8. A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady-state response. Express your answer in the form $R\cos(\omega t - \delta)$.

9. In this problem we ask you to supply some of the details in the analysis of a forced damped oscillator.

a. Derive equations (10), (11), and (12) for the steady-state solution of equation (8).

b. Derive the expression in equation (13) for Rk/F_0.

c. Show that ω^2_{max} and R_{max} are given by equations (14) and (15), respectively.

d. Verify that Rk/F_0, ω/ω_0, and $\Gamma = \gamma^2/(mk)$ are all dimensionless quantities.

10. Find the velocity of the steady-state response given by equation (10). Then show that the velocity is maximum when $\omega = \omega_0$.

11. Find the solution of the initial value problem

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

N **12.** A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of 5×10^3 Ω, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12 V battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

N **13.** Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

a. Find the solution $u(t)$ for $\omega \ne 1$.

G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

14. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 1, \quad u'(0) = 1.$$

a. Find the solution for $\omega \ne 1$.

G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 13; that is, describe the effect of the nonzero initial conditions.

G **15.** For the initial value problem in Problem 13, plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. (Recall that such a plot is called a phase plot.) Use a t interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 16 through 18 deal with the initial value problem

$$u'' + \frac{1}{8}u' + 4u = F(t), \quad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

G **a.** Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.

G **b.** Draw the phase plot of the solution; that is, plot u' versus u.

16. $F(t) = 3\cos(t/4)$

17. $F(t) = 3\cos(2t)$

18. $F(t) = 3\cos(6t)$

G **19.** A spring-mass system with a hardening spring (Problem 24 of Section 3.7) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$u'' + u + \frac{1}{5}u^3 = \cos\omega t, \quad u(0) = 0, \quad u'(0) = 0.$$

a. Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?

b. Plot the solution for several values of ω between $1/2$ and 2. Describe how the solution changes as ω increases.

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CHAPTER **4**

Higher-Order Linear Differential Equations

The theoretical structure and methods of solution developed in the preceding chapter for second-order linear equations extend directly to linear equations of third and higher order. In this chapter we briefly review this generalization, taking particular note of those instances where new phenomena may appear, because of the greater variety of situations that can occur for equations of higher order.

4.1 General Theory of n^{th} Order Linear Differential Equations

An n^{th} order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t). \tag{1}$$

We assume that the functions $P_0, \ldots, P_n$, and G are continuous real-valued functions on some interval I: $\alpha < t < \beta$, and that P_0 is nowhere zero in this interval. Then, dividing equation (1) by $P_0(t)$, we obtain

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t). \tag{2}$$

The linear differential operator L of order n defined by equation (2) is similar to the second-order operator introduced in Chapter 3. The mathematical theory associated with equation (2) is completely analogous to that for the second-order linear equation; for this reason we simply state the results for the n^{th} order problem. The proofs of most of the results are also similar to those for the second-order equation and are usually left as exercises.

Since equation (2) involves the n^{th} derivative of y with respect to t, it will, so to speak, require n integrations to solve equation (2). Each of these integrations introduces an arbitrary constant. Hence we expect that to obtain a unique solution it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3}$$

where t_0 may be any point in the interval I and $y_0, y_0', \ldots, y_0^{(n-1)}$ are any prescribed real constants. The following theorem, which is similar to Theorem 3.2.1, guarantees that the initial value problem (2), (3) has a solution and that it is unique.

Theorem 4.1.1

If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3), where t_0 is any point in I. This solution exists throughout the interval I.

For this expression to be zero throughout an interval, it is certainly sufficient to require that

$$k_1 + 2k_2 + 3k_3 = 0, \quad k_2 + 4k_4 = 0, \quad -k_3 + k_4 = 0.$$

These three equations, with four unknowns, have many solutions. For instance, if $k_4 = 1$, then $k_3 = 1$, $k_2 = -4$, and $k_1 = 5$. If we use these values for the coefficients in equation (11), then these functions satisfy the linear relation

$$5f_1(t) - 4f_2(t) + f_3(t) + f_4(t) = 0$$

for each value of t. Thus the given functions are linearly dependent on every interval.

The concept of linear independence provides an alternative characterization of fundamental sets of solutions of the homogeneous equation (4). Suppose that the functions $y_1, \dots, y_n$ are solutions of equation (4) on an interval I, and consider the equation

$$k_1 y_1(t) + \cdots + k_n y_n(t) = 0. \tag{12}$$

By differentiating equation (12) repeatedly, we obtain the additional $n-1$ equations

$$\begin{aligned} k_1 y_1'(t) + \cdots + k_n y_n'(t) &= 0, \\ &\vdots \\ k_1 y_1^{(n-1)}(t) + \cdots + k_n y_n^{(n-1)}(t) &= 0. \end{aligned} \tag{13}$$

The system consisting of equations (12) and (13) is a system of n linear algebraic equations for the n unknowns $k_1, \dots, k_n$. The determinant of coefficients for this system is the Wronskian $W[y_1, \dots, y_n](t)$ of $y_1, \dots, y_n$. This leads to the following theorem.

Theorem 4.1.3

If $y_1(t), \dots, y_n(t)$ form a fundamental set of solutions of the homogeneous n^{th} order linear differential equation (4)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval I, then $y_1(t), \dots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \dots, y_n(t)$ are linearly independent solutions of equation (4) on I, then they form a fundamental set of solutions on I.

To prove this theorem, first suppose that $y_1(t), \dots, y_n(t)$ form a fundamental set of solutions of the homogeneous differential equation (4) on I. Then the Wronskian $W[y_1, \dots, y_n](t) \neq 0$ for every t in I. Hence the system (12), (13) has only the solution $k_1 = \cdots = k_n = 0$ for every t in I. Thus $y_1(t), \dots, y_n(t)$ cannot be linearly dependent on I and must therefore be linearly independent there.

To demonstrate the converse, let $y_1(t), \dots, y_n(t)$ be linearly independent on I. To show that they form a fundamental set of solutions, we need to show that their Wronskian is never zero in I. Suppose that this is not true; then there is at least one point t_0 where the Wronskian is zero. At this point the system (12), (13) has a nonzero solution; let us denote it by $k_1^*, \dots, k_n^*$. Now form the linear combination

$$\phi(t) = k_1^* y_1(t) + \cdots + k_n^* y_n(t). \tag{14}$$

Then $y = \phi(t)$ satisfies the initial value problem

$$L[y] = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0, \dots, y^{(n-1)}(t_0) = 0. \tag{15}$$

The function ϕ satisfies the differential equation because it is a linear combination of solutions; it satisfies the initial conditions because these are just the equations in the system (12), (13) evaluated at t_0. However, the function $y(t) = 0$ for all t in I is also a solution of this initial value problem, and by Theorem 4.1.1, the solution to the initial value problem (15) is unique. Thus $\phi(t) = 0$ for all t in I. Consequently, $y_1(t), \dots, y_n(t)$ are linearly dependent on I, which is a contradiction. Hence the assumption that there is a point where the Wronskian is zero is untenable. Therefore, the Wronskian is never zero on I, as was to be proved.

Note that for a set of functions $f_1, \dots, f_n$ that are not solutions of the homogeneous linear differential equation (4), the converse part of Theorem 4.1.3 is not necessarily true. They may be linearly independent on I even though the Wronskian is zero at some points, or even every point, but with different sets of constants $k_1, \dots, k_n$ at different points. See Problem 18 for an example.

The Nonhomogeneous Equation. Now consider the nonhomogeneous equation (2)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t).$$

If Y_1 and Y_2 are any two solutions of equation (2), then it follows immediately from the linearity of the operator L that

$$L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0.$$

Hence the difference of any two solutions of the nonhomogeneous equation (2) is a solution of the homogeneous differential equation (4). Since any solution of the homogeneous equation can be expressed as a linear combination of a fundamental set of solutions $y_1, \dots, y_n$, it follows that any solution of the nonhomogeneous differential equation (2) can be written as

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t), \tag{16}$$

where Y is some particular solution of the nonhomogeneous differential equation (2). The linear combination (16) is called the **general solution** of the nonhomogeneous equation (2).

Thus the primary problem is to determine a fundamental set of solutions $\{y_1, \dots, y_n\}$ of the homogeneous n^{th} order linear differential equation (4). If the coefficients are constants, this is a fairly simple problem; it is discussed in the next section. If the coefficients are not constants, it is usually necessary to use numerical methods such as those in Chapter 8 or series methods similar to those in Chapter 5. These tend to become more cumbersome as the order of the equation increases.

To find a particular solution $Y(t)$ in equation (16), the methods of undetermined coefficients and variation of parameters are again available. They are discussed and illustrated in Sections 4.3 and 4.4, respectively.

The method of reduction of order (Section 3.4) also applies to n^{th} order linear differential equations. If y_1 is one solution of equation (4), then the substitution $y = v(t)y_1(t)$ leads to a linear differential equation of order $n-1$ for v' (see Problem 19 for the case when $n = 3$). However, if $n \geq 3$, the reduced equation is itself at least of second order, and only rarely will it be significantly simpler than the original equation. Thus, in practice, reduction of order is seldom useful for equations of higher than second order.

Problems

In each of Problems 1 through 4, determine intervals in which solutions are sure to exist.

1. $y^{(4)} + 4y''' + 3y = t$
2. $t(t-1)y^{(4)} + e^t y'' + 4t^2 y = 0$
3. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$
4. $(x^2 - 4)y^{(6)} + x^2 y''' + 9y = 0$

In each of Problems 5 through 7, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

5. $f_1(t) = 2t - 3, \; f_2(t) = t^2 + 1, \; f_3(t) = 2t^2 - t$
6. $f_1(t) = 2t - 3, \; f_2(t) = 2t^2 + 1, \; f_3(t) = 3t^2 + t$
7. $f_1(t) = 2t - 3, \; f_2(t) = t^2 + 1, \; f_3(t) = 2t^2 - t,$ $f_4(t) = t^2 + t + 1$

In each of Problems 8 through 11, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

8. $y^{(4)} + y'' = 0; \quad 1, \; t, \; \cos t, \; \sin t$
9. $y''' + 2y'' - y' - 2y = 0; \quad e^t, \; e^{-t}, \; e^{-2t}$
10. $xy''' - y'' = 0; \quad 1, \; x, \; x^3$
11. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0; \quad x, \; x^2, \; 1/x$
12. a. Show that $W[5, \sin^2 t, \cos(2t)] = 0$ for all t by directly evaluating the Wrosnkian.
    b. Establish the same result without direct evaluation of the Wronskian.

13. Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n-times-differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if $y_1, y_2, \ldots, y_n$ are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

14. Let the linear differential operator L be defined by

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny,$$

where $a_0, a_1, \ldots, a_n$ are real constants.

a. Find $L[t^n]$.
b. Find $L[e^{rt}]$.
c. Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

15. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher-order equations. We first outline the procedure for the third-order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.

a. If $W = W[y_1, y_2, y_3]$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

b. Substitute for y_1''', y_2''', and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

c. Show that

$$W[y_1, y_2, y_3](t) = c\exp\left(-\int p_1(t)\,dt\right).$$

It follows that W is either always zero or nowhere zero on I.

d. Generalize this argument to the n^{th} order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish **Abel's formula**

$$W[y_1, \ldots, y_n](t) = c\exp\left(-\int p_1(t)\,dt\right) \quad (17)$$

for this case.

In each of Problems 16 and 17, use Abel's formula (17) to find the Wronskian of a fundamental set of solutions of the given differential equation.

16. $y''' + 2y'' - y' - 3y = 0$

17. $ty''' + 2y'' - y' + ty = 0$

18. Let $f(t) = t^2|t|$ and $g(t) = t^3$.

a. Show that the functions $f(t)$ and $g(t)$ are linearly dependent on $0 < t < 1$.
b. Show that $f(t)$ and $g(t)$ are linearly dependent on $-1 < t < 0$.
c. Show that $f(t)$ and $g(t)$ are linearly independent on $-1 < t < 1$.
d. Show that $W[f, g](t)$ is zero for all t in $-1 < t < 1$.
e. Explain why the results in c and d do not contradict Theorem 4.1.3.

19. Show that if y_1 is a solution of

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0,$$

then the substitution $y = y_1(t)v(t)$ leads to the following second-order equation for v':

$$y_1v''' + (3y_1' + p_1y_1)v'' + (3y_1'' + 2p_1y_1' + p_2y_1)v' = 0.$$

In each of Problems 20 and 21, use the method of reduction of order (Problem 19) to solve the given differential equation.

20. $(2 - t)y''' + (2t - 3)y'' - ty' + y = 0, \quad t < 2; \quad y_1(t) = e^t$

21. $t^2(t + 3)y''' - 3t(t + 2)y'' + 6(1 + t)y' - 6y = 0, \quad t > 0;$ $y_1(t) = t^2, \quad y_2(t) = t^3$

4.2 Homogeneous Differential Equations with Constant Coefficients

Consider the n^{th} order linear homogeneous differential equation

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0, \quad (1)$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$. From our knowledge of second-order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of equation (1) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \quad (2)$$

for all r, where

$$Z(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n. \quad (3)$$

For those values of r for which $Z(r) = 0$, it follows that $L[e^{rt}] = 0$ and $y = e^{rt}$ is a solution of equation (1). The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (1). Since $a_0 \neq 0$, we know that $Z(r)$ is a polynomial of degree n and therefore has n zeros,[1] say, $r_1, r_2, \ldots, r_n$, some of which may be equal and some of which may be complex-valued. Hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n). \quad (4)$$

Real and Unequal Roots. If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt}$ of equation (1). If these functions are linearly independent, then the general solution of the homogeneous n^{th} order linear differential equation (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}. \quad (5)$$

One way to establish the linear independence of $e^{r_1t}, e^{r_2t}, \ldots, e^{r_nt}$ is to evaluate their Wronskian determinant; another way is outlined in Problem 30.

EXAMPLE 1

Find the general solution of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0. \quad (6)$$

Also find the solution that satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = -1. \quad (7)$$

Plot its graph and determine the behavior of the solution as $t \to \infty$.

Solution:

Assuming that $y = e^{rt}$, we must determine r by solving the polynomial equation

$$r^4 + r^3 - 7r^2 - r + 6 = 0. \quad (8)$$

The roots of this equation are $r_1 = 1, r_2 = -1, r_3 = 2$, and $r_4 = -3$. Therefore, the general solution of differential equation (6) is

$$y = c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-3t}. \quad (9)$$

The initial conditions (7) require that $c_1, \ldots, c_4$ satisfy the four equations

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1, \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0, \\ c_1 + c_2 + 4c_3 + 9c_4 &= -2, \\ c_1 - c_2 + 8c_3 - 27c_4 &= -1. \end{aligned} \quad (10)$$

[1] An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss (1777–1855) in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

In conclusion, we note that the problem of finding all the roots of a polynomial equation may not be entirely straightforward, even with computer assistance. In particular, it may be difficult to determine whether two roots are equal or merely very close together. Recall that the form of the general solution is different in these two cases.

If the constants $a_0, a_1, \dots, a_n$ in equation (1) are complex numbers, the solution of equation (1) is still of the form (4). In this case, however, the roots of the characteristic equation are, in general, complex numbers, and it is no longer true that the complex conjugate of a root is also a root. The corresponding solutions are complex-valued.

Problems

In each of Problems 1 through 4, express the given complex number in polar form $R(\cos\theta + i\sin\theta) = Re^{i\theta}$.

1. $1 + i$
2. $-1 + \sqrt{3}i$
3. -3
4. $\sqrt{3} - i$

In each of Problems 5 through 7, follow the procedure in Example 4 to determine the indicated roots of the given complex number.

5. $1^{1/3}$
6. $(1 - i)^{1/2}$
7. $(2(\cos(\pi/3) + i\sin(\pi/3)))^{1/2}$

In each of Problems 8 through 19, find the general solution of the given differential equation.

8. $y''' - y'' - y' + y = 0$
9. $y''' - 3y'' + 3y' - y = 0$
10. $y^{(4)} - 4y''' + 4y'' = 0$
11. $y^{(6)} + y = 0$
12. $y^{(6)} - 3y^{(4)} + 3y'' - y = 0$
13. $y^{(6)} - y'' = 0$
14. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0$
15. $y^{(8)} + 8y^{(4)} + 16y = 0$
16. $y^{(4)} + 2y'' + y = 0$
17. $y''' + 5y'' + 6y' + 2y = 0$
18. (N) $y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$
19. (N) $12y^{(4)} + 31y''' + 75y'' + 37y' + 5y = 0$

In each of Problems 20 through 25, find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

20. (G) $y''' + y' = 0$; $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$
21. (G) $y^{(4)} + y = 0$; $y(0) = 0$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 0$
22. (G) $y^{(4)} - 4y''' + 4y'' = 0$; $y(1) = -1$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 0$
23. (G) $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0$; $y(0) = -2$, $y'(0) = 0$, $y''(0) = -2$, $y'''(0) = 0$
24. (G) $4y''' + y' + 5y = 0$; $y(0) = 2$, $y'(0) = 1$, $y''(0) = -1$
25. (G) $6y''' + 5y'' + y' = 0$; $y(0) = -2$, $y'(0) = 2$, $y''(0) = 0$

26. (G) **a.** Verify that $y(t) = 3e^{-t} + \frac{1}{2}\cos t - \sin t$ is the solution to $y^{(4)} - y = 0$, $y(0) = \frac{7}{2}$, $y'(0) = -4$, $y''(0) = \frac{5}{2}$, $y'''(0) = -2$.
 (N) **b.** Find the solution to $y^{(4)} - y = 0$, $y(0) = \frac{7}{2}$, $y'(0) = -4$, $y''(0) = \frac{5}{2}$, $y'''(0) = -\frac{15}{8}$.
 Note: These are the initial value problems considered in Example 2.

27. Show that the general solution of $y^{(4)} - y = 0$ can be written as

$$y = c_1\cos t + c_2\sin t + c_3\cosh t + c_4\sinh t.$$

Determine the solution satisfying the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 1$. Why is it convenient to use the solutions $\cosh t$ and $\sinh t$ rather than e^t and e^{-t}?

28. Consider the equation $y^{(4)} - y = 0$.
 a. Use Abel's formula (Problem 15d of Section 4.1) to find the Wronskian of a fundamental set of solutions of the given equation.
 b. Determine the Wronskian of the solutions e^t, e^{-t}, $\cos t$, and $\sin t$.
 c. Determine the Wronskian of the solutions $\cosh t$, $\sinh t$, $\cos t$, and $\sin t$.

29. Consider the spring-mass system, shown in Figure 4.2.4, consisting of two unit masses suspended from springs with spring constants 3 and 2, respectively. Assume that there is no damping in the system.
 a. Show that the displacements u_1 and u_2 of the masses from their respective equilibrium positions satisfy the equations

$$u_1'' + 5u_1 = 2u_2, \quad u_2'' + 2u_2 = 2u_1. \tag{22}$$

 b. Solve the first of equations (22) for u_2 and substitute into the second equation, thereby obtaining the following fourth-order equation for u_1:

$$u_1^{(4)} + 7u_1'' + 6u_1 = 0. \tag{23}$$

 Find the general solution of equation (23).
 c. Suppose that the initial conditions are

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 2, \quad u_2'(0) = 0. \tag{24}$$

 Use the first of equations (22) and the initial conditions (24) to obtain values for $u_1''(0)$ and $u_1'''(0)$. Then show that the solution of equation (23) that satisfies the four initial conditions on u_1 is $u_1(t) = \cos t$. Show that the corresponding solution u_2 is $u_2(t) = 2\cos t$.

 d. Now suppose that the initial conditions are

$$u_1(0) = -2, \quad u_1'(0) = 0, \quad u_2(0) = 1, \quad u_2'(0) = 0. \tag{25}$$

 Proceed as in part c to show that the corresponding solutions are $u_1(t) = -2\cos\left(\sqrt{6}\,t\right)$ and $u_2(t) = \cos\left(\sqrt{6}\,t\right)$.
 e. Observe that the solutions obtained in parts c and d describe two distinct modes of vibration. In the first, the frequency of the motion is 1, and the two masses move in phase, both moving up or down together; the second mass moves twice as far as the first. The second motion has frequency $\sqrt{6}$, and the masses move out of phase with each other, one moving down while the other is moving up, and vice versa. In this mode the first mass moves twice as far as the second. For other initial conditions, not proportional to either of equations (24) or (25), the motion of the masses is a combination of these two modes.

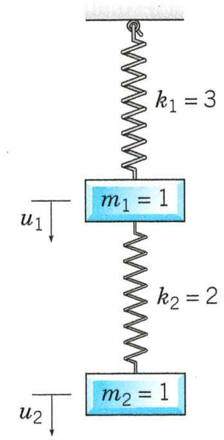

FIGURE 4.2.4 A two-spring, two-mass system.

30. In this problem we outline one way to show that if $r_1, \dots, r_n$ are all real and different, then $e^{r_1 t}, \dots, e^{r_n t}$ are linearly independent on $-\infty < t < \infty$. To do this, we consider the linear relation

$$c_1e^{r_1t} + \dots + c_ne^{r_nt} = 0, \quad -\infty < t < \infty \tag{26}$$

and show that all the constants are zero.
 a. Multiply equation (26) by e^{-r_1t} and differentiate with respect to t, thereby obtaining

$$c_2(r_2 - r_1)e^{(r_2-r_1)t} + \dots + c_n(r_n - r_1)e^{(r_n-r_1)t} = 0.$$

 b. Multiply the result of part a by $e^{-(r_2-r_1)t}$ and differentiate with respect to t to obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3-r_2)t} + \dots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n-r_2)t} = 0.$$

 c. Continue the procedure from parts a and b, eventually obtaining

$$c_n(r_n - r_{n-1})\cdots(r_n - r_1)e^{(r_n-r_{n-1})t} = 0.$$

 Hence $c_n = 0$, and therefore,

$$c_1e^{r_1t} + \dots + c_{n-1}e^{r_{n-1}t} = 0.$$

 d. Repeat the preceding argument to show that $c_{n-1} = 0$. In a similar way it follows that $c_{n-2} = \dots = c_1 = 0$. Thus the functions $e^{r_1t}, \dots, e^{r_nt}$ are linearly independent.

31. In this problem we indicate one way to show that if $r = r_1$ is a root of multiplicity s of the characteristic polynomial $Z(r)$, then $e^{r_1t}, te^{r_1t}, \dots, t^{s-1}e^{r_1t}$ are solutions of equation (1). This problem extends to n^{th} order equations the method for second-order equations given in Problem 17 of Section 3.4. We start from equation (2) in the text

$$L[e^{rt}] = e^{rt}Z(r) \tag{27}$$

and differentiate repeatedly with respect to r, setting $r = r_1$ after each differentiation.
 a. Recall that if r_1 is a root of multiplicity s, then $Z(r) = (r - r_1)^s q(r)$, where $q(r)$ is a polynomial of degree $n - s$ and $q(r_1) \neq 0$. Show that $Z(r_1), Z'(r_1), \dots, Z^{(s-1)}(r_1)$ are all zero, but $Z^{(s)}(r_1) \neq 0$.
 b. By differentiating equation (27) repeatedly with respect to r, show that

$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}],$$
$$\vdots$$
$$\frac{\partial^{s-1}}{\partial r^{s-1}}L[e^{rt}] = L[t^{s-1}e^{rt}].$$

 c. Show that $e^{r_1t}, te^{r_1t}, \dots, t^{s-1}e^{r_1t}$ are solutions of equation (27).

4.3 The Method of Undetermined Coefficients

A particular solution Y of the nonhomogeneous n^{th} order linear differential equation with constant coefficients

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \dots + a_{n-1}y' + a_ny = g(t) \tag{1}$$

can be obtained by the method of undetermined coefficients, provided the nonhomogeneous term $g(t)$ is of an appropriate form. Although the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when it is applicable.

You should keep in mind that the amount of algebra required to calculate the coefficients may be quite substantial for higher-order equations, especially if the nonhomogeneous term is even moderately complicated. A computer algebra system can be extremely helpful in executing these algebraic calculations.

The method of undetermined coefficients can be used whenever it is possible to guess the correct form for $Y(t)$. However, this is usually impossible for differential equations not having constant coefficients, or for nonhomogeneous terms other than the type described previously. For more complicated problems we can use the method of variation of parameters, which is discussed in the next section.

Problems

In each of Problems 1 through 6, determine the general solution of the given differential equation.

1. $y''' - y'' - y' + y = 2e^{-t} + 3$
2. $y^{(4)} - y = 3t + \cos t$
3. $y''' + y'' + y' + y = e^{-t} + 4t$
4. $y^{(4)} - 4y'' = t^2 + e^t$
5. $y^{(4)} + 2y'' + y = 3 + \cos 2t$
6. $y^{(6)} + y''' = t$

In each of Problems 7 through 9, find the solution of the given initial-value problem. Then plot a graph of the solution.

7. $y''' + 4y' = t;\quad y(0) = y'(0) = 0,\quad y''(0) = 1$
8. $y^{(4)} + 2y'' + y = 3t + 4;\quad y(0) = y'(0) = 0,$ $y''(0) = y'''(0) = 1$
9. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t};$ $y(0) = 3,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = 2$

In each of Problems 10 through 13, determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

10. $y''' - 2y'' + y' = t^3 + 2e^t$
11. $y''' - y' = te^{-t} + 2\cos t$
12. $y^{(4)} - y''' - y'' + y' = t^2 + 4 + t\sin t$
13. $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$
14. Consider the nonhomogeneous n^{th} order linear differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t), \tag{10}$$

where $a_0, \ldots, a_n$ are constants. Verify that if $g(t)$ is of the form

$$e^{\alpha t}(b_0 t^m + \cdots + b_m),$$

then the substitution $y = e^{\alpha t}u(t)$ reduces equation (10) to the form

$$k_0 u^{(n)} + k_1 u^{(n-1)} + \cdots + k_n u = b_0 t^m + \cdots + b_m, \tag{11}$$

where $k_0, \ldots, k_n$ are constants. Determine k_0 and k_n in terms of the a's and α. Thus the problem of determining a particular solution of the original equation is reduced to the simpler problem of determining a particular solution of an equation with constant coefficients and a polynomial for the nonhomogeneous term.

Method of Annihilators. In Problems 15 through 17, we consider another way of arriving at the proper form of $Y(t)$ for use in the method of undetermined coefficients. The procedure is based on the observation that exponential, polynomial, or sinusoidal terms (or sums and products of such terms) can be viewed as solutions of certain linear homogeneous differential equations with constant coefficients. It is convenient to use the symbol D for $\dfrac{d}{dt}$. Then, for example, e^{-t} is a solution of $(D+1)y = 0$; the differential operator $D+1$ is said to *annihilate*, or to be an *annihilator* of, e^{-t}. In the same way, $D^2 + 4$ is an annihilator of $\sin 2t$ or $\cos 2t$, $(D-3)^2 = D^2 - 6D + 9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

15. Show that linear differential operators with constant coefficients obey the commutative law. That is, show that

$$(D-a)(D-b)f = (D-b)(D-a)f$$

for any twice-differentiable function f and any constants a and b. The result extends at once to any finite number of factors.

16. Consider the problem of finding the form of a particular solution $Y(t)$ of

$$(D-2)^3(D+1)Y = 3e^{2t} - te^{-t}, \tag{12}$$

where the left-hand side of the equation is written in a form corresponding to the factorization of the characteristic polynomial.

a. Show that $D-2$ and $(D+1)^2$, respectively, are annihilators of the terms on the right-hand side of equation (12), and that the combined operator $(D-2)(D+1)^2$ annihilates both terms on the right-hand side of equation (12) simultaneously.

b. Apply the operator $(D-2)(D+1)^2$ to equation (12) and use the result of Problem 15 to obtain

$$(D-2)^4(D+1)^3 Y = 0. \tag{13}$$

Thus Y is a solution of the homogeneous equation (13). By solving equation (13), show that

$$Y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}, \tag{14}$$

where $c_1, \ldots, c_7$ are constants, as yet undetermined.

c. Observe that e^{2t}, te^{2t}, t^2e^{2t}, and e^{-t} are solutions of the homogeneous equation corresponding to equation (12); hence these terms are not useful in solving the nonhomogeneous equation. Therefore, choose c_1, c_2, c_3, and c_5 to be zero in equation (14), so that

$$Y(t) = c_4 t^3 e^{2t} + c_6 te^{-t} + c_7 t^2 e^{-t}. \tag{15}$$

This is the form of the particular solution Y of equation (12). The values of the coefficients c_4, c_6, and c_7 can be found by substituting from equation (15) in the differential equation (12).

Summary of the Method of Annihilators. Suppose that

$$L(D)y = g(t), \tag{16}$$

where $L(D)$ is a linear differential operator with constant coefficients, and $g(t)$ is a sum or product of exponential, polynomial, or sinusoidal terms. To find the form of a particular solution of equation (16), you can proceed as follows:

a. Find a differential operator $H(D)$ with constant coefficients that annihilates $g(t)$—that is, an operator such that $H(D)g(t) = 0$.

b. Apply $H(D)$ to equation (16), obtaining

$$H(D)L(D)y = 0, \tag{17}$$

which is a homogeneous equation of higher-order.

c. Solve equation (17).

d. Eliminate from the solution found in step c the terms that also appear in the solution of $L(D)y = 0$. The remaining terms constitute the correct form of a particular solution of equation (16).

17. Use the method of annihilators to find the form of a particular solution $Y(t)$ for each of the equations in Problems 10 through 13. Do not evaluate the coefficients.

4.4 The Method of Variation of Parameters

The method of variation of parameters for determining a particular solution of the nonhomogeneous n^{th} order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{1}$$

is a direct extension of the method for the second-order differential equation (see Section 3.6). As before, to use the method of variation of parameters, it is first necessary to solve the corresponding homogeneous differential equation. In general, this may be difficult unless the coefficients are constants. However, the method of variation of parameters is still more general than the method of undetermined coefficients in that it leads to an expression for the particular solution for *any* continuous function g, whereas the method of undetermined coefficients is restricted in practice to a limited class of functions g.

Suppose then that we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ of the homogeneous equation. Then the general solution of the homogeneous equation is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t). \tag{2}$$

The method of variation of parameters for determining a particular solution of equation (1) rests on the possibility of determining n functions $u_1, u_2, \ldots, u_n$ such that $Y(t)$ is of the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t). \tag{3}$$

Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that Y satisfy equation (1). The other $n-1$ conditions are chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining Y if we must solve high order differential equations for $u_1, \ldots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \ldots, u_n$. From equation (3) we obtain

$$Y' = (u_1 y_1' + u_2 y_2' + \cdots + u_n y_n') + (u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n), \tag{4}$$

where we have omitted the independent variable t on which each function in equation (4) depends. Thus the first condition that we impose is that

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0. \tag{5}$$

It follows that the expression (4) for Y' reduces to

$$Y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'. \tag{6}$$

We continue this process by calculating the successive derivatives $Y'', \ldots, Y^{(n-1)}$. After each differentiation we set equal to zero the sum of terms involving derivatives of $u_1, \ldots, u_n$. In this way we obtain $n-2$ further conditions similar to equation (5); that is,

$$u_1' y_1^{(m)} + u_2' y_2^{(m)} + \cdots + u_n' y_n^{(m)} = 0, \quad m = 1, 2, \ldots, n-2. \tag{7}$$

As a result of these conditions, it follows that the expressions for $Y'', \ldots, Y^{(n-1)}$ reduce to

$$Y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 2, 3, \ldots, n-1. \tag{8}$$

Problems

In each of Problems 1 through 4, use the method of variation of parameters to determine the general solution of the given differential equation.

1. $y''' + y' = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
2. $y''' - y' = t$
3. $y''' - 2y'' - y' + 2y = e^{4t}$
4. $y''' - y'' + y' - y = e^{-t}\sin t$

In each of Problems 5 and 6, find the general solution of the given differential equation. Leave your answer in terms of one or more integrals.

5. $y''' - y'' + y' - y = \sec t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$
6. $y''' - y' = \csc t, \quad 0 < t < \pi$

In each of Problems 7 and 8, find the solution of the given initial-value problem. Then plot a graph of the solution.

G 7. $y''' - y'' + y' - y = \sec t; \quad y(0) = 2, \ y'(0) = -1, \ y''(0) = 1$

G 8. $y''' - y' = \tan t; \quad y\left(\frac{\pi}{4}\right) = 2, \ y'\left(\frac{\pi}{4}\right) = 1, \ y''\left(\frac{\pi}{4}\right) = -1$

9. Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \quad x > 0,$$

determine a particular solution.

10. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - y'' + y' - y = g(t).$$

11. Find a formula involving integrals for a particular solution of the differential equation

$$y^{(4)} - y = g(t).$$

Hint: The functions $\sin t, \cos t, \sinh t$, and $\cosh t$ form a fundamental set of solutions of the homogeneous equation.

12. Find a formula involving integrals for a particular solution of the differential equation

$$y''' - 3y'' + 3y' - y = g(t).$$

If $g(t) = t^{-2}e^t$, determine $Y(t)$.

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CHAPTER 5

Series Solutions of Second-Order Linear Equations

Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation. So far, we have given a systematic procedure for constructing fundamental solutions only when the equation has constant coefficients. To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: we assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

5.1 Review of Power Series

In this chapter we discuss the use of power series to construct fundamental sets of solutions of second-order linear differential equations whose coefficients are functions of the independent variable. We begin by summarizing very briefly the pertinent results about power series that we need. Readers who are familiar with power series may go on to Section 5.2. Those who need more details than are presented here should consult a book on calculus.

1. A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to *converge at a point x* if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x - x_0)^n$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and not for others.

2. The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to *converge absolutely at a point x* if the associated power series

$$\sum_{n=0}^{\infty} |a_n(x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n$$

converges. It can be shown that if the power series converges absolutely, then the power series also converges; however, the converse is not necessarily true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test: If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0|L,$$