

Chapter 4

equation can be expressed as a linear combination of a fundamental set of solutions y_1, \dots, y_n , it follows that any solution of Eq. (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \quad (16)$$

where Y is some particular solution of the nonhomogeneous equation (2). The linear combination (16) is called the general solution of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions y_1, \dots, y_n of the homogeneous equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

To find a particular solution $Y(t)$ in Eq. (16), the methods of undetermined coefficients and variation of parameters are again available. They are discussed and illustrated in Sections 4.3 and 4.4, respectively.

The method of reduction of order (Section 3.4) also applies to n th order linear equations. If y_1 is one solution of Eq. (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n - 1$ for v' (see Problem 26 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

PROBLEMS

In each of Problems 1 through 6, determine intervals in which solutions are sure to exist.

- $y^{(4)} + 4y''' + 3y = t$
- $ty''' + (\sin t)y'' + 3y = \cos t$
- $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
- $y''' + ty'' + t^2 y' + t^3 y = \ln t$
- $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
- $(x^2-4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 7 through 10, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

- $f_1(t) = 2t - 3, f_2(t) = t^2 + 1, f_3(t) = 2t^2 - t$
- $f_1(t) = 2t - 3, f_2(t) = 2t^2 + 1, f_3(t) = 3t^2 + t$
- $f_1(t) = 2t - 3, f_2(t) = t^2 + 1, f_3(t) = 2t^2 - t, f_4(t) = t^2 + t + 1$
- $f_1(t) = 2t - 3, f_2(t) = t^3 + 1, f_3(t) = 2t^2 - t, f_4(t) = t^2 + t + 1$

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

- $y''' + y' = 0; \quad 1, \cos t, \sin t$
- $y^{(4)} + y'' = 0; \quad 1, t, \cos t, \sin t$
- $y''' + 2y'' - y' - 2y = 0; \quad e^t, e^{-t}, e^{-2t}$
- $y^{(4)} + 2y''' + y'' = 0; \quad 1, t, e^{-t}, te^{-t}$
- $xy''' - y'' = 0; \quad 1, x, x^3$
- $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \quad x, x^2, 1/x$
- Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t . Can you establish this result without direct evaluation of the Wronskian?
- Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n -times-differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if y_1, y_2, \dots, y_n are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \dots + c_ny_n$ is also a solution of $L[y] = 0$.

19. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_ny,$$

where a_0, a_1, \dots, a_n are real constants.

- (a) Find $L[t^n]$.
 (b) Find $L[e^x]$.
 (c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?
20. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2 , and y_3 be solutions of this equation on an interval I .

- (a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

- (b) Substitute for $y_1''', y_2''',$ and y_3''' from the differential equation; multiply the first row by p_3 , multiply the second row by p_2 , and add these to the last row to obtain

$$W' = -p_1(t)W.$$

- (c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp \left[- \int p_1(t) dt \right].$$

It follows that W is either always zero or nowhere zero on I .

- (d) Generalize this argument to the n th order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = 0$$

with solutions y_1, \dots, y_n . That is, establish Abel's formula

$$W(y_1, \dots, y_n)(t) = c \exp \left[- \int p_1(t) dt \right]$$

for this case.

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. $y''' + 2y'' - y' - 3y = 0$

22. $y^{(4)} + y = 0$

23. $ty''' + 2y'' - y' + ty = 0$

24. $t^2y^{(4)} + ty''' + y'' - 4y = 0$

25. (a) Show that the functions $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$.
 (b) Show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.
 (c) Show that $W(f, g)(t)$ is zero for all t in $-1 < t < 1$.
26. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second order equation for v :

$$y_1 v''' + (3y_1' + p_1 y_1) v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1) v' = 0.$$

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

27. $(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \quad y_1(t) = e^t$
 28. $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Equations with Constant Coefficients

Consider the n th order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \quad (1)$$

where a_0, a_1, \dots, a_n are real constants and $a_0 \neq 0$. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (1) for suitable values of r . Indeed,

$$L[e^{rt}] = e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) = e^{rt} Z(r) \quad (2)$$

for all r , where

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n. \quad (3)$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of Eq. (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1). Since $a_0 \neq 0$, we know that $Z(r)$ is a polynomial of degree n and therefore has n zeros,¹ say, r_1, r_2, \dots, r_n , some of which may be equal. Hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2) \cdots (r - r_n). \quad (4)$$

¹An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss (1777–1855) in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

In determining the roots of the characteristic equation, it may be necessary to compute the cube roots, the fourth roots, or even higher roots of a (possibly complex) number. This can usually be done most conveniently by using Euler's formula $e^{it} = \cos t + i \sin t$ and the algebraic laws given in Section 3.3. This is illustrated in the following example.

**EXAMPLE
4**

Find the general solution of

$$y^{(4)} + y = 0. \quad (20)$$

The characteristic equation is

$$r^4 + 1 = 0.$$

To solve the equation, we must compute the fourth roots of -1 . Now -1 , thought of as a complex number, is $-1 + 0i$. It has magnitude 1 and polar angle π . Thus

$$-1 = \cos \pi + i \sin \pi = e^{i\pi}.$$

Moreover, the angle is determined only up to a multiple of 2π . Thus

$$-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi + 2m\pi)},$$

where m is zero or any positive or negative integer. Thus

$$(-1)^{1/4} = e^{i(\pi/4 + m\pi/2)} = \cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right).$$

The four fourth roots of -1 are obtained by setting $m = 0, 1, 2,$ and 3 ; they are

$$\frac{1+i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}.$$

It is easy to verify that, for any other value of m , we obtain one of these four roots. For example, corresponding to $m = 4$, we obtain $(1+i)/\sqrt{2}$. The general solution of Eq. (20) is

$$y = e^{t/\sqrt{2}} \left(c_1 \cos \frac{t}{\sqrt{2}} + c_2 \sin \frac{t}{\sqrt{2}} \right) + e^{-t/\sqrt{2}} \left(c_3 \cos \frac{t}{\sqrt{2}} + c_4 \sin \frac{t}{\sqrt{2}} \right). \quad (21)$$

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. For instance, it may be difficult to determine whether two roots are equal or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants a_0, a_1, \dots, a_n in Eq. (1) are complex numbers, the solution of Eq. (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

PROBLEMS

In each of Problems 1 through 6, express the given complex number in the form $R(\cos \theta + i \sin \theta) = Re^{i\theta}$.

- | | |
|-------------------|---------------------|
| 1. $1 + i$ | 2. $-1 + \sqrt{3}i$ |
| 3. -3 | 4. $-i$ |
| 5. $\sqrt{3} - i$ | 6. $-1 - i$ |

(b) Solve the first of Eqs. (i) for u_2 and substitute into the second equation, thereby obtaining the following fourth order equation for u_1 :

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0. \quad (\text{ii})$$

Find the general solution of Eq. (ii).

(c) Suppose that the initial conditions are

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 2, \quad u_2'(0) = 0. \quad (\text{iii})$$

Use the first of Eqs. (i) and the initial conditions (iii) to obtain values for $u_1''(0)$ and $u_1^{(4)}(0)$. Then show that the solution of Eq. (ii) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2 \cos t$.

(d) Now suppose that the initial conditions are

$$u_1(0) = -2, \quad u_1'(0) = 0, \quad u_2(0) = 1, \quad u_2'(0) = 0. \quad (\text{iv})$$

Proceed as in part (c) to show that the corresponding solutions are $u_1(t) = -2 \cos \sqrt{6}t$ and $u_2(t) = \cos \sqrt{6}t$.

(e) Observe that the solutions obtained in parts (c) and (d) describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together; the second mass moves twice as far as the first. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up, and vice versa. In this mode the first mass moves twice as far as the second. For other initial conditions, not proportional to either of Eqs. (iii) or (iv), the motion of the masses is a combination of these two modes.

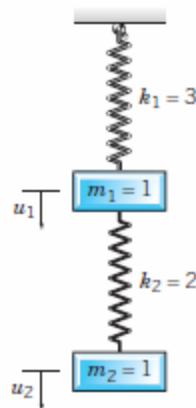


FIGURE 4.2.4 A two-spring, two-mass system.

40. In this problem we outline one way to show that if r_1, \dots, r_n are all real and different, then $e^{r_1 t}, \dots, e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1 e^{r_1 t} + \dots + c_n e^{r_n t} = 0, \quad -\infty < t < \infty \quad (\text{i})$$

and show that all the constants are zero.

(a) Multiply Eq. (i) by $e^{-r_1 t}$ and differentiate with respect to t , thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \dots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0.$$

(b) Multiply the result of part (a) by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

(c) Continue the procedure from parts (a) and (b), eventually obtaining

$$c_n(r_n - r_{n-1}) \cdots (r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

Hence $c_n = 0$, and therefore,

$$c_1e^{r_1t} + \cdots + c_{n-1}e^{r_{n-1}t} = 0.$$

(d) Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \cdots = c_1 = 0$. Thus the functions $e^{r_1t}, \dots, e^{r_{n-1}t}$ are linearly independent.

41. In this problem we indicate one way to show that if $r = r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1t}, te^{r_1t}, \dots, t^{s-1}e^{r_1t}$ are solutions of Eq. (1). This problem extends to n th order equations the method for second order equations given in Problem 22 of Section 3.4. We start from Eq. (2) in the text

$$L[e^{r_1t}] = e^{r_1t} Z(r) \tag{i}$$

and differentiate repeatedly with respect to r , setting $r = r_1$ after each differentiation.

(a) Observe that if r_1 is a root of multiplicity s , then $Z(r) = (r - r_1)^s q(r)$, where $q(r)$ is a polynomial of degree $n - s$ and $q(r_1) \neq 0$. Show that $Z(r_1), Z'(r_1), \dots, Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1) \neq 0$.

(b) By differentiating Eq. (i) repeatedly with respect to r , show that

$$\begin{aligned} \frac{\partial}{\partial r} L[e^{r_1t}] &= L \left[\frac{\partial}{\partial r} e^{r_1t} \right] = L[te^{r_1t}], \\ &\vdots \\ \frac{\partial^{s-1}}{\partial r^{s-1}} L[e^{r_1t}] &= L[t^{s-1}e^{r_1t}]. \end{aligned}$$

(c) Show that $e^{r_1t}, te^{r_1t}, \dots, t^{s-1}e^{r_1t}$ are solutions of Eq. (1).

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous n th order linear equation with constant coefficients

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator L is applied to a polynomial $A_0t^m + A_1t^{m-1} + \cdots + A_m$, an

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

PROBLEMS

In each of Problems 1 through 8, determine the general solution of the given differential equation.

- | | |
|--|---------------------------------------|
| 1. $y''' - y'' - y' + y = 2e^{-t} + 3$ | 2. $y^{(4)} - y = 3t + \cos t$ |
| 3. $y''' + y'' + y' + y = e^{-t} + 4t$ | 4. $y''' - y' = 2 \sin t$ |
| 5. $y^{(4)} - 4y'' = t^2 + e^t$ | 6. $y^{(4)} + 2y'' + y = 3 + \cos 2t$ |
| 7. $y^{(6)} + y''' = t$ | 8. $y^{(4)} + y''' = \sin 2t$ |

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

9. $y''' + 4y' = t$; $y(0) = y'(0) = 0$, $y''(0) = 1$
10. $y^{(4)} + 2y'' + y = 3t + 4$; $y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$
11. $y''' - 3y'' + 2y' = t + e^t$; $y(0) = 1$, $y'(0) = -\frac{1}{4}$, $y''(0) = -\frac{3}{2}$
12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12 \sin t - e^{-t}$; $y(0) = 3$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 2$

In each of Problems 13 through 18, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

- | | |
|--|--|
| 13. $y''' - 2y'' + y' = t^3 + 2e^t$ | 14. $y''' - y' = te^{-t} + 2 \cos t$ |
| 15. $y^{(4)} - 2y'' + y = e^t + \sin t$ | 16. $y^{(4)} + 4y'' = \sin 2t + te^t + 4$ |
| 17. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t \sin t$ | 18. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t} \sin t$ |

19. Consider the nonhomogeneous n th order linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t), \quad (\text{i})$$

where a_0, \dots, a_n are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t} (b_0 t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t} u(t)$ reduces Eq. (i) to the form

$$k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_n u = b_0 t^m + \cdots + b_m, \quad (\text{ii})$$

where k_0, \dots, k_n are constants. Determine k_0 and k_n in terms of the a 's and α . Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 20 through 22, we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for d/dt . Then, for example, e^{-t} is a solution of $(D + 1)y = 0$; the differential operator $D + 1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t} . In the same way, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D - 3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t} , and so forth.

20. Show that linear differential operators with constant coefficients obey the commutative law. That is, show that

$$(D - a)(D - b)f = (D - b)(D - a)f$$

for any twice-differentiable function f and any constants a and b . The result extends at once to any finite number of factors.

21. Consider the problem of finding the form of a particular solution $Y(t)$ of

$$(D - 2)^3(D + 1)Y = 3e^{2t} - te^{-t}, \quad (\text{i})$$

where the left side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

(a) Show that $D - 2$ and $(D + 1)^2$, respectively, are annihilators of the terms on the right side of Eq. (i), and that the combined operator $(D - 2)(D + 1)^2$ annihilates both terms on the right side of Eq. (i) simultaneously.

(b) Apply the operator $(D - 2)(D + 1)^2$ to Eq. (i) and use the result of Problem 20 to obtain

$$(D - 2)^4(D + 1)^3Y = 0. \quad (\text{ii})$$

Thus Y is a solution of the homogeneous equation (ii). By solving Eq. (ii), show that

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t}, \quad (\text{iii})$$

where c_1, \dots, c_7 are constants, as yet undetermined.

(c) Observe that e^{2t} , te^{2t} , t^2e^{2t} , and e^{-t} are solutions of the homogeneous equation corresponding to Eq. (i); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose c_1, c_2, c_3 , and c_5 to be zero in Eq. (iii), so that

$$Y(t) = c_4t^3e^{2t} + c_6te^{-t} + c_7t^2e^{-t}. \quad (\text{iv})$$

This is the form of the particular solution Y of Eq. (i). The values of the coefficients c_4, c_6 , and c_7 can be found by substituting from Eq. (iv) in the differential equation (i).

Summary. Suppose that

$$L(D)y = g(t), \quad (\text{v})$$

where $L(D)$ is a linear differential operator with constant coefficients, and $g(t)$ is a sum or product of exponential, polynomial, or sinusoidal terms. To find the form of a particular solution of Eq. (v), you can proceed as follows:

(a) Find a differential operator $H(D)$ with constant coefficients that annihilates $g(t)$ —that is, an operator such that $H(D)g(t) = 0$.

(b) Apply $H(D)$ to Eq. (v), obtaining

$$H(D)L(D)y = 0, \quad (\text{vi})$$

which is a homogeneous equation of higher order.

(c) Solve Eq. (vi).

(d) Eliminate from the solution found in step (c) the terms that also appear in the solution of $L(D)y = 0$. The remaining terms constitute the correct form of a particular solution of Eq. (v).

22. Use the method of annihilators to find the form of a particular solution $Y(t)$ for each of the equations in Problems 13 through 18. Do not evaluate the coefficients.

and

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}.$$

Substituting these results in Eq. (13), we have

$$\begin{aligned} Y(t) &= e^t \int_0^t \frac{g(s)(-1-2s)}{4e^s} ds + te^t \int_0^t \frac{g(s)(2)}{4e^s} ds + e^{-t} \int_0^t \frac{g(s)e^{2s}}{4e^s} ds \\ &= \frac{1}{4} \int_0^t \{e^{t-s}[-1+2(t-s)] + e^{-(t-s)}\} g(s) ds. \end{aligned} \quad (15)$$

Depending on the specific function $g(t)$, it may or may not be possible to evaluate the integrals in Eq. (15) in terms of elementary functions.

PROBLEMS

In each of Problems 1 through 6, use the method of variation of parameters to determine the general solution of the given differential equation.

- $y''' + y' = \tan t$, $-\pi/2 < t < \pi/2$
- $y''' - y' = t$
- $y''' - 2y'' - y' + 2y = e^{4t}$
- $y''' + y' = \sec t$, $-\pi/2 < t < \pi/2$
- $y''' - y'' + y' - y = e^{-t} \sin t$
- $y^{(4)} + 2y'' + y = \sin t$

In each of Problems 7 and 8, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

- $y''' - y'' + y' - y = \sec t$, $-\pi/2 < t < \pi/2$
- $y''' - y' = \csc t$, $0 < t < \pi$

In each of Problems 9 through 12, find the solution of the given initial value problem. Then plot a graph of the solution.

- $y''' + y' = \sec t$; $y(0) = 2$, $y'(0) = 1$, $y''(0) = -2$
- $y^{(4)} + 2y'' + y = \sin t$; $y(0) = 2$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 1$
- $y''' - y'' + y' - y = \sec t$; $y(0) = 2$, $y'(0) = -1$, $y''(0) = 1$
- $y''' - y' = \csc t$; $y(\pi/2) = 2$, $y'(\pi/2) = 1$, $y''(\pi/2) = -1$

13. Given that x , x^2 , and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

14. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

15. Find a formula involving integrals for a particular solution of the differential equation

$$y^{(4)} - y = g(t).$$

Hint: The functions $\sin t$, $\cos t$, $\sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

16. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

17. Find a formula involving integrals for a particular solution of the differential equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = g(x), \quad x > 0.$$

Hint: Verify that x, x^2 , and x^3 are solutions of the homogeneous equation.

REFERENCES

- Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice-Hall, 1961; New York: Dover, 1989).
Coddington, E. A. and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mathematics, 1997).