Problems

In each of Problems 1 through 4, use the method of variation of parameters to determine the general solution of the given differential equation.

1. $y''' + y' = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
2. $y''' - y' = t$
3. $y''' - 2y'' - y' + 2y = e^{4t}$
4. $y''' - y'' + y' - y = e^{-t}\sin t$

In each of Problems 5 and 6, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

5. $y''' - y'' + y' - y = \sec t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
6. $y''' - y' = \csc t, \quad 0 < t < \pi$

In each of Problems 7 and 8, find the solution of the given initial-value problem. Then plot a graph of the solution.

G 7. $y''' - y'' + y' - y = \sec t; \quad y(0) = 2, \ y'(0) = -1, \ y''(0) = 1$

G 8. $y''' - y' = \tan t; \quad y\left(\frac{\pi}{4}\right) = 2, \ y'\left(\frac{\pi}{4}\right) = 1, \ y''\left(\frac{\pi}{4}\right) = -1$

9. Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

10. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

11. Find a formula involving integrals for a particular solution of the differential equation

$$y^{(4)} - y = g(t).$$

Hint: The functions $\sin t$, $\cos t$, $\sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

12. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

References

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CHAPTER 5

Series Solutions of Second-Order Linear Equations

Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation. So far, we have given a systematic procedure for constructing fundamental solutions only when the equation has constant coefficients. To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: we assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

5.1 Review of Power Series

In this chapter we discuss the use of power series to construct fundamental sets of solutions of second-order linear differential equations whose coefficients are functions of the independent variable. We begin by summarizing very briefly the pertinent results about power series that we need. Readers who are familiar with power series may go on to Section 5.2. Those who need more details than are presented here should consult a book on calculus.

1. A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to *converge at a point x* if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x - x_0)^n$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and not for others.

2. The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to *converge absolutely at a point x* if the associated power series

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n$$

converges. It can be shown that if the power series converges absolutely, then the power series also converges; however, the converse is not necessarily true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test: If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0|L,$$

Next, shift the index down by 1 and start counting 1 higher. Thus

$$\sum_{n=0}^{\infty}(r+n)a_n x^{r+n+1} = \sum_{n=1}^{\infty}(r+n-1)a_{n-1}x^{r+n}. \tag{7}$$

Again, you can easily verify that the two series in equation (7) are identical and that both are exactly the same as the expression (5).

EXAMPLE 6

Assume that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \tag{8}$$

for all x, and determine what this implies about the coefficients a_n.

Solution:

We want to use statement 10 to equate corresponding coefficients in the two series. In order to do this, we must first rewrite equation (8) so that the series display the same power of x in their generic terms. For instance, in the series on the left-hand side of equation (8), we can replace n by $n+1$ and start counting 1 lower. Thus equation (8) becomes

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n. \tag{9}$$

According to statement 10, we conclude that

$$(n+1)a_{n+1} = a_n, \quad n = 0, 1, 2, 3, \ldots$$

or

$$a_{n+1} = \frac{a_n}{n+1}, \quad n = 0, 1, 2, 3, \ldots. \tag{10}$$

Hence, choosing successive values of n in equation (10), we have

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!},$$

and so forth. In general,

$$a_n = \frac{a_0}{n!}, \quad n = 1, 2, 3, \ldots. \tag{11}$$

Thus the relation (8) determines all the following coefficients in terms of a_0. Finally, using the coefficients given by equation (11), we obtain

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{a_0}{n!}x^n = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0 e^x,$$

where we have followed the usual convention that $0! = 1$, and recalled that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all values of x. (See Problem 8.)

Problems

In each of Problems 1 through 6, determine the radius of convergence of the given power series.

1. $\sum_{n=0}^{\infty}(x-3)^n$
2. $\sum_{n=0}^{\infty}\frac{n}{2^n}x^n$
3. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$
4. $\sum_{n=0}^{\infty}2^n x^n$
5. $\sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n}$
6. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2(x+2)^n}{3^n}$

In each of Problems 7 through 13, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

7. $\sin x, \quad x_0 = 0$
8. $e^x, \quad x_0 = 0$
9. $x, \quad x_0 = 1$
10. $x^2, \quad x_0 = -1$
11. $\ln x, \quad x_0 = 1$
12. $\frac{1}{1-x}, \quad x_0 = 0$
13. $\frac{1}{1-x}, \quad x_0 = 2$
14. Let $y = \sum_{n=0}^{\infty} n x^n$.
    a. Compute y' and write out the first four terms of the series.
    b. Compute y'' and write out the first four terms of the series.
15. Let $y = \sum_{n=0}^{\infty} a_n x^n$.
    a. Compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term.
    b. Show that if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1.
    c. Show that $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$, $n = 0, 1, 2, 3, \ldots$.

In each of Problems 16 and 17, verify the given equation.

16. $\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$
17. $\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1})x^k$

In each of Problems 18 through 22, rewrite the given expression as a single power series whose generic term involves x^n.

18. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$
19. $x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$
20. $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$
21. $\sum_{n=1}^{\infty} n a_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n$
22. $x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$
23. Determine the a_n so that the equation
$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$
is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

5.2 Series Solutions Near an Ordinary Point, Part I

In Chapter 3 we described methods of solving second-order linear differential equations with constant coefficients. We now consider methods of solving second-order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient a_n in terms of a_0 and a_1. What is essential is that we can determine *as many coefficients as we want.* Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package we can also produce plots such as those shown in the figures in this section.

Problems

In each of Problems 1 through 11:

a. Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation that the coefficients must satisfy.
b. Find the first four nonzero terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).
c. By evaluating the Wronskian $W[y_1, y_2](x_0)$, show that y_1 and y_2 form a fundamental set of solutions.
d. If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$
2. $y'' + 3y' = 0, \quad x_0 = 0$
3. $y'' - xy' - y = 0, \quad x_0 = 0$
4. $y'' - xy' - y = 0, \quad x_0 = 1$
5. $y'' + k^2x^2y = 0, \quad x_0 = 0,$ k a constant
6. $(1 - x)y'' + y = 0, \quad x_0 = 0$
7. $y'' + xy' + 2y = 0, \quad x_0 = 0$
8. $xy'' + y' + xy = 0, \quad x_0 = 1$
9. $(3 - x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$
10. $2y'' + xy' + 3y = 0, \quad x_0 = 0$
11. $2y'' + (x + 1)y' + 3y = 0, \quad x_0 = 2$

In each of Problems 12 through 14:

a. Find the first five nonzero terms in the solution of the given initial-value problem.
G **b.** Plot the four-term and the five-term approximations to the solution on the same axes.
c. From the plot in part b, estimate the interval in which the four-term approximation is reasonably accurate.

12. $y'' - xy' - y = 0, \quad y(0) = 2, \ y'(0) = 1;$ see Problem 3
13. $y'' + xy' + 2y = 0, \quad y(0) = 4, \ y'(0) = -1;$ see Problem 7
14. $(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, \ y'(0) = 2$
15. **a.** By making the change of variable $x - 1 = t$ and assuming that y has a Taylor series in powers of t, find two series solutions of
$$y'' + (x - 1)^2y' + (x^2 - 1)y = 0$$
in powers of $x - 1$.
b. Show that you obtain the same result by assuming that y has a Taylor series in powers of $x - 1$ and also expressing the coefficient $x^2 - 1$ in powers of $x - 1$.
16. Prove equation (10).
17. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converge for all x; see equation (20) of the text.
18. **The Hermite Equation.** The equation
$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty,$$
where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.
a. Find the first four nonzero terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.
b. Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6,$ 8, and 10. Note that each polynomial is determined only up to a multiplicative constant.
c. The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), H_1(x), \ldots, H_5(x)$.
19. Consider the initial-value problem $y' = \sqrt{1 - y^2}$, $y(0) = 0$.
a. Show that $y = \sin x$ is the solution of this initial-value problem.
b. Look for a solution of the initial-value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

In each of Problems 20 through 23, plot several partial sums in a series solution of the given initial-value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4 (except that we do not know an explicit formula for the actual solution).

G 20. $y'' + xy' + 2y = 0, \quad y(0) = 0, \ y'(0) = 1;$ see Problem 7
G 21. $(4 - x^2)y'' + 2y = 0, \quad y(0) = 0, \ y'(0) = 1$
G 22. $y'' + x^2y = 0, \quad y(0) = 1, \ y'(0) = 0;$ see Problem 5
G 23. $(1 - x)y'' + xy' - 2y = 0, \quad y(0) = 0, \ y'(0) = 1$

[5]Charles Hermite (1822–1901) was an influential French analyst and algebraist. An inspiring teacher, he was professor at the École Polytechnique and the Sorbonne. He introduced the Hermite functions in 1864 and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

5.3 Series Solutions Near an Ordinary Point, Part II

In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that equation (1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$\phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{2}$$

that converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the a_n can be determined by directly substituting the series (2) for y in equation (1).

Let us now consider how we might justify the statement that if x_0 is an ordinary point of equation (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this, we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of equation (1) of the form (2). By differentiating equation (2) m times and setting x equal to x_0, we obtain

$$m!a_m = \phi^{(m)}(x_0). \tag{3}$$

Hence, to compute a_n in the series (2), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$ from the differential equation (1).

Suppose that $y = \phi(x)$ is a solution of equation (1) satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$. Then $a_0 = y_0$ and $a_1 = y_0'$. If we are solely interested in finding a solution of equation (1) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \ldots,$ we turn to equation (1) with the goal of finding a formula for $\phi''(x), \phi'''(x), \ldots$. Since ϕ is a solution of equation (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which P is nonzero, we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{4}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Observe that, at $x = x_0$, the right-hand side of equation (4) is known, thus allowing us to compute $\phi''(x_0)$: Setting x equal to x_0 in equation (4) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Hence, using equation (3) with $m = 2$, we find that a_2 is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{5}$$

To determine a_3, we differentiate equation (4) and then set x equal to x_0, obtaining

$$\begin{aligned} 3!a_3 = \phi'''(x_0) &= -\big(p(x)\phi'(x) + q(x)\phi(x)\big)'\big|_{x=x_0} \\ &= -2!p(x_0)a_2 - \big(p'(x_0) + q(x_0)\big)a_1 - q'(x_0)a_0. \end{aligned} \tag{6}$$

Substituting for a_2 from equation (5) gives a_3 in terms of a_1 and a_0.

EXAMPLE 3

What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? about $x = 1$?

Solution:

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions $x = 1 \pm i$. The distance in the complex plane from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ is $\sqrt{2}$; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$.

The distance in the complex plane from $x = 1$ to either $x = 1 + i$ or $x = 1 - i$ is 1; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n (x-1)^n$ about $x = 1$ is 1.

According to Theorem 5.3.1, the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of x and $x - 1$, respectively, since in each problem $P(x) = 1$ and hence is never zero.

A series solution may converge for a wider range of x than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

EXAMPLE 4

Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

Solution:

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of P, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and possibly for larger values of x. Indeed, it can be shown that if α is a positive integer, one of the series solutions terminates after a finite number of terms, that is, one solution is a polynomial, and hence converges not just for $|x| < 1$ but for all x. For example, if $\alpha = 1$, the polynomial solution is $y = x$. See Problems 17 through 23 at the end of this section for a further discussion of the Legendre equation.

EXAMPLE 5

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0 \qquad (10)$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

Solution:

Again P, Q, and R are polynomials, and P has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n \left(x + \frac{1}{2}\right)^n$ converges at least for $\left|x + \frac{1}{2}\right| < \frac{\sqrt{5}}{2}$.

An interesting observation that we can make about equation (10) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y'_0$ are given. Since $1 + x^2 \neq 0$ for all x, we know from Theorem 3.2.1 that there exists a unique solution of the initial-value problem on $-\infty < x < \infty$. On the other hand, Theorem 5.3.1 only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0$, $a_1 = y'_0$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all x.

EXAMPLE 6

Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

Solution:

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x. Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x. Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x.

Problems

In each of Problems 1 through 3, determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{(4)}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial-value problem.

1. $y'' + xy' + y = 0; \quad y(0) = 1, \quad y'(0) = 0$
2. $x^2 y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0$
3. $y'' + x^2 y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$

In each of Problems 4 through 6, determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

4. $y'' + 4y' + 6xy = 0; \quad x_0 = 0, \quad x_0 = 4$
5. $(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4, \quad x_0 = 0$
6. $(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$
7. Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 11 of Section 5.2.
8. **The Chebyshev Equation.** The Chebyshev[7] differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

a. Determine two solutions in powers of x for $|x| < 1$, and show that they form a fundamental set of solutions.

b. Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the **Chebyshev polynomials**. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \le x \le 1$.

c. Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2, 3$.

For each of the differential equations in Problems 9 through 11, find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

9. $y'' + (\sin x)y = 0$
10. $e^x y'' + xy = 0$
11. $(\cos x)y'' + xy' - 2y = 0$
12. Let $y = x$ and $y = x^2$ be solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point? Prove your answer.

First-Order Equations. The series methods discussed in this section are directly applicable to the first-order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 13 through 16, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problem 17 involves a nonhomogeneous differential equation to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

[7] Pafnuty L. Chebyshev (1821–1894), the most influential nineteenth-century Russian mathematician, was for 35 years professor at the University of St. Petersburg, which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began in about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

13. $y' - y = 0$

14. $y' - xy = 0$

15. $(1 - x)y' = y$

16. $y' - y = x^2$

The Legendre Equation. Problems 17 through 23 deal with the Legendre[8] equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 4, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1 + \gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

17. Show that two solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4$$
$$+ \sum_{m=3}^{\infty}(-1)^m \frac{\alpha \cdots (\alpha - 2m + 2)(\alpha + 1) \cdots (\alpha + 2m - 1)}{(2m)!}x^{2m},$$

$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3$$
$$+ \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5$$
$$+ \sum_{m=3}^{\infty}(-1)^m$$
$$\times \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}.$$

18. Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0, 2$, and 4. Show that if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1, 3$, and 5.

19. The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

a. Using the results of Problem 18, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.

G **b.** Plot the graphs of $P_0(x), \ldots, P_5(x)$ for $-1 \le x \le 1$.

N **c.** Find the zeros of $P_0(x), \ldots, P_5(x)$.

20. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi\frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \quad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

21. Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

This formula, known as Rodrigues's formula,[9] is true for all positive integers n.

22. Show that the Legendre equation can also be written as

$$\left((1 - x^2)y'\right)' = -\alpha(\alpha + 1)y.$$

Then it follows that

$$\left((1 - x^2)P_n'(x)\right)' = -n(n + 1)P_n(x)$$

and

$$\left((1 - x^2)P_m'(x)\right)' = -m(m + 1)P_m(x).$$

By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \ \text{ if } n \ne m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the preceding integral is $2/(2n + 1)$.

23. Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, P_2, \ldots, P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of Problem 22, show that

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx.$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

[9] Benjamin Olinde Rodrigues (1795–1851) published this result as part of his doctoral thesis from the University of Paris in 1815. He then became a banker and social reformer but retained an interest in mathematics. Unfortunately, his later papers were not appreciated until the late twentieth century.

5.4 Euler Equations; Regular Singular Points

In this section we will begin to consider how to solve equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a singular point x_0. Recall that if the functions P, Q, and R are polynomials having no factors common to all three of them, then the singular points of equation (1) are the points for which $P(x) = 0$.

Euler Equations. A relatively simple differential equation that has a singular point is the **Euler equation**[10]

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0, \tag{2}$$

where α and β are real constants. Then $P(x) = x^2$, $Q(x) = \alpha x$, and $R(x) = \beta$. If $\beta \ne 0$, then $P(x)$, $Q(x)$, and $R(x)$ have no common factors, so the only singular point of equation (2) is $x = 0$; all other points are ordinary points. For convenience we first consider the interval $x > 0$; later we extend our results to the interval $x < 0$.

Observe that $(x^r)' = rx^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$. Hence, if we assume that equation (2) has a solution of the form

$$y = x^r, \tag{3}$$

then we obtain

$$\begin{aligned} L[x^r] &= x^2(x^r)'' + \alpha x(x^r)' + \beta x^r \\ &= x^2r(r-1)x^{r-2} + \alpha x(rx^{r-1}) + \beta x^r \\ &= x^r(r(r-1) + \alpha r + \beta). \end{aligned} \tag{4}$$

If r is a root of the quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0, \tag{5}$$

then $L[x^r]$ is zero, and $y = x^r$ is a solution of equation (2). The roots of equation (5) are

$$r_1, r_2 = \frac{-(\alpha - 1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}, \tag{6}$$

and the quadratic polynomial $F(r)$ defined in equation (5) can also be written as $F(r) = (r - r_1)(r - r_2)$. Mirroring the treatment of second-order linear differential equations with constant coefficients, we consider separately the cases in which the roots are real and different, real but equal, and complex conjugates. Indeed, the entire discussion of Euler equations is similar to the treatment of second-order linear equations with constant coefficients in Chapter 3, with e^{rx} replaced by x^r.

Real, Distinct Roots. If $F(r) = 0$ has real roots r_1 and r_2, with $r_1 \ne r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions of equation (2). Since

$$W[x^{r_1}, x^{r_2}] = (r_2 - r_1)x^{r_1+r_2-1}$$

is nonzero for $r_1 \ne r_2$ and $x > 0$, it follows that the general solution of equation (2) is

$$y = c_1x^{r_1} + c_2x^{r_2}, \quad x > 0. \tag{7}$$

Note that if r is not a rational number, then x^r is defined by $x^r = e^{r\ln x}$.

EXAMPLE 1

Solve

$$2x^2y'' + 3xy' - y = 0, \quad x > 0. \tag{8}$$

[10] This equation is sometimes called the Cauchy–Euler equation or the equidimensional equation. Euler studied it in about 1740, but its solution was known to Johann Bernoulli before 1700.

and

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{2x(x-2)} = 0.$$

Since these limits are finite, $x = 0$ is a regular singular point.

For $x = 2$ we have

$$\lim_{x\to 2}(x-2)\,p(x) = \lim_{x\to 2}(x-2)\frac{3}{2(x-2)^2} = \lim_{x\to 2}\frac{3}{2(x-2)},$$

so the limit does not exist; hence $x = 2$ is an irregular singular point.

EXAMPLE 6

Determine the singular points of

$$\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)\,y' + (\sin x)\,y = 0$$

and classify them as regular or irregular.

Solution:

The only singular point is $x = \frac{\pi}{2}$. To study it, we consider the functions

$$\left(x - \frac{\pi}{2}\right)p(x) = \left(x - \frac{\pi}{2}\right)\frac{Q(x)}{P(x)} = \frac{\cos x}{x - \pi/2}$$

and

$$\left(x - \frac{\pi}{2}\right)^2 q(x) = \left(x - \frac{\pi}{2}\right)^2 \frac{R(x)}{P(x)} = \sin x.$$

Starting from the Taylor series for $\cos x$ about $x = \frac{\pi}{2}$, we find that

$$\frac{\cos x}{x - \pi/2} = -1 + \frac{(x-\pi/2)^2}{3!} - \frac{(x-\pi/2)^4}{5!} + \cdots,$$

which converges for all x. Similarly, $\sin x$ is analytic at $x = \frac{\pi}{2}$. Therefore, we conclude that $\frac{\pi}{2}$ is a regular singular point for this equation.

Problems

In each of Problems 1 through 8, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' - xy' + y = 0$
5. $x^2y'' + 6xy' - y = 0$
6. $2x^2y'' - 4xy' + 6y = 0$
7. $x^2y'' - 5xy' + 9y = 0$
8. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$

In each of Problems 9 through 11, find the solution of the given initial-value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

G 9. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$

G 10. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$

G 11. $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$

In each of Problems 12 through 23, find all singular points of the given equation and determine whether each one is regular or irregular.

12. $xy'' + (1-x)y' + xy = 0$
13. $x^2(1-x)^2y'' + 2xy' + 4y = 0$

14. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$
15. $x^2(1-x^2)y'' + \left(\frac{2}{x}\right)y' + 4y = 0$
16. $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$
17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (Bessel equation)
18. $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$
19. $x(3-x)y'' + (x+1)y' - 2y = 0$
20. $xy'' + e^xy' + (3\cos x)y = 0$
21. $y'' + (\ln|x|)y' + 3xy = 0$
22. $(\sin x)y'' + xy' + 4y = 0$
23. $(x\sin x)y'' + 3y' + xy = 0$
24. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + \frac{5}{2}y = 0$ approach zero as $x \to 0$.
25. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.
26. Find γ so that the solution of the initial-value problem $x^2y'' - 2y = 0$, $y(1) = 1$, $y'(1) = \gamma$ is bounded as $x \to 0$.
27. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that:
    a. All solutions approach zero as $x \to 0$.
    b. All solutions are bounded as $x \to 0$.
    c. All solutions approach zero as $x \to \infty$.
    d. All solutions are bounded as $x \to \infty$.
    e. All solutions are bounded both as $x \to 0$ and as $x \to \infty$.
28. Using the method of reduction of order, show that if r_1 is a repeated root of
$$r(r-1) + \alpha r + \beta = 0,$$
then x^{r_1} and $x^{r_1}\ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.
29. Verify that $W[x^\lambda\cos(\mu\ln x), x^\lambda\sin(\mu\ln x)] = \mu x^{2\lambda-1}$.

In each of Problems 30 and 31, show that the point $x = 0$ is a regular singular point. In each problem try to find solutions of the form $\sum_{n=0}^{\infty} a_nx^n$. Show that (except for constant multiples) there is only one nonzero solution of this form in Problem 30 and that there are no nonzero solutions of this form in Problem 31. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

30. $2xy'' + 3y' + xy = 0$
31. $2x^2y'' + 3xy' - (1+x)y = 0$
32. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point x_0 is finite. In more advanced work in differential equations, it is often necessary to consider the point at infinity. This is done by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$. Show that, for the differential equation
$$P(x)y'' + Q(x)y' + R(x)y = 0,$$
the point at infinity is an ordinary point if
$$\frac{1}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4P(1/\xi)}$$
have Taylor series expansions about $\xi = 0$. Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both
$$\frac{\xi}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2P(1/\xi)}$$
do have such expansions.

In each of Problems 33 through 37, use the results of Problem 32 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

33. $y'' + y = 0$
34. $x^2y'' + xy' - 4y = 0$
35. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ (Legendre equation)
36. $y'' - 2xy' + \lambda y = 0$ (Hermite equation)
37. $y'' - xy = 0$ (Airy equation)

5.5 Series Solutions Near a Regular Singular Point, Part I

We now consider the question of solving the general second-order linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point $x = x_0$. For convenience we assume that $x_0 = 0$. If $x_0 \neq 0$, the equation can be transformed into one for which the regular singular point is at the origin by letting $x - x_0$ equal t.

The assumption that $x = 0$ is a regular singular point of equation (1) means that $xQ(x)/P(x) = xp(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0$ and are analytic at $x = 0$. Thus they have convergent power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_nx^n, \tag{2}$$

For each root r_1 and r_2 of the indicial equation, we use the recurrence relation (17) to determine a set of coefficients $a_1, a_2, \ldots$. For $r = r_1 = 1$, equation (17) becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n}, \quad n \ge 1.$$

Thus

$$a_1 = -\frac{a_0}{3 \cdot 1},$$

$$a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)},$$

and

$$a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}.$$

In general, we have

$$a_n = \frac{(-1)^n}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!} a_0, \quad n \ge 4. \tag{18}$$

If we multiply both the numerator and denominator of the right-hand side of equation (18) by $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n = 2^n n!$, we can rewrite a_n as

$$a_n = \frac{(-1)^n 2^n}{(2n+1)!} a_0, \quad n \ge 1.$$

Hence, if we omit the constant multiplier a_0, one solution of equation (8) is

$$y_1(x) = x\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right), \quad x > 0. \tag{19}$$

To determine the radius of convergence of the series in equation (19), we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$$

for all x. Thus the series converges for all x.

Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly. From equation (17) we have

$$a_n = -\frac{a_{n-1}}{2n\left(n - \frac{1}{2}\right)} = -\frac{a_{n-1}}{n(2n-1)}, \quad n \ge 1.$$

Hence

$$a_1 = -\frac{a_0}{1 \cdot 1},$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)},$$

$$a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)},$$

and, in general,

$$a_n = \frac{(-1)^n}{n!(1 \cdot 3 \cdot 5 \cdots (2n-1))} a_0, \quad n \ge 4. \tag{20}$$

Just as in the case of the first root r_1, we multiply the numerator and denominator by $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n = 2^n n!$. Then we have

$$a_n = \frac{(-1)^n 2^n}{(2n)!} a_0, \quad n \ge 1.$$

Again omitting the constant multiplier a_0, we obtain the second solution

$$y_2(x) = x^{1/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right), \quad x > 0. \tag{21}$$

As before, we can show that the series in equation (21) converges for all x. Since y_1 and y_2 behave like x and $x^{1/2}$, respectively, near $x = 0$, they are linearly independent and so they form a fundamental set of solutions. Hence the general solution of equation (8) is

$$y = c_1 y_1(x) + c_2 y_2(x), \quad x > 0.$$

The preceding example illustrates that if $x = 0$ is a regular singular point, then sometimes there are two solutions of the form (7) in the neighborhood of this point. Similarly, if there is a regular singular point at $x = x_0$, then there may be two solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{22}$$

that are valid near $x = x_0$. However, just as an Euler equation may not have two solutions of the form $y = x^r$, so a more general equation with a regular singular point may not have two solutions of the form (7) or (22). In particular, we show in the next section that if the roots r_1 and r_2 of the indicial equation are equal or differ by an integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22); if r_1 and r_2 differ by an integer, this solution corresponds to the larger value of r. If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.6 and 5.7.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of x. However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If P, Q, and R are polynomials, it is often much better to work directly with equation (1) than with equation (3). This avoids the necessity of expressing $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ as power series. For example, it is more convenient to consider the equation

$$x(1+x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2 y'' + \frac{2x}{1+x} y' + \frac{x^2}{1+x} y = 0,$$

which would entail expanding $\frac{2x}{1+x}$ and $\frac{x^2}{1+x}$ in power series.

Problems

In each of Problems 1 through 6:

a. Show that the given differential equation has a regular singular point at $x = 0$.

b. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation.

c. Find the series solution ($x > 0$) corresponding to the larger root.

d. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$
2. $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$
3. $xy'' + y = 0$
4. $xy'' + y' - y = 0$
5. $x^2y'' + xy' + (x-2)y = 0$
6. $xy'' + (1-x)y' - y = 0$

7. The Legendre equation of order α is

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

The solution of this equation near the ordinary point $x = 0$ was discussed in Problems 17 and 18 of Section 5.3. In Example 4 of Section 5.4, it was shown that $x = \pm 1$ are regular singular points.

a. Determine the indicial equation and its roots for the point $x = 1$.

b. Find a series solution in powers of $x - 1$ for $x - 1 > 0$. *Hint:* Write $1 + x = 2 + (x - 1)$ and $x = 1 + (x - 1)$. Alternatively, make the change of variable $x - 1 = t$ and determine a series solution in powers of t.

8. The Chebyshev equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant; see Problem 8 of Section 5.3.

a. Show that $x = 1$ and $x = -1$ are regular singular points, and find the exponents at each of these singularities.

b. Find two solutions about $x = 1$.

9. The Laguerre[13] differential equation is

$$xy'' + (1-x)y' + \lambda y = 0.$$

a. Show that $x = 0$ is a regular singular point.

b. Determine the indicial equation, its roots, and the recurrence relation.

c. Find one solution (for $x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized, this polynomial is known as the **Laguerre polynomial**, $L_m(x)$.

10. The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0.$$

a. Show that $x = 0$ is a regular singular point.

b. Show that the roots of the indicial equation are $r_1 = r_2 = 0$.

c. Show that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

The function J_0 is known as the **Bessel function of the first kind of order zero**.

d. Show that the series for $J_0(x)$ converges for all x.

11. Referring to Problem 10, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.
Hint: If $y_2(x) = J_0(x)v(x)$, then

$$y_2(x) = J_0(x)\int \frac{dx}{x\big(J_0(x)\big)^2}.$$

Find the first term in the series expansion of $\dfrac{1}{x\big(J_0(x)\big)^2}$.

12. The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

a. Show that $x = 0$ is a regular singular point.

b. Show that the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$.

c. Show that one solution for $x > 0$ is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!\,n!\,2^{2n}}.$$

The function J_1 is known as the **Bessel function of the first kind of order one**.

d. Show that the series for $J_1(x)$ converges for all x.

e. Show that it is impossible to determine a second solution of the form

$$x^{-1}\sum_{n=0}^{\infty} b_n x^n, \quad x > 0.$$

[13]Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879. He is also known for an algorithm for calculating roots of polynomial equations.

5.6 Series Solutions Near a Regular Singular Point, Part II

Now let us consider the general problem of determining a solution of the equation

$$L[y] = x^2y'' + x(xp(x))y' + \big(x^2q(x)\big)y = 0, \tag{1}$$

where

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

and both series converge in an interval $|x| < \rho$ for some $\rho > 0$. The point $x = 0$ is a regular singular point, and the corresponding Euler equation is

$$x^2y'' + p_0xy' + q_0y = 0. \tag{3}$$

We seek a solution of equation (1) for $x > 0$ and assume that it has the form

$$y = \phi(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \tag{4}$$

where $a_0 \neq 0$, and we have written $y = \phi(r, x)$ to emphasize that ϕ depends on r as well as x. It follows that

$$y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2}. \tag{5}$$

Then, substituting from equations (2), (4), and (5) in equation (1) gives

$$\begin{aligned}
L[\phi](r,x) &= a_0r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots + a_n(r+n)(r+n-1)x^{r+n} + \cdots \\
&\quad + (p_0 + p_1x + \cdots + p_nx^n + \cdots)\big(a_0rx^r + a_1(r+1)x^{r+1} + \cdots + a_n(r+n)x^{r+n} + \cdots\big) \\
&\quad + (q_0 + q_1x + \cdots + q_nx^n + \cdots)\big(a_0x^r + a_1x^{r+1} + \cdots + a_nx^{r+n} + \cdots\big) \\
&= 0.
\end{aligned}$$

Multiplying the infinite series together and then collecting terms, we obtain

$$\begin{aligned}
L[\phi](r,x) &= a_0F(r)x^r + \big[a_1F(r+1) + a_0(p_1r + q_1)\big]x^{r+1} \\
&\quad + \big[a_2F(r+2) + a_0(p_2r + q_2) + a_1\big(p_1(r+1) + q_1\big)\big]x^{r+2} \\
&\quad + \cdots + \big[a_nF(r+n) + a_0(p_nr + q_n) + a_1\big(p_{n-1}(r+1) + q_{n-1}\big) \\
&\qquad + \cdots + a_{n-1}\big(p_1(r+n-1) + q_1\big)\big]x^{r+n} + \cdots = 0,
\end{aligned}$$

or, in a more compact form,

$$\begin{aligned}
L[\phi] &= a_0F(r)x^r \\
&\quad + \sum_{n=1}^{\infty}\left(F(r+n)a_n + \sum_{k=0}^{n-1} a_k\big((r+k)p_{n-k} + q_{n-k}\big)\right)x^{r+n} = 0,
\end{aligned} \tag{6}$$

where

$$F(r) = r(r-1) + p_0r + q_0. \tag{7}$$

For equation (6) to be satisfied for all $x > 0$, the coefficient of each power of x must be zero.

Since $a_0 \neq 0$, the term involving x^r yields the equation $F(r) = 0$. This equation is called the *indicial equation*; note that it is exactly the equation we would obtain in looking for solutions $y = x^r$ of the Euler equation (3). Let us denote the roots of the indicial equation by r_1 and r_2 with $r_1 \geq r_2$ if the roots are real. If the roots are complex, the designation of the roots is immaterial. Only for these values of r can we expect to find solutions of equation (1) of the form (4). The roots r_1 and r_2 are called the *exponents at the singularity*; they determine the qualitative nature of the solution in the neighborhood of the singular point.

Setting the coefficient of x^{r+n} in equation (6) equal to zero gives the **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k\big((r+k)p_{n-k} + q_{n-k}\big) = 0, \quad n \geq 1. \tag{8}$$

Equation (8) shows that, in general, a_n depends on the value of r and all the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$. It also shows that we can successively compute $a_1, a_2, \ldots, a_n, \ldots$ in terms of a_0 and the coefficients in the series for $xp(x)$ and $x^2q(x)$, provided that $F(r+1), F(r+2), \ldots, F(r+n), \ldots$ are not zero. The only values of r for which $F(r) = 0$ are $r = r_1$ and $r = r_2$; since $r_1 \geq r_2$, it follows that $r_1 + n$ is not equal to r_1 or r_2 for $n \geq 1$. Consequently, $F(r_1 + n) \neq 0$ for $n \geq 1$. Hence we can always determine one solution of equation (1) in the form (4), namely,

$$y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right), \quad x > 0. \tag{9}$$

Setting $r = r_1$ in equation (15), we find that $L[\phi](r_1, x) = 0$; hence, as we already know, $y_1(x)$ given by equation (9) is one solution of equation (1). But more important, it also follows from equation (15), just as for the Euler equation, that

$$L\left[\frac{\partial \phi}{\partial r}\right](r_1, x) = a_0 \frac{\partial}{\partial r}\left(x^r (r - r_1)^2\right)\Big|_{r=r_1}$$
$$= a_0\left((r - r_1)^2 x^r \ln x + 2(r - r_1)x^r\right)\Big|_{r=r_1} = 0. \qquad (16)$$

Hence, a second solution of equation (1) is

$$y_2(x) = \frac{\partial \phi(r, x)}{\partial r}\Big|_{r=r_1} = \frac{\partial}{\partial r}\left(x^r\left(a_0 + \sum_{n=1}^{\infty} a_n(r)x^n\right)\right)\Big|_{r=r_1}$$
$$= (x^{r_1} \ln x)\left(a_0 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right) + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n$$
$$= y_1(x) \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n, \quad x > 0, \qquad (17)$$

where $a_n'(r_1)$ denotes $\dfrac{da_n}{dr}$ evaluated at $r = r_1$.

Although equation (17) provides an explicit expression for a second solution $y_2(x)$, it may turn out that it is difficult to determine $a_n(r)$ as a function of r from the recurrence relation (8) and then to differentiate the resulting expression with respect to r. An alternative is simply to assume that y has the *form* of equation (17). That is, assume that

$$y = y_1(x) \ln x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n, \quad x > 0, \qquad (18)$$

where $y_1(x)$ has already been found. The coefficients b_n are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of x equal to zero. A third possibility is to use the method of reduction of order to find $y_2(x)$ once $y_1(x)$ is known.

Roots r_1 and r_2 Differing by an Integer N. For this case the derivation of the second solution is considerably more complicated and will not be given here. The form of this solution is stated in equation (24) in the following theorem. The coefficients $c_n(r_2)$ in equation (24) are given by

$$c_n(r_2) = \frac{d}{dr}[(r - r_2)a_n(r)]\Big|_{r=r_2}, \quad n = 1, 2, \ldots, \qquad (19)$$

where $a_n(r)$ is determined from the recurrence relation (8) with $a_0 = 1$. Further, the coefficient a in equation (24) is

$$a = \lim_{r \to r_2}(r - r_2)a_N(r). \qquad (20)$$

If $a_N(r_2)$ is finite, then $a = 0$ and there is no logarithmic term in y_2. A full derivation of formulas (19) and (20) may be found in Coddington (Chapter 4).

In practice, the best way to determine whether a is zero in the second solution is simply to try to compute the a_n corresponding to the root r_2 and to see whether it is possible to determine $a_N(r_2)$. If so, there is no further problem. If not, we must use the form (24) with $a \neq 0$.

When $r_1 - r_2 = N$, there are again three ways to find a second solution. First, we can calculate a and $c_n(r_2)$ directly by substituting the expression (24) for y in equation (1). Second, we can calculate $c_n(r_2)$ and a of equation (24) using the formulas (19) and (20). If this is the planned procedure, then in calculating the solution corresponding to $r = r_1$, be sure to obtain the general formula for $a_n(r)$ rather than just $a_n(r_1)$. The third alternative is to use the method of reduction of order.

The following theorem summarizes the results that we have obtained in this section.

Theorem 5.6.1

Consider the differential equation (1)

$$x^2 y'' + x(xp(x))y' + (x^2 q(x))y = 0,$$

where $x = 0$ is a regular singular point. Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r - 1) + p_0 r + q_0 = 0,$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right), \qquad (21)$$

where the $a_n(r_1)$ are given by the recurrence relation (8) with $a_0 = 1$ and $r = r_1$.

CASE 1 If $r_1 - r_2$ is not zero or a positive integer, then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form

$$y_2(x) = |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right). \qquad (22)$$

The $a_n(r_2)$ are also determined by the recurrence relation (8) with $a_0 = 1$ and $r = r_2$. The power series in equations (21) and (22) converge at least for $|x| < \rho$.

CASE 2 If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x) \ln |x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n. \qquad (23)$$

CASE 3 If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = a y_1(x) \ln |x| + |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right). \qquad (24)$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, and $c_n(r_2)$ and the constant a can be determined by substituting the form of the series solutions for y in equation (1). The constant a may turn out to be zero, in which case there is no logarithmic term in the solution (24). Each of the series in equations (23) and (24) converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$.

In all three cases, the two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the given differential equation.

Problems

In each of Problems 1 through 8:

a. Find all the regular singular points of the given differential equation.

b. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$
2. $x^2 y'' - x(2 + x)y' + (2 + x^2)y = 0$
3. $y'' + 4xy' + 6y = 0$
4. $2x(x + 2)y'' + y' - xy = 0$
5. $x^2 y'' + \frac{1}{2}(x + \sin x)y' + y = 0$
6. $x^2(1 - x)y'' - (1 + x)y' + 2xy = 0$
7. $(x - 2)^2(x + 2)y'' + 2xy' + 3(x - 2)y = 0$
8. $(4 - x^2)y'' + 2xy' + 3y = 0$

In each of Problems 9 through 12:

a. Show that $x = 0$ is a regular singular point of the given differential equation.

b. Find the exponents at the singular point $x = 0$.

c. Find the first three nonzero terms in each of two solutions (not multiples of each other) about $x = 0$.

9. $xy'' + y' - y = 0$
10. $xy'' + 2xy' + 6e^x y = 0$ (see Problem 1)

11. $xy'' + y = 0$

12. $x^2y'' + (\sin x)y' - (\cos x)y = 0$

13. **a.** Show that

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$.

b. Determine the roots of the indicial equation at $x = 1$.

c. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{r+n}$ corresponding to the larger root. You can assume $x - 1 > 0$.

d. What would you expect the radius of convergence of the series to be?

14. In several problems in mathematical physics, it is necessary to study the differential equation

$$x(1-x)y'' + (\gamma - (1 + \alpha + \beta)x)y' - \alpha\beta y = 0, \qquad (25)$$

where α, β, and γ are constants. This equation is known as the **hypergeometric equation**.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are 0 and $1 - \gamma$.

b. Show that $x = 1$ is a regular singular point and that the roots of the indicial equation are 0 and $\gamma - \alpha - \beta$.

c. Assuming that $1 - \gamma$ is not a positive integer, show that, in the neighborhood of $x = 0$, one solution of equation (25) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots.$$

What would you expect the radius of convergence of this series to be?

d. Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left(1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right).$$

e. Show that the point at infinity is a regular singular point and that the roots of the indicial equation are α and β. See Problem 32 of Section 5.4.

15. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0,$$

where α and β are real constants and $\alpha \neq 0$.

a. Show that $x = 0$ is an irregular singular point.

b. By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear and that, consequently, there is only one formal solution of the assumed form.

c. Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of β/α, show that the formal series solution has a zero radius of convergence and so does not represent an actual solution in any interval.

16. Consider the differential equation

$$y'' + \frac{\alpha}{x^s}y' + \frac{\beta}{x^t}y = 0, \qquad (26)$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and s and t are positive integers that for the moment are arbitrary.

a. Show that if $s > 1$ or $t > 2$, then the point $x = 0$ is an irregular singular point.

b. Try to find a solution of equation (26) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad x > 0. \qquad (27)$$

Show that if $s = 2$ and $t = 2$, then there is only one possible value of r for which there is a formal solution of equation (26) of the form (27).

c. Show that if $s = 1$ and $t = 3$, then there are no solutions of equation (26) of the form (27).

d. Show that the maximum values of s and t for which the indicial equation is quadratic in r [and hence we can hope to find two solutions of the form (27)] are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a "weak singularity," or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution, we point out that although it is sometimes possible to obtain a formal series solution of the form (27) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 15 for an example.

5.7 Bessel's Equation

In this section we illustrate the discussion in Section 5.6 by considering three special cases of Bessel's[14] equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad (1)$$

[14]Friedrich Wilhelm Bessel (1784–1846) left school at the age of 14 to embark on a career in the import-export business but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of equation (1). He is also famous for making, in 1838, the first accurate determination of the distance from the earth to a star.

where ν is a constant. It is easy to show that $x = 0$ is a regular singular point of equation (1). We have

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{1}{x} = 1,$$

$$q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{x^2 - \nu^2}{x^2} = -\nu^2.$$

Thus the indicial equation is

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0,$$

with the roots $r = \pm\nu$. We will consider the three cases $\nu = 0$, $\nu = \frac{1}{2}$, and $\nu = 1$ for the interval $x > 0$. Bessel functions will reappear in Sections 11.4 and 11.5.

Bessel Equation of Order Zero. In this case $\nu = 0$, so differential equation (1) reduces to

$$L[y] = x^2y'' + xy' + x^2y = 0, \qquad (2)$$

and the roots of the indicial equation are equal: $r_1 = r_2 = 0$. Substituting

$$y = \phi(r, x) = a_0 x^r + \sum_{n=1}^{\infty} a_n x^{r+n} \qquad (3)$$

in equation (2), we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} a_n((r+n)(r+n-1) + (r+n))x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0(r(r-1) + r)x^r + a_1((r+1)r + (r+1))x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}(a_n((r+n)(r+n-1) + (r+n)) + a_{n-2})x^{r+n} = 0. \end{aligned} \qquad (4)$$

As we have already noted, the roots of the indicial equation $F(r) = r(r-1) + r = 0$ are $r_1 = 0$ and $r_2 = 0$. The recurrence relation is

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1) + (r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad n \geq 2. \qquad (5)$$

To determine $y_1(x)$, we set r equal to 0. Then, from equation (4), it follows that for the coefficient of x^{r+1} to be zero we must choose $a_1 = 0$. Hence, from equation (5), $a_3 = a_5 = a_7 = \cdots = 0$. Further,

$$a_n(0) = -\frac{a_{n-2}(0)}{n^2}, \quad n = 2, 4, 6, 8, \ldots,$$

or, letting $n = 2m$, we obtain

$$a_{2m}(0) = -\frac{a_{2m-2}(0)}{(2m)^2}, \quad m = 1, 2, 3, \ldots.$$

Thus

$$a_2(0) = -\frac{a_0}{2^2}, \quad a_4(0) = \frac{a_0}{2^4 2^2}, \quad a_6(0) = -\frac{a_0}{2^6(3 \cdot 2)^2},$$

and, in general,

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \quad m = 1, 2, 3, \ldots. \qquad (6)$$

Hence

$$y_1(x) = a_0\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right), \quad x > 0. \qquad (7)$$

choose $c_2 = 1/2^2$. Then we obtain

$$c_4 = \frac{-1}{2^4 \cdot 2}\left(\frac{3}{2}+1\right) = \frac{-1}{2^4 2!}\left(\left(1+\frac{1}{2}\right)+1\right)$$
$$= \frac{(-1)}{2^4 \cdot 2!}(H_2 + H_1).$$

It is possible to show that the solution of the recurrence relation (31) is

$$c_{2m} = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m} m!(m-1)!}, \quad m = 1, 2, \ldots$$

with the understanding that $H_0 = 0$. Thus

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left(1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m!(m-1)!} x^{2m}\right), \quad x > 0. \tag{32}$$

The calculation of $y_2(x)$ using the alternative procedure (see equations (19) and (20) of Section 5.6) in which we determine the $c_n(r_2)$ is slightly easier. In particular, the latter procedure yields the general formula for c_{2m} without the necessity of solving a recurrence relation of the form (31) (see Problem 10). In this regard, you may also wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 9.

The second solution of equation (23), the Bessel function of the second kind of order one, Y_1, is usually taken to be a certain linear combination of J_1 and y_2. Following Copson (Chapter 12), Y_1 is defined as

$$Y_1(x) = \frac{2}{\pi}\left(-y_2(x) + (\gamma - \ln 2)J_1(x)\right), \tag{33}$$

where γ is defined in equation (12). The general solution of equation (23) for $x > 0$ is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that although J_1 is analytic at $x = 0$, the second solution Y_1 becomes unbounded in the same manner as $1/x$ as $x \to 0$. The graphs of J_1 and Y_1 are shown in Figure 5.7.5.

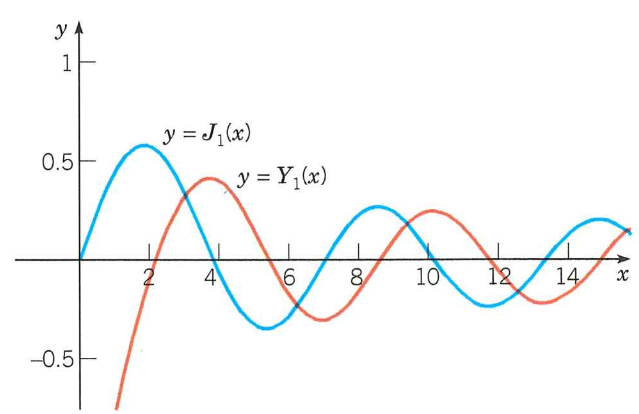

FIGURE 5.7.5 The Bessel functions of order one: $y = J_1(x)$ (blue) and $y = Y_1(x)$ (red).

Problems

In each of Problems 1 through 3, show that the given differential equation has a regular singular point at $x = 0$, and determine two solutions for $x > 0$.

1. $x^2y'' + 2xy' + xy = 0$
2. $x^2y'' + 3xy' + (1+x)y = 0$
3. $x^2y'' + xy' + 2xy = 0$
4. Find two solutions (not multiples of each other) of the Bessel equation of order $\frac{3}{2}$

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0, \quad x > 0.$$

5. Show that the Bessel equation of order one-half

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, \quad x > 0$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

6. Show directly that the series for $J_0(x)$, equation (7), converges absolutely for all x.

7. Show directly that the series for $J_1(x)$, equation (27), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

8. Consider the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0,$$

where ν is real and positive.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are ν and $-\nu$.

b. Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^{\nu}\left(1 - \frac{1}{1!(1+\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+\nu)(2+\nu)}\left(\frac{x}{2}\right)^4\right.$$
$$\left. + \sum_{m=3}^{\infty} \frac{(-1)^m}{m!(1+\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

c. If 2ν is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu}\left(1 - \frac{1}{1!(1-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\nu)(2-\nu)}\left(\frac{x}{2}\right)^4\right.$$
$$\left. + \sum_{m=3}^{\infty} \frac{(-1)^m}{m!(1-\nu)\cdots(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.

d. Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution, provided only that ν is not an integer.

9. In this section we showed that one solution of Bessel's equation of order zero

$$L[y] = x^2y'' + xy' + x^2y = 0$$

is J_0, where $J_0(x)$ is given by equation (7) with $a_0 = 1$. According to Theorem 5.6.1, a second solution has the form $(x > 0)$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

a. Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} n b_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \tag{34}$$

b. Substituting the series representation for $J_0(x)$ in equation (34), show that

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty}(n^2 b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}. \tag{35}$$

c. Note that only even powers of x appear on the right-hand side of equation (35). Show that $b_1 = b_3 = b_5 = \cdots = 0$, $b_2 = \frac{1}{2^2(1!)^2}$, and that

$$(2n)^2 b_{2n} + b_{2n-2} = -2\frac{(-1)^n(2n)}{2^{2n}(n!)^2}, \quad n = 2, 3, 4, \ldots.$$

Deduce that

$$b_4 = -\frac{1}{2^2 4^2}\left(1+\frac{1}{2}\right) \text{ and } b_6 = \frac{1}{2^2 4^2 6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right).$$

The general solution of the recurrence relation is $b_{2n} = \frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}$. Substituting for b_n in the expression for $y_2(x)$, we obtain the solution given in equation (10).

10. Find a second solution of Bessel's equation of order one by computing the $c_n(r_2)$ and a of equation (24) of Section 5.6 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use equation (24) of this section to show that $a_1(-1)$ and $a_1'(-1)$ are 0. Then show that $c_1(-1) = 0$ and, from the recurrence relation, that $c_n(-1) = 0$ for $n = 3, 5, \ldots$. Finally, use equation (25) to show that

$$a_2(r) = -\frac{a_0}{(r+1)(r+3)},$$
$$a_4(r) = \frac{a_0}{(r+1)(r+3)(r+3)(r+5)},$$

and that

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+1)\cdots(r+2m-1)(r+3)\cdots(r+2m+1)}, \quad m \geq 3.$$

Then show that

$$c_{2m}(-1) = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m}m!(m-1)!}, \quad m \geq 1.$$

11. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of

$$x^2y'' + \left(\alpha^2\beta^2x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right)y = 0, \quad x > 0$$

is given by $y = x^{1/2}f(\alpha x^\beta)$, where $f(\xi)$ is a solution of the Bessel equation of order ν.

12. Using the result of Problem 11, show that the general solution of the Airy equation

$$y'' - xy = 0, \quad x > 0$$

is $y = x^{1/2}\left(c_1f_1\left(\frac{2}{3}ix^{3/2}\right) + c_2f_2\left(\frac{2}{3}ix^{3/2}\right)\right)$, where $f_1(\xi)$ and $f_2(\xi)$ are a fundamental set of solutions of the Bessel equation of order one-third.

13. It can be shown that J_0 has infinitely many zeros for $x > 0$. In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.7.1). Let λ_j, $j = 1, 2, 3, \ldots$, denote the zeros of J_0; it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that $y = J_0(\lambda_j x)$ satisfies the differential equation

$$y'' + \frac{1}{x}y' + \lambda_j^2 y = 0, \quad x > 0.$$

Hence show that

$$\int_0^1 xJ_0(\lambda_i x)J_0(\lambda_j x)\,dx = 0 \quad \text{if } \lambda_i \neq \lambda_j.$$

This important property of $J_0(\lambda_i x)$, which is known as the **orthogonality property**, is useful in solving boundary value problems.

Hint: Write the differential equation for $J_0(\lambda_i x)$. Multiply it by $xJ_0(\lambda_j x)$ and subtract that result from $xJ_0(\lambda_i x)$ times the differential equation for $J_0(\lambda_j x)$. Then integrate from 0 to 1.

References

Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice-Hall, 1961; New York: Dover, 1989).

Coddington, E. A., and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mathematics, 1997).

Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable* (Oxford: Oxford University Press, 1935).

K. Knopp, *Theory and Applications of Infinite Series* (NewYork: Hafner, 1951).

Proofs of Theorems 5.3.1 and 5.6.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, Chapters 5 and 6 of Coddington and Carlson, or Chapters 3 and 4 of

Rainville, E. D., *Intermediate Differential Equations* (2nd ed.) (New York: Macmillan, 1964).

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 32 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of

Coddington, E. A., and Levinson, N., *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955; Malabar, FL: Krieger, 1984).

Fuller discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. One text dealing with special functions such as the Legendre polynomials and the Bessel functions is

Hochstadt, H., *Special Functions of Mathematical Physics* (New York: Holt, 1961).

An excellent compilation of formulas, graphs, and tables of Bessel functions, Legendre functions, and other special functions of mathematical physics may be found in

Abramowitz, M., and Stegun, I. A. (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover, 1965); originally published by the National Bureau of Standards, Washington, DC, 1964.

The digital successor to Abramowitz and Stegun is

Digital Library of Mathematical Functions. Released August 29, 2011. National Institute of Standards and Technology from http://dlmf.nist.gov/.

CHAPTER 6

The Laplace Transform

Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods described in Chapter 3 are often rather awkward to use. Another method that is especially well suited to these problems, although useful much more generally, is based on the Laplace transform. In this chapter we describe how this important method works, emphasizing problems typical of those that arise in engineering applications.

6.1 Definition of the Laplace Transform

Improper Integrals. Since the Laplace transform involves an integral from zero to infinity, a knowledge of improper integrals of this type is necessary to appreciate the subsequent development of the properties of the transform. We provide a brief review of such improper integrals here. If you are already familiar with improper integrals, you may wish to skip over this review. On the other hand, if improper integrals are new to you, then you should probably consult a calculus book, where you will find many more details and examples.

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt, \tag{1}$$

where A is a positive real number. If the definite integral from a to A exists for each $A > a$, and if the limit of these values as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or to fail to exist. The following examples illustrate both possibilities.

EXAMPLE 1

Does the improper integral $\displaystyle\int_1^\infty \frac{dt}{t}$ diverge or converge?

Solution:

From equation (1) we have

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty}\int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A.$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.