \sin(bt) = \frac{1}{2i}(e^{ibt} - e^{-ibt}).$$

In each of Problems 8 through 11, use the linearity of the Laplace transform to find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

8. $f(t) = \sin(bt)$

9. $f(t) = \cos(bt)$

10. $f(t) = e^{at}\sin(bt)$

11. $f(t) = e^{at}\cos(bt)$

In each of Problems 12 through 15, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

12. $f(t) = te^{at}$

13. $f(t) = t\sin(at)$

14. $f(t) = t^ne^{at}$

15. $f(t) = t^2\sin(at)$

In each of Problems 16 through 18, find the Laplace transform of the given function.

16. $f(t) = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & \pi \leq t < \infty \end{cases}$

17. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < \infty \end{cases}$

18. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2-t, & 1 \leq t < 2 \\ 0, & 2 \leq t < \infty \end{cases}$

In each of Problems 19 through 21, determine whether the given integral converges or diverges.

19. $\displaystyle\int_0^{\infty} (t^2+1)^{-1}dt$

20. $\displaystyle\int_0^{\infty} te^{-t}dt$

21. $\displaystyle\int_1^{\infty} t^{-2}e^{t}dt$

22. Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.

23. The Gamma Function. The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^{\infty} e^{-x}x^p\,dx. \tag{7}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

  a. Show that, for $p > 0$,
  $$\Gamma(p+1) = p\Gamma(p).$$
  b. Show that $\Gamma(1) = 1$.
  c. If p is a positive integer n, show that
  $$\Gamma(n+1) = n!.$$

  Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

  d. Show that, for $p > 0$,
  $$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}.$$

  Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \leq 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.

24. Consider the Laplace transform of t^p, where $p > -1$.

a. Referring to Problem 23, show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx$$
$$= \frac{\Gamma(p)}{s^{p+1}}, \quad s > 0.$$

b. Let p be a positive integer n in part a; show that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0.$$

c. Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx, \quad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{s}}, \quad s > 0.$$

d. Show that

$$\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}, \quad s > 0.$$

6.2 Solution of Initial Value Problems

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. The usefulness of the Laplace transform for this purpose rests primarily on the fact that the transform of f' is related in a simple way to the transform of f. The relationship is expressed in the following theorem.

Theorem 6.2.1

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{1}$$

To prove this theorem, we consider the integral

$$\int_0^A e^{-st} f'(t)dt,$$

whose limit as $A \to \infty$, if it exists, is the Laplace transform of f'. To calculate this limit we first need to write the integral in a suitable form. If f' has points of discontinuity in the interval $0 \le t \le A$, let them be denoted by $t_1, t_2, \dots, t_k$. Then we can write the integral as

$$\int_0^A e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \cdots + \int_{t_k}^A e^{-st} f'(t)dt.$$

Integrating each term on the right by parts yields

$$\int_0^A e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_k}^{A}$$
$$+ s\left[\int_0^{t_1} e^{-st} f(t)dt + \int_{t_1}^{t_2} e^{-st} f(t)dt + \cdots + \int_{t_k}^A e^{-st} f(t)dt\right].$$

Since f is continuous, the contributions of the integrated terms at $t_1, t_2, \dots, t_k$ cancel. Further, the integrals on the right-hand side can be combined into a single integral so that we obtain

$$\int_0^A e^{-st} f'(t)dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)dt. \tag{2}$$

Now we let $A \to \infty$ in equation (2). The integral on the right-hand side of this equation approaches $\mathcal{L}\{f(t)\}$. Further, for $A \ge M$, we have $|f(A)| \le Ke^{aA}$; consequently, $|e^{-sA} f(A)| \le Ke^{-(s-a)A}$. Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Thus the right-hand side of equation (2) has the limit $s\mathcal{L}\{f(t)\} - f(0)$. Consequently, the left-hand side of equation (2) also has a limit, and as noted above, this limit is $\mathcal{L}\{f'(t)\}$. Therefore, for $s > a$, we conclude that

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

which completes the proof of Theorem 6.2.1.

If f' and f'' satisfy the same conditions that are imposed on f and f', respectively, in Theorem 6.2.1, then it follows that the Laplace transform of f'' also exists for $s > a$ and is given by

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0)\\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).\end{aligned} \tag{3}$$

Indeed, provided the function f and its derivatives satisfy suitable conditions, an expression for the transform of the n^{th} derivative $f^{(n)}$ can be derived by n successive applications of this theorem. The result is given in the following corollary.

Corollary 6.2.2

Suppose that the functions $f, f', \dots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}, |f'(t)| \le Ke^{at}, \dots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{4}$$

We now show how the Laplace transform can be used to solve initial value problems. It is most useful for problems involving nonhomogeneous differential equations, as we will demonstrate in later sections of this chapter. However, we begin by looking at some homogeneous equations, which are a bit simpler.

EXAMPLE 1

Find the solution of the differential equation

$$y'' - y' - 2y = 0 \tag{5}$$

that satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{6}$$

Solution:

This initial value problem is easily solved by the methods of Section 3.1. The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0,$$

Consequently, $a = 2$, $c = 0$, $b = \frac{5}{3}$, and $d = -\frac{2}{3}$, from which it follows that

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}. \tag{24}$$

From lines 5 and 6 of Table 6.2.1, the solution of the given initial value problem is

$$y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t). \tag{25}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y^{(4)} - y = 0, \tag{26}$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0. \tag{27}$$

Solution:

In this problem we need to assume that the solution $y(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 4$. The Laplace transform of the differential equation (26) is

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y(s) = 0.$$

Then, using the initial conditions (27) and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4-1}. \tag{28}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}, \tag{29}$$

and it follows that

$$(as+b)(s^2+1) + (cs+d)(s^2-1) = s^2 \tag{30}$$

for all s. In this problem we use a combination of substituting values of s and equating coefficients of like powers of s. First, setting $s = 1$ and $s = -1$, respectively, in equation (30), we obtain the pair of equations

$$2(a+b) = 1, \quad 2(-a+b) = 1,$$

and therefore $a = 0$ and $b = \frac{1}{2}$. If we set $s = 0$ in equation (30), then $b - d = 0$, so $d = \frac{1}{2}$. Finally, equating the coefficients of the cubic terms on each side of equation (30), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1}, \tag{31}$$

and from lines 7 and 5 of Table 6.2.1, the solution of the initial value problem (26), (27) is

$$y(t) = \frac{1}{2}(\sinh t + \sin t). \tag{32}$$

We conclude by noting that we could have looked for a partial fraction expansion of $Y(s)$ in the form

$$Y(s) = \frac{a}{s-1} + \frac{b}{s+1} + \frac{cs+d}{s^2+1}.$$

We used the form in equation (29) because Table 6.2.1 includes entries for both $1/(s^2 \pm 1)$ and $s/(s^2 \pm 1)$.

The most important elementary applications of the Laplace transform are in the study of mechanical vibrations and in the analysis of electric circuits; the governing equations were derived in Section 3.7. A vibrating spring-mass system has the equation of motion

$$m\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + ku = F(t), \tag{33}$$

where m is the mass, γ the damping coefficient, k the spring constant, and $F(t)$ the applied external force. The equation that describes an electric circuit containing an inductance L, a resistance R, and a capacitance C (an LRC circuit) is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \tag{34}$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is the applied voltage. In terms of the current $I(t) = dQ(t)/dt$, we can differentiate equation (34) and write

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}(t). \tag{35}$$

Suitable initial conditions on u, Q, or I must also be prescribed.

We have noted previously, in Section 3.7, that equation (33) for the spring-mass system and equations (34) or (35) for the electric circuit are identical mathematically, differing only in the interpretation of the constants and variables appearing in them. There are other physical problems that also lead to the same differential equation. Thus, once the mathematical problem is solved, its solution can be interpreted in terms of whichever corresponding physical problem is of immediate interest.

In the problem lists following this and other sections in this chapter are numerous initial-value problems for second-order linear differential equations with constant coefficients. Many can be interpreted as models of particular physical systems, but usually we do not point this out explicitly.

Problems

In each of Problems 1 through 7, find the inverse Laplace transform of the given function.

1. $F(s) = \dfrac{3}{s^2+4}$
2. $F(s) = \dfrac{4}{(s-1)^3}$
3. $F(s) = \dfrac{2}{s^2+3s-4}$
4. $F(s) = \dfrac{2s+2}{s^2+2s+5}$
5. $F(s) = \dfrac{2s-3}{s^2-4}$
6. $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$
7. $F(s) = \dfrac{1-2s}{s^2+4s+5}$

In each of Problems 8 through 16, use the Laplace transform to solve the given initial value problem.

8. $y'' - y' - 6y = 0;\quad y(0) = 1,\ y'(0) = -1$
9. $y'' + 3y' + 2y = 0;\quad y(0) = 1,\ y'(0) = 0$
10. $y'' - 2y' + 2y = 0;\quad y(0) = 0,\ y'(0) = 1$
11. $y'' - 2y' + 4y = 0;\quad y(0) = 2,\ y'(0) = 0$
12. $y'' + 2y' + 5y = 0;\quad y(0) = 2,\ y'(0) = -1$
13. $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0;\quad y(0) = 0,$ $y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 1$
14. $y^{(4)} - y = 0;\quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,$ $y'''(0) = 0$
15. $y'' + \omega^2 y = \cos(2t),\ \omega^2 \neq 4;\quad y(0) = 1,\ y'(0) = 0$
16. $y'' - 2y' + 2y = e^{-t};\quad y(0) = 0,\ y'(0) = 1$

In each of Problems 17 through 19, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 16 through 18 in Section 6.1.

17. $y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \quad y(0) = 1,\ y'(0) = 0$
18. $y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \quad y(0) = 0,\ y'(0) = 0$
19. $y'' + y = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty; \end{cases} \quad y(0) = 0,\ y'(0) = 0$

20. The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.

a. Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term-by-term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \qquad s > 1.$$

b. Let

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Show that $f(t)$ is continuous for all real values of t. Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term-by-term, verify that

$$\mathcal{L}\{f(t)\} = \arctan\left(\frac{1}{s}\right), \qquad s > 1.$$

c. The Bessel function of the first kind of order zero, J_0, has the Taylor series (see Section 5.7)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term-by-term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \qquad s > 0.$$

Problems 21 through 27 are concerned with differentiation of the Laplace transform.

21. Let

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.

a. Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

b. Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

In each of Problems 22 through 25, use the result of Problem 21 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

22. $f(t) = te^{at}$

23. $f(t) = t^2 \sin(bt)$

24. $f(t) = t^n e^{at}$

25. $f(t) = te^{at}\sin(bt)$

26. Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

a. Show that $Y(s)$ satisfies

$$(1+s^2)Y'(s) + sY(s) = 0.$$

b. Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

c. Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term-by-term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

27. For each of the following initial value problems, use the results of Problem 21 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{y(t)\}$, where $y(t)$ is the solution of the given initial value problem.

a. $y'' - ty = 0; \quad y(0) = 1, \ y'(0) = 0$ (Airy's equation)

b. $(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0; \quad y(0) = 0, \ y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first-order in part a, but of second-order in part b. This is due to the fact that t appears at most to the first power in the equation of part a, whereas it appears to the second power in that of part b. This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

28. Suppose that

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = \frac{F(s)}{s}.$$

29. In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = \frac{P(s)}{Q(s)},$$

where $Q(s)$ is a polynomial of degree n with n distinct zeros $r_1, \ldots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \cdots + \frac{A_n}{s-r_n}, \qquad (36)$$

where the coefficients $A_1, \ldots, A_n$ must be determined.

a. Show that

$$A_k = \frac{P(r_k)}{Q'(r_k)}, \qquad k = 1, \ldots, n.$$

Hint: One way to do this is to multiply equation (36) by $s - r_k$ and then to take the limit as $s \to r_k$. Note that limits are used because it is not appropriate to simply evaluate equation (36) multiplied by $s - r_k$ because equation (36) is not defined at each root of $Q(s)$.

b. Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}.$$

6.3 Step Functions

In Section 6.2 we outlined the general procedure involved in solving initial value problems by means of the Laplace transform. Some of the most interesting elementary applications of the transform method occur in the solution of linear differential equations with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. In this section and the remaining sections in Chapter 6, we develop some additional properties of the Laplace transform that are useful in the solution of such problems. Unless a specific statement is made to the contrary, all functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large.

To deal effectively with functions having jump discontinuities, it is very helpful to introduce a function known as the **unit step function** or **Heaviside function**. This function will be denoted by u_c and is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases} \qquad (1)$$

Since the Laplace transform involves values of t in the interval $[0, \infty)$, we are also interested only in nonnegative values of c. The graph of $y = u_c(t)$ is shown in Figure 6.3.1. We have somewhat arbitrarily assigned the value one to u_c at $t = c$. However, for a piecewise continuous function such as u_c, the value at a discontinuity point is usually irrelevant. The step can also be negative. For instance, Figure 6.3.2 shows the graph of $y = 1 - u_c(t)$.

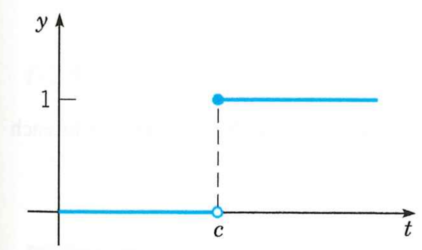

FIGURE 6.3.1 Graph of $y = u_c(t)$.

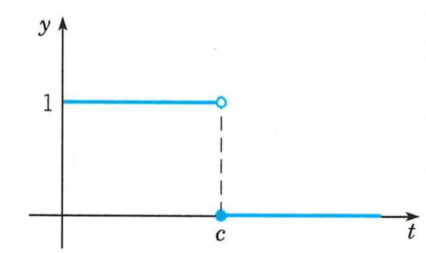

FIGURE 6.3.2 Graph of $y = 1 - u_c(t)$.

If we associate the value 1 with "on" and 0 with "off," then the function $u_c(t)$ represents a switch that is turned on at time c. Likewise, $1 - u_c(t)$ represents a switch being turned off at time c.

EXAMPLE 1

Sketch the graph of $y = h(t)$, where

$$h(t) = u_\pi(t) - u_{2\pi}(t), \qquad t \geq 0.$$

Theorem 6.3.2

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad s > a + c. \tag{7}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{8}$$

According to Theorem 6.3.2, multiplication of $f(t)$ by e^{ct} results in a translation of the transform $F(s)$ a distance c in the positive s direction, and conversely. To prove this theorem, we evaluate $\mathcal{L}\{e^{ct} f(t)\}$. Thus

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\,dt = \int_0^\infty e^{-(s-c)t} f(t)\,dt$$
$$= F(s-c),$$

which is equation (7). The restriction $s > a + c$ follows from the observation that, according to hypothesis (ii) of Theorem 6.1.2, $|f(t)| \leq Ke^{at}$; hence $|e^{ct} f(t)| \leq Ke^{(a+c)t}$. Equation (8) is obtained by taking the inverse transform of equation (7), and the proof is complete.

The principal application of Theorem 6.3.2 is in the evaluation of certain inverse transforms, as illustrated by Example 5.

EXAMPLE 5

Find the inverse Laplace transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}.$$

Solution:

First, to avoid dealing with the complex-valued roots of the denominator $s^2 - 4s + 5$, we complete the square in the denominator:

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2),$$

where $F(s) = (s^2 + 1)^{-1}$. Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows from Theorem 6.3.2 that

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t} \sin t.$$

The results of this section are often useful in solving differential equations, particularly those that have discontinuous forcing functions. The next section is devoted to examples illustrating this point.

Problems

In each of Problems 1 through 4, sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
3. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
4. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 5 through 8:

a. Sketch the graph of the given function.
b. Express $f(t)$ in terms of the unit step function $u_c(t)$.

5. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$

6. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$

7. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$

8. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 9 through 12, find the Laplace transform of the given function.

9. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$

10. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$

11. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

12. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$

In each of Problems 13 through 16, find the inverse Laplace transform of the given function.

13. $F(s) = \dfrac{3!}{(s-2)^4}$

14. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$

15. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$

16. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$

17. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.
    a. Show that if c is a positive constant, then
    $$\mathcal{L}\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right), \quad s > ca.$$
    b. Show that if k is a positive constant, then
    $$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right).$$
    c. Show that if a and b are constants with $a > 0$, then
    $$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right).$$

In each of Problems 18 through 20, use the results of Problem 17 to find the inverse Laplace transform of the given function.

18. $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$

19. $F(s) = \dfrac{2s+1}{4s^2 + 4s + 5}$

20. $F(s) = \dfrac{1}{9s^2 - 12s + 3}$

In each of Problems 21 through 23, find the Laplace transform of the given function. In Problem 23, assume that term-by-term integration of the infinite series is permissible.

21. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$

22. $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

23. $f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$. See Figure 6.3.8.

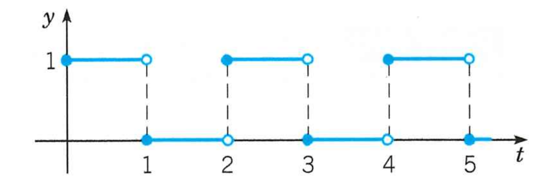

FIGURE 6.3.8 The function $f(t)$ in Problem 23; a square wave.

24. Let f satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be **periodic with period** T on $0 \leq t < \infty$. Show that
$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

In each of Problems 25 through 28, use the result of Problem 24 to find the Laplace transform of the given function.

25. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & 1 \leq t < 2; \end{cases} \quad f(t+2) = f(t).$
    Compare with Problem 23.

26. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2; \end{cases} \quad f(t+2) = f(t).$
    See Figure 6.3.9.

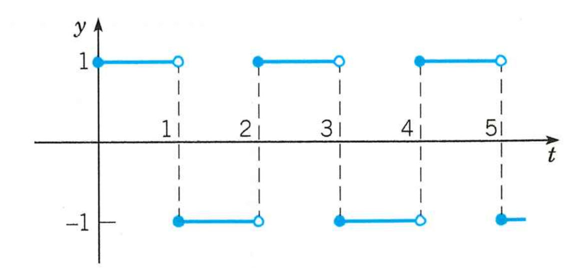

FIGURE 6.3.9 The function $f(t)$ in Problem 26; a square wave.

27. $f(t) = t, \quad 0 \le t < 1; \quad f(t+1) = f(t).$
See Figure 6.3.10.

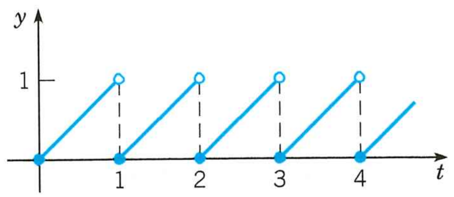

FIGURE 6.3.10 The function $f(t)$ in Problem 27; a sawtooth wave.

28. $f(t) = \sin t, \quad 0 \le t < \pi; \quad f(t+\pi) = f(t).$
See Figure 6.3.11.

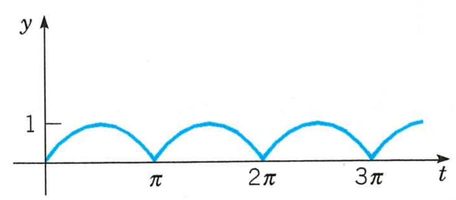

FIGURE 6.3.11 The function $f(t)$ in Problem 28; a rectified sine wave.

29. **a.** If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$. Sketch the graph of $y = f(t)$. Compare with Problem 21.
b. Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function f is defined in part a. Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$. Use your expression for $\mathcal{L}\{g(t)\}$ to find an explicit formula for $g(t)$. *Hint:* See Problem 28 in Section 6.2.
c. Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part b. Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$. Use your expression for $\mathcal{L}\{h(t)\}$ to find an explicit formula for $h(t)$.

30. Consider the function p defined by

$$p(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2; \end{cases} \qquad p(t+2) = p(t).$$

a. Sketch the graph of $y = p(t)$.
b. Find $\mathcal{L}\{p(t)\}$ by noting that p is the periodic extension of the function h in Problem 29c; then use the result of Problem 24.
c. Find $\mathcal{L}\{p(t)\}$ by noting that

$$p(t) = \int_0^t f(t)\,dt,$$

where f is the function in Problem 26; then use Theorem 6.2.1.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or **forcing function**, is discontinuous.

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \text{ or } t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \quad y'(0) = 0. \tag{3}$$

This problem governs the charge on the capacitor in a simple electric circuit with a unit voltage pulse for $5 \le t < 20$. Alternatively, y may represent the response of a damped oscillator subject to the applied force $g(t)$.

Solution:

The Laplace transform of equation (1) is

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$
$$= \frac{1}{s}\left(e^{-5s} - e^{-20s}\right).$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = \left(e^{-5s} - e^{-20s}\right)H(s), \tag{5}$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}. \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Observe that we have used Theorem 6.3.1 to write the inverse transforms of $e^{-5s}H(s)$ and $e^{-20s}H(s)$, respectively. Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients, we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$H(s) = \frac{1}{2}s - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{1}{2}s - \frac{1}{2}\,\frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$
$$= \frac{1}{2}s - \frac{1}{2}\left(\frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1}{\sqrt{15}}\,\frac{\frac{\sqrt{15}}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right). \tag{9}$$

Then, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \frac{1}{2} - \frac{1}{2}\left(e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{1}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)\right). \tag{10}$$

In Figure 6.4.1 the graph of $y(t)$ from equations (7) and (10) shows that the solution consists of three distinct parts. For $0 < t < 5$, the differential equation is

$$2y'' + y' + 2y = 0, \tag{11}$$

and the initial conditions are given by equation (3). Since the initial conditions impart no energy to the system, and since there is no external forcing, the system remains at rest; that is, $y = 0$ for $0 < t < 5$. This can be confirmed by solving equation (11) subject to the initial conditions (3). In particular, evaluating the solution and its derivative at $t = 5$, or, more precisely, as t approaches 5 from below, we have

$$y(5) = 0, \quad y'(5) = 0. \tag{12}$$

Once $t > 5$, the differential equation becomes

$$2y'' + y' + 2y = 1, \tag{13}$$

Thus the solution of the initial value problem (17), (18), (19) is

$$y(t) = \frac{1}{5}\big(u_5(t)h(t-5) - u_{10}(t)h(t-10)\big), \tag{24}$$

where $h(t)$ is the inverse transform of $H(s)$.

The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}, \tag{25}$$

and it then follows from lines 3 and 5 of Table 6.2.1 that

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t). \tag{26}$$

The graph of $y(t)$ is shown in Figure 6.4.3. Observe that it has the qualitative form that we indicated earlier. To find the amplitude of the eventual steady oscillation, it is sufficient to locate one of the maximum or minimum points for $t > 10$. Setting the derivative of the solution (24) equal to zero, we find that the first maximum is located approximately at $(10.642, 0.2979)$, so the amplitude of the oscillation is approximately $0.2979 - 0.25 = 0.0479$.

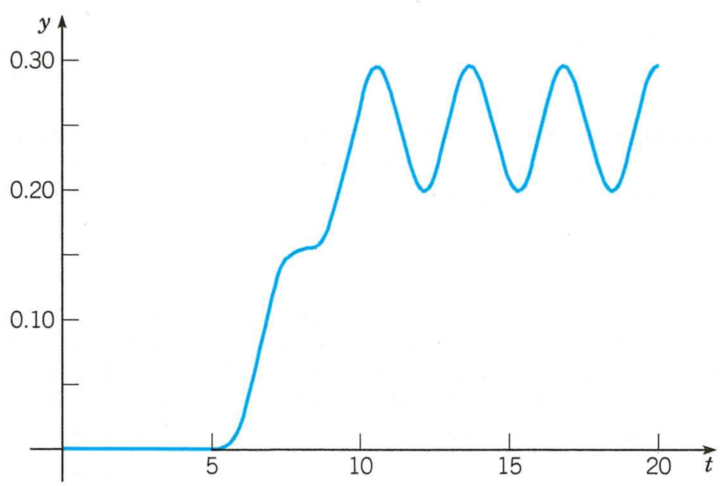

FIGURE 6.4.3 Solution of the initial value problem (12), (13), (14).

Note that in this example, the forcing function g is continuous, but g' is discontinuous at $t = 5$ and $t = 10$. It follows that the solution $y(t)$ and its first two derivatives are continuous everywhere, but $y'''(t)$ has discontinuities at $t = 5$ and at $t = 10$ that match the discontinuities in g' at those points.

Problems

In each of Problems 1 through 8:

a. Sketch the graph of the forcing function on an appropriate interval.

b. Find the solution of the given initial value problem.

G **c.** Plot the graph of the solution.

d. Explain how the graphs of the forcing function and the solution are related.

1. $y'' + y = f(t); \quad y(0) = 0, \ y'(0) = 1;$
$f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$

2. $y'' + 2y' + 2y = h(t); \quad y(0) = 0, \ y'(0) = 1;$
$h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \ \text{ or } \ t \ge 2\pi \end{cases}$

3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \quad y(0) = 0, \ y'(0) = 0$

4. $y'' + 3y' + 2y = f(t); \quad y(0) = 0, \ y'(0) = 0;$
$f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$

5. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)\left(t - \frac{\pi}{2}\right); \quad y(0) = 0, \ y'(0) = 0$

6. $y'' + y' + \frac{5}{4}y = g(t); \quad y(0) = 0, \ y'(0) = 0;$
$g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

7. $y'' + 4y = u_\pi(t) - u_{3\pi}(t); \quad y(0) = 0, \ y'(0) = 0$

8. $y^{(4)} + 5y'' + 4y = 1 - u_\pi(t); \quad y(0) = 0, \ y'(0) = 0,$
$y''(0) = 0, \ y'''(0) = 0$

9. Find an expression involving $u_c(t)$ for a function f that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$.

10. Find an expression involving $u_c(t)$ for a function g that ramps up from zero at $t = t_0$ to the value h at $t = t_0 + k$ and then ramps back down to zero at $t = t_0 + 2k$.

11. A certain spring-mass system satisfies the initial value problem

$$u'' + \frac{1}{4}u' + u = kg(t), \quad u(0) = 0, \ u'(0) = 0,$$

where $g(t) = u_{3/2}(t) - u_{5/2}(t)$ and $k > 0$ is a parameter.

a. Sketch the graph of $g(t)$. Observe that it is a pulse of unit magnitude extending over one time unit.

b. Solve the initial value problem.

G **c.** Plot the solution for $k = 1/2$, $k = 1$, and $k = 2$. Describe the principal features of the solution and how they depend on k.

N **d.** Find, to two decimal places, the smallest value of k for which the solution $u(t)$ reaches the value 2.

N **e.** Suppose $k = 2$. Find the time τ after which $|u(t)| < 0.1$ for all $t > \tau$.

12. Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$$f(t) = \frac{1}{k}\big(u_5(t)(t-5) - u_{5+k}(t)(t-5-k)\big)/k.$$

a. Sketch the graph of $f(t)$ and describe how it depends on k. For what value of k is $f(t)$ identical to $g(t)$ in the example?

b. Solve the initial value problem

$$y'' + 4y = f(t), \quad y(0) = 0, \ y'(0) = 0.$$

G **c.** The solution in part b depends on k, but for sufficiently large t, the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on k. Then confirm your conclusion by plotting the solution for a few different values of k.

Resonance and Beats. In Section 3.8 we observed that an undamped harmonic oscillator (such as a spring-mass system) with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In Problems 13 through 17 we explore the effect of some nonsinusoidal periodic forcing functions.

13. Consider the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \ y'(0) = 0,$$

where

$$f(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

a. Draw the graph of $f(t)$ on an interval such as $0 \le t \le 6\pi$.

b. Find the solution of the initial value problem.

G **c.** Let $n = 15$. Plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does.

d. Investigate how the solution changes as n increases. What happens as $n \to \infty$?

14. Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \quad y(0) = 0, \ y'(0) = 0,$$

where $f(t)$ is the same as in Problem 13.

G **a.** Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

b. Estimate the amplitude and frequency of the steady-state part of the solution.

c. Compare the results of part b with those from Section 3.8 for a sinusoidally forced oscillator.

15. Consider the initial value problem

$$y'' + y = g(t), \quad y(0) = 0, \ y'(0) = 0,$$

where

$$g(t) = u_0(t) + \sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

a. Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 13a.

b. Find the solution of the initial value problem.

G **c.** Let $n = 15$. Plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 13.

d. Investigate how the solution changes as n increases. What happens as $n \to \infty$?

16. Consider the initial value problem

$$y'' + 0.1y' + y = g(t), \quad y(0) = 0, \ y'(0) = 0,$$

where $g(t)$ is the same as in Problem 15.

G **a.** Plot the graph of the solution. Use a large enough value of n and a long enough t-interval so that the transient part of the solution has become negligible and the steady state is clearly shown.

N **b.** Estimate the amplitude and frequency of the steady-state part of the solution.

c. Compare the results of part b with those from Problem 15 and from Section 3.8 for a sinusoidally forced oscillator.

17. Consider the initial value problem

$$y'' + y = h(t), \quad y(0) = 0, \ y'(0) = 0,$$

where

$$h(t) = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t).$$

Observe that this problem is identical to Problem 15, except that the frequency of the forcing term has been increased somewhat.

a. Find the solution of this initial value problem.

G **b.** Let $n \ge 33$ and plot the solution for $0 \le t \le 90$ or longer. Your plot should show a clearly recognizable beat.

N **c.** From the graph in part b, estimate the "slow period" and the "fast period" for this oscillator.

d. For a sinusoidally forced oscillator, it was shown in Section 3.8 that the "slow frequency" is given by $\frac{1}{2}|\omega - \omega_0|$, where ω_0 is the natural frequency of the system and ω is the forcing frequency. Similarly, the "fast frequency" is $\frac{1}{2}(\omega + \omega_0)$. Use these expressions to calculate the "fast period" and the "slow period" for the oscillator in this problem. How well do the results compare with your estimates from part c?

Then from equation (11) it follows that

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}. \tag{13}$$

Equation (13) defines $\mathcal{L}\{\delta(t-t_0)\}$ for any $t_0 > 0$. We extend this result, to allow t_0 to be zero, by letting $t_0 \to 0^+$ on the right-hand side of equation (13); thus

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0^+} e^{-st_0} = 1. \tag{14}$$

It is reassuring to see that the Laplace transform formulas derived in equations (13) and (14) are consistent with the Laplace transform of a horizontally shifted function:

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}\mathcal{L}\{\delta(t)\} = e^{-st_0}.$$

In a similar way, it is possible to define the integral of the product of the delta function and any continuous function f. We have

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\,dt. \tag{15}$$

Using the definition (4) of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)\,dt = \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt$$

$$= \frac{1}{2\tau}\cdot 2\tau \cdot f(t^*) = f(t^*),$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence $t^* \to t_0$ as $\tau \to 0^+$, and it follows from equation (15) that

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0). \tag{16}$$

The following example illustrates the use of the delta function in solving an initial value problem with an impulsive forcing function.

EXAMPLE 1

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \tag{17}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{18}$$

Solution:

This initial value problem arises from the study of the same electric circuit or mechanical oscillator as in Example 1 of Section 6.4. The only difference is in the forcing term.

To solve the given problem, we first take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2}\,\frac{1}{\left(s+\frac{1}{4}\right)^2 + \frac{15}{16}}. \tag{19}$$

By Theorem 6.3.2, or from line 9 of Table 6.2.1,

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right\} = \frac{4}{\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right). \tag{20}$$

Hence, by Theorem 6.3.1, we have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right) \tag{21}$$

which is the formal solution of the given problem. It is also possible to write $y(t)$ in the form

$$y = \begin{cases} 0, & t < 5, \\ \dfrac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\left(\dfrac{\sqrt{15}}{4}(t-5)\right), & t \ge 5. \end{cases} \tag{22}$$

The graph of equation (22) is shown in Figure 6.5.3. Since the initial conditions at $t = 0$ are homogeneous and there is no external excitation until $t = 5$, there is no response in the interval $0 < t < 5$. The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely. The response is continuous at $t = 5$ despite the singularity in the forcing function at that point. However, the first derivative of the solution has a jump discontinuity at $t = 5$, and the second derivative has an infinite discontinuity there. This is required by the differential equation (17), since a singularity on one side of the equation must be balanced by a corresponding singularity on the other side.

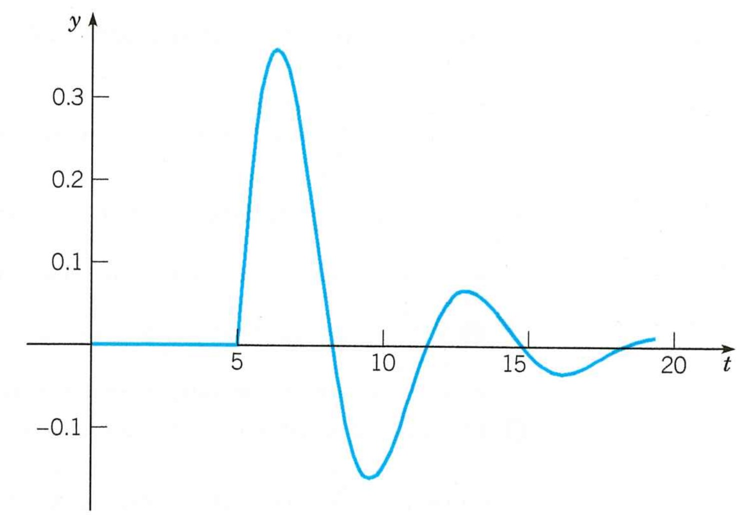

FIGURE 6.5.3 Solution of the initial value problem (17), (18): $2y'' + y' + 2y = \delta(t-5)$, $y(0) = 0$, $y'(0) = 0$.

In dealing with problems that involve impulsive forcing, the use of the delta function usually simplifies the mathematical calculations, often quite significantly. However, if the actual excitation extends over a short, but nonzero, time interval, then an error will be introduced by modeling the excitation as taking place instantaneously. This error may be negligible, but in a practical problem it should not be dismissed without consideration. In Problem 12 you are asked to investigate this issue for a simple harmonic oscillator.

Problems

In each of Problems 1 through 8:

a. Find the solution of the given initial value problem.

b. Plot a graph of the solution.

1. $y'' + 2y' + 2y = \delta(t-\pi)$; $y(0) = 1$, $y'(0) = 0$
2. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$; $y(0) = 0$, $y'(0) = 0$
3. $y'' + 3y' + 2y = \delta(t-5) + u_{10}(t)$; $y(0) = 0$, $y'(0) = 1/2$
4. $y'' + 2y' + 3y = \sin t + \delta(t-3\pi)$; $y(0) = 0$, $y'(0) = 0$
5. $y'' + y = \delta(t-2\pi)\cos t$; $y(0) = 0$, $y'(0) = 1$
6. $y'' + 4y = 2\delta(t-\pi/4)$; $y(0) = 0$, $y'(0) = 0$
7. $y'' + 2y' + 2y = \cos t + \delta(t-\pi/2)$; $y(0) = 0$, $y'(0) = 0$
8. $y^{(4)} - y = \delta(t-1)$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$

9. Consider again the system in Example 1 of this section, in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle—that is, when the response first returns to equilibrium moving in the positive direction.

N **a.** Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.

G **b.** Solve the resulting initial value problem, and plot its solution to confirm that it behaves in the specified manner.

N **10.** Consider the initial value problem

$$y'' + \gamma y' + y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0,$$

where γ is the damping coefficient (or resistance).

G **a.** Let $\gamma = \frac{1}{2}$. Find the solution of the initial value problem and plot its graph.

b. Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.

G **c.** Let $\gamma = \frac{1}{4}$ and repeat parts a and b.

d. Determine how t_1 and y_1 vary as γ decreases. What are the values of t_1 and y_1 when $\gamma = 0$?

11. Consider the initial value problem

$$y'' + \gamma y' + y = k\delta(t-1), \quad y(0) = 0, \quad y'(0) = 0,$$

where k is the magnitude of an impulse at $t = 1$, and γ is the damping coefficient (or resistance).

G **a.** Let $\gamma = \dfrac{1}{2}$. Find the value of k for which the response has a peak value of 2; call this value k_1.

G **b.** Repeat part (a) for $\gamma = \dfrac{1}{4}$.

c. Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?

12. Consider the initial value problem

$$y'' + y = f_k(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f_k(t) = \dfrac{1}{2k}\left(u_{4-k}(t) - u_{4+k}(t)\right)$ with $0 < k \le 1$.

a. Find the solution $y = \phi(t, k)$ of the initial value problem.

b. Calculate $\lim\limits_{k\to 0^+} \phi(t,k)$ from the solution found in part a.

c. Observe that $\lim\limits_{k\to 0^+} f_k(t) = \delta(t-4)$. Find the solution $\phi_0(t)$ of the given initial value problem with $f_k(t)$ replaced by $\delta(t-4)$. Is it true that $\phi_0(t) = \lim\limits_{k\to 0^+} \phi(t,k)$?

G **d.** Plot $\phi\left(t, \frac{1}{2}\right)$, $\phi\left(t, \frac{1}{4}\right)$, and $\phi_0(t)$ on the same axes. Describe the relation between $\phi(t,k)$ and $\phi_0(t)$.

Problems 13 through 16 deal with the effect of a sequence of impulses on an undamped oscillator. Suppose that

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

For each of the following choices for $f(t)$:

a. Try to predict the nature of the solution without solving the problem.

G **b.** Test your prediction by finding the solution and plotting its graph.

c. Determine what happens after the sequence of impulses ends.

13. $f(t) = \sum\limits_{k=1}^{20} \delta(t - k\pi)$

14. $f(t) = \sum\limits_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi)$

15. $f(t) = \sum\limits_{k=1}^{15} \delta(t - (2k-1)\pi)$

16. $f(t) = \sum\limits_{k=1}^{40} (-1)^{k+1}\delta\left(t - \dfrac{11}{4}k\right)$

17. The position of a certain lightly damped oscillator satisfies the initial value problem

$$y'' + 0.1y' + y = \sum_{k=1}^{20} (-1)^{k+1}\delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 14.

a. Try to predict the nature of the solution without solving the problem.

G **b.** Test your prediction by finding the solution and drawing its graph.

c. Determine what happens after the sequence of impulses ends.

G **18.** Proceed as in Problem 17 for the oscillator satisfying

$$y'' + 0.1y' + y = \sum_{k=1}^{15} \delta(t - (2k-1)\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Observe that, except for the damping term, this problem is the same as Problem 15.

19. a. By the method of variation of parameters, show that the solution of the initial value problem

$$y'' + 2y' + 2y = f(t); \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau)\, d\tau.$$

b. Show that if $f(t) = \delta(t - \pi)$, then the solution of part a reduces to

$$y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi).$$

c. Use a Laplace transform to solve the given initial value problem with $f(t) = \delta(t-\pi)$, and confirm that the solution agrees with the result of part b.

6.6 The Convolution Integral

Sometimes it is possible to identify a Laplace transform $H(s)$ as the product of two other Laplace transforms $F(s)$ and $G(s)$, the latter transforms corresponding to known functions f and g, respectively. In this event, we might anticipate that $H(s)$ would be the transform of the product of f and g. However, this is not the case; in other words, the Laplace transform cannot be commuted with ordinary multiplication. On the other hand, if an appropriately defined "generalized product" is introduced, then the situation changes, as stated in the following theorem.

Theorem 6.6.1 | Convolution Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \ge 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau = \int_0^t f(\tau)g(t-\tau)\, d\tau. \tag{2}$$

The function h is known as the **convolution of** f **and** g; the integrals in equation (2) are called **convolution integrals**.

The equality of the two integrals in equation (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem, let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in equation (2); the second integral in equation (2) is denoted as $(g * f)(t)$.

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \quad \text{(commutative law)} \tag{4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \quad \text{(distributive law)} \tag{5}$$

$$(f * g) * h = f * (g * h) \quad \text{(associative law)} \tag{6}$$

$$f * 0 = 0 * f = 0. \quad \text{(zero property)} \tag{7}$$

In equation (7) the zeros denote not the number 0 but the function that has the value 0 for each value of t. The proofs of these properties are left to you as exercises.

However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t-\tau)\cdot 1\, d\tau = \int_0^t f(t-\tau)\, d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned}(f * 1)(t) &= \int_0^t \cos(t-\tau)\, d\tau = -\sin(t-\tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t.\end{aligned}$$

Clearly, $(f * 1)(t) \ne f(t)$ in this case. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

The initial value problem (15), (16) is often referred to as an input-output problem. The coefficients a, b, and c describe the properties of some physical system, and $g(t)$ is the input to the system. The values y_0 and y_0' describe the initial state, and the solution y is the output at time t.

Taking the Laplace transform of equation (20) and using initial conditions (21), we obtain

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_0' = G(s).$$

If we let

$$\Phi(s) = \frac{(as+b)y_0 + ay_0'}{as^2 + bs + c} \quad \text{and} \quad \Psi(s) = \frac{G(s)}{as^2 + bs + c}, \tag{22}$$

then we can write

$$Y(s) = \Phi(s) + \Psi(s). \tag{23}$$

Consequently,

$$y(t) = \phi(t) + \psi(t), \tag{24}$$

where $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$. Observe that $\phi(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_0', \tag{25}$$

obtained from equations (20) and (21) by setting $g(t)$ equal to zero. Similarly, $\psi(t)$ is the solution of

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{26}$$

in which the initial values y_0 and y_0' are each replaced by zero.

Once specific values of a, b, and c are given, we can use Table 6.2.1 to find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$, possibly in conjunction with a translation or a partial fraction expansion. To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, it is convenient to write $\Psi(s)$ as

$$\Psi(s) = H(s)G(s), \tag{27}$$

where $H(s) = (as^2 + bs + c)^{-1}$. The function H is known as the **transfer function**[6] and depends only on the properties of the system under consideration; that is, $H(s)$ is determined entirely by the coefficients a, b, and c. On the other hand, $G(s)$ depends only on the external excitation $g(t)$ that is applied to the system. By the Convolution Theorem (Theorem 6.6.1) we can write

$$\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)\,d\tau, \tag{28}$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function.

To obtain a better understanding of the significance of $h(t)$, we consider the case in which $G(s) = 1$; consequently, $g(t) = \delta(t)$ and $\Psi(s) = H(s)$. This means that $y = h(t)$ is the solution of the initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{29}$$

obtained from equation (26) by replacing $g(t)$ by $\delta(t)$. Thus $h(t)$ is the response of the system to a unit impulse applied at $t = 0$, and it is natural to call $h(t)$ the **impulse response** of the system. Equation (28) then says that $\psi(t)$ is the convolution of the impulse response and the forcing function.

Referring to Example 2, we note that the transfer function is $H(s) = 1/(s^2+4)$ and the impulse response is $h(t) = \frac{1}{2}\sin(2t)$. Also, the first two terms on the right-hand side of equation (19) constitute the function $\phi(t)$, the solution of the corresponding homogeneous equation that satisfies the given initial conditions.

[6] This terminology arises from the fact that $H(s)$ is the ratio of the transforms of the output and the input of the problem (20).

Problems

1. Prove the commutative, distributive, and associative properties of the convolution integral.

a. $f * g = g * f$
b. $f * (g_1 + g_2) = f * g_1 + f * g_2$
c. $f * (g * h) = (f * g) * h$

2. Find an example different from the one in the text showing that $(f * 1)(t)$ need not be equal to $f(t)$.

3. Show, by means of the example $f(t) = \sin t$, that $f * f$ is not necessarily nonnegative.

In each of Problems 4 through 6, find the Laplace transform of the given function.

4. $f(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\,d\tau$

5. $f(t) = \int_0^t e^{-(t-\tau)} \sin\tau\,d\tau$

6. $f(t) = \int_0^t \sin(t-\tau)\cos\tau\,d\tau$

In each of Problems 7 through 9, find the inverse Laplace transform of the given function by using the convolution theorem.

7. $F(s) = \dfrac{1}{s^4(s^2+1)}$

8. $F(s) = \dfrac{s}{(s+1)(s^2+4)}$

9. $F(s) = \dfrac{1}{(s+1)^2(s^2+4)}$

10. **a.** If $f(t) = t^m$ and $g(t) = t^n$, where m and n are positive integers, show that

$$f * g = t^{m+n+1}\int_0^1 u^m(1-u)^n\,du.$$

b. Use the convolution theorem to show that

$$\int_0^1 u^m(1-u)^n\,du = \frac{m!\,n!}{(m+n+1)!}.$$

c. Extend the result of part b to the case where m and n are positive numbers but not necessarily integers.

In each of Problems 11 through 15, express the solution of the given initial value problem in terms of a convolution integral.

11. $y'' + \omega^2 y = g(t); \quad y(0) = 0, \quad y'(0) = 1$

12. $4y'' + 4y' + 17y = g(t); \quad y(0) = 0, \quad y'(0) = 0$

13. $y'' + y' + \frac{5}{4}y = 1 - u_\pi(t); \quad y(0) = 1, \quad y'(0) = -1$

14. $y'' + 3y' + 2y = \cos(\alpha t); \quad y(0) = 1, \quad y'(0) = 0$

15. $y^{(4)} + 5y'' + 4y = g(t); \quad y(0) = 1, \quad y'(0) = 0,$ $y''(0) = 0, \quad y'''(0) = 0$

16. Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\,d\xi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as **Volterra integral equations**[7]. Take the Laplace transform of the given integral equation and obtain an expression for $\mathcal{L}\{\phi(t)\}$ in terms of the transforms $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{k(t)\}$ of the given functions f and k. The inverse transform of $\mathcal{L}\{\phi(t)\}$ is the solution of the original integral equation.

17. Consider the Volterra integral equation (see Problem 16)

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = \sin(2t). \tag{30}$$

a. Solve the integral equation (30) by using the Laplace transform.

b. By differentiating equation (30) twice, show that $\phi(t)$ satisfies the differential equation

$$\phi''(t) + \phi(t) = -4\sin(2t).$$

Show also that the initial conditions are

$$\phi(0) = 0, \quad \phi'(0) = 2.$$

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

In each of Problems 18 and 19:

a. Solve the given Volterra integral equation by using the Laplace transform.

b. Convert the integral equation into an initial value problem, as in Problem 17b.

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

18. $\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = 1$

19. $\phi(t) + 2\int_0^t \cos(t-\xi)\phi(\xi)\,d\xi = e^{-t}$

There are also equations, known as **integro-differential equations**, in which both derivatives and integrals of the unknown function appear. In each of Problems 20 and 21:

a. Solve the given integro-differential equation by using the Laplace transform.

b. By differentiating the integro-differential equation a sufficient number of times, convert it into an initial value problem.

c. Solve the initial value problem in part b, and verify that the solution is the same as the one in part a.

20. $\phi'(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi = t, \quad \phi(0) = 0$

21. $\phi'(t) - \frac{1}{2}\int_0^t (t-\xi)^2\phi(\xi)\,d\xi = -t, \quad \phi(0) = 1$

[7] See the footnote about **Vito Volterra** in Section 9.5.

22. The Tautochrone. A problem of interest in the history of mathematics is that of finding the **tautochrone**[8]—the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its starting point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens (1629–1695) in 1673 by geometric methods, and later by Leibniz and Jakob Bernoulli using analytic arguments. Bernoulli's solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved. The geometric configuration is shown in Figure 6.6.2. The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{1/2}. \tag{31}$$

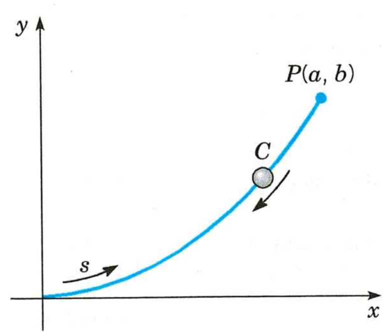

FIGURE 6.6.2 The tautochrone.

Then it follows from the **principle of conservation of energy** that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b-y}}\,dy. \tag{32}$$

a. Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of equation (32) in this case, and using the convolution theorem, Theorem 6.6.1, show that

$$F(s) = \sqrt{\frac{2g}{\pi}}\,\frac{T_0}{\sqrt{s}}; \tag{33}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi}\,\frac{T_0}{\sqrt{y}}. \tag{34}$$

Hint: See Problem 24 of Section 6.1.

b. Combining equations (32) and (34), show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}, \tag{35}$$

where $\alpha = gT_0^2/\pi^2$.

c. Use the substitution $y = 2\alpha \sin^2(\theta/2)$ to solve equation (35), and show that

$$x = \alpha(\theta + \sin\theta), \quad y = \alpha(1 - \cos\theta). \tag{36}$$

Equations (36) can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

[8] The word "tautochrone" comes from the Greek words *tauto*, which means "same," and *chronos*, which means "time."

References

The books listed below contain additional information on the Laplace transform and its applications.

Churchill, R. V., *Operational Mathematics* (3rd ed.) (New York: McGraw-Hill, 1971).

Doetsch, G., *Introduction to the Theory and Application of the Laplace Transform* (trans. W. Nader) (New York: Springer, 1974).

Kaplan, W., *Operational Methods for Linear Systems* (Reading, MA: Addison-Wesley, 1962).

Kuhfittig, P. K. F., *Introduction to the Laplace Transform* (New York: Plenum, 1978).

Miles, J. W., *Integral Transforms in Applied Mathematics* (Oxford: Cambridge University Press, 2008).

Rainville, E. D., *The Laplace Transform: An Introduction* (New York: Macmillan, 1963).

Each of the books just mentioned contains a table of transforms. Extensive tables are also available. See, for example,

Erdelyi, A. (ed.), *Tables of Integral Transforms* (Vol. 1) (New York: McGraw-Hill, 1954).

Roberts, G. E., and Kaufman, H., *Table of Laplace Transforms* (Philadelphia: Saunders, 1966).

A further discussion of generalized functions can be found in

Lighthill, M. J., *An Introduction to Fourier Analysis and Generalized Functions* (Cambridge, UK: Cambridge University Press, 1958).

CHAPTER 7

Systems of First-Order Linear Equations

Many physical problems involve a number of separate but interconnected components. For example, the current and voltage in an electrical network, each mass in a mechanical system, each element (or compound) in a chemical system, or each species in a biological system have this character. In these and similar cases, the corresponding mathematical problem consists of a *system* of two or more differential equations, which can always be written as first-order differential equations. In this chapter we focus on systems of first-order *linear* differential equations and, in particular, differential equations having constant coefficients, utilizing some of the elementary aspects of linear algebra to unify the presentation. In many respects this chapter follows the same lines as the treatment of second-order linear differential equations in Chapter 3.

7.1 Introduction

Systems of simultaneous ordinary differential equations arise naturally in problems involving several dependent variables, each of which is a function of the same single independent variable. We will denote the independent variable by t and will let $x_1, x_2, x_3, \ldots$ represent dependent variables that are functions of t. Differentiation[1] with respect to t will be denoted by, for example, $\dfrac{dx_1}{dt}$ or x_1'.

Let us begin by considering the spring–mass system in Figure 7.1.1. The two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constrained by the three springs whose constants are k_1, k_2, and k_3, respectively. We regard motion and displacement to the right as being positive.

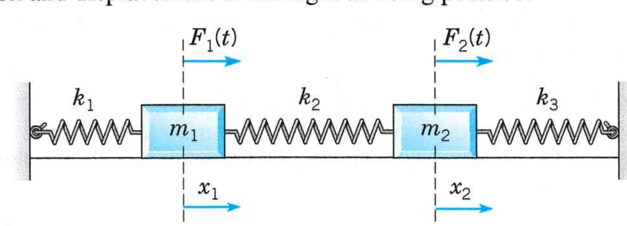

FIGURE 7.1.1 A two-mass, three-spring system.

Using arguments similar to those in Section 3.7, we find the following equations for the coordinates x_1 and x_2 of the two masses:

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= k_2(x_2 - x_1) - k_1x_1 + F_1(t) = -(k_1 + k_2)x_1 + k_2x_2 + F_1(t), \\ m_2\frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2(x_2 - x_1) + F_2(t) = k_2x_1 - (k_2 + k_3)x_2 + F_2(t). \end{aligned} \tag{1}$$

See Problem 14 for a full derivation of the system of differential equations (1).

Next, consider the parallel *LRC* circuit shown in Figure 7.1.2. Let V be the voltage drop across the capacitor and I the current through the inductor. Then, referring to Section 3.7 and

[1] In some treatments you will see differentiation with respect to time represented with a dot over the function, as in $\dot{x}_1 = \dfrac{dx_1}{dt}$ and $\ddot{x}_1 = \dfrac{d^2x_1}{dt^2}$. We reserve this notation for a specific purpose, which will be introduced in Section 9.6.