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A valuable compendium of methods for solving differential equations is

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For further discussion and examples of nonlinear phenomena, including bifurcations and chaos, see

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Devaney, R. L., *Chaos, Fractals, and Dynamics* (Reading, MA: Addison-Wesley, 1990).

CHAPTER 3

Second-Order Linear Differential Equations

Linear differential equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and of these methods is understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second-order equations. The second reason to study second-order linear differential equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second-order linear differential equations. We illustrate this at the end of this chapter with a discussion of the oscillations of some basic mechanical and electrical systems.

3.1 Homogeneous Differential Equations with Constant Coefficients

Many second-order ordinary differential equations have the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and dy/dt. In equation (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite equation (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of equation (3), we sometimes see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide equation (4) by $P(t)$ and thereby obtain equation (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \quad q(t) = \frac{R(t)}{P(t)}, \quad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

whose roots are $r = \frac{3}{2}$ and $r = \frac{1}{2}$. Therefore, the general solution of the differential equation is

$$y = c_1 e^{3t/2} + c_2 e^{t/2}. \tag{33}$$

Applying the initial conditions, we obtain the following two equations for c_1 and c_2:

$$c_1 + c_2 = 2, \quad \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, so the solution of the initial value problem (32) is

$$y = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}. \tag{34}$$

Figure 3.1.2 shows the graph of the solution.

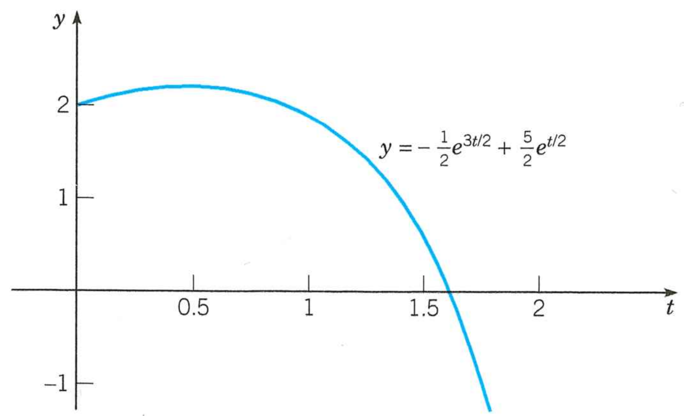

FIGURE 3.1.2 Solution of the initial value problem (32): $4y'' - 8y' + 3y = 0,\ y(0) = 2,\ y'(0) = 1/2$.

EXAMPLE 5

The solution (31) of the initial value problem (28) initially increases (because its initial slope is positive), but eventually approaches zero (because both terms involve negative exponential functions). Therefore, the solution must have a maximum point, and the graph in Figure 3.1.1 confirms this. Determine the location of this maximum point.

Solution:

The coordinates of the maximum point can be estimated from the graph, but to find them more precisely, we seek the point where the solution has a horizontal tangent line. By differentiating the solution (31), $y = 9e^{-2t} - 7e^{-3t}$, with respect to t, we obtain

$$y' = -18e^{-2t} + 21e^{-3t}. \tag{35}$$

Setting y' equal to zero and multiplying by e^{3t}, we find that the critical value t_m satisfies $e^t = 7/6$; hence

$$t_m = \ln(7/6) \cong 0.15415. \tag{36}$$

The corresponding maximum value y_m is given by

$$y_m = 9e^{-2t_m} - 7e^{-3t_m} = \frac{108}{49} \cong 2.20408. \tag{37}$$

In this example the initial slope is 3, but the solution of the given differential equation behaves in a similar way for any other positive initial slope. In Problem 19 you are asked to determine how the coordinates of the maximum point depend on the initial slope.

Returning to the equation $ay'' + by' + cy = 0$ with arbitrary coefficients, recall that when $r_1 \neq r_2$, its general solution (18) is the sum of two exponential functions. Therefore, the solution has a relatively simple geometrical behavior: as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else exhibits unbounded growth (when at least one exponent is positive). These two cases are illustrated by the solutions of Examples 3 and 4, which are shown in Figures 3.1.1 and 3.1.2, respectively. Note that whether a growing solution approaches $+\infty$ or $-\infty$ as $t \to \infty$ is determined by the sign of the coefficient of the exponential for the larger root of the characteristic equation. (See Problem 21.) There is also a third case that occurs less often: the solution approaches a constant when one exponent is zero and the other is negative.

In Sections 3.3 and 3.4, respectively, we return to the problem of solving the equation $ay''+by'+cy = 0$ when the roots of the characteristic equation either are complex conjugates or are real and equal. In the meantime, in Section 3.2, we provide a systematic discussion of the mathematical structure of the solutions of all second-order linear homogeneous equations.

Problems

In each of Problems 1 through 6, find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$
2. $y'' + 3y' + 2y = 0$
3. $6y'' - y' - y = 0$
4. $y'' + 5y' = 0$
5. $4y'' - 9y = 0$
6. $y'' - 2y' - 2y = 0$

In each of Problems 7 through 12, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

G 7. $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$

G 8. $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$

G 9. $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$

G 10. $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

G 11. $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$

G 12. $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$

13. Find a differential equation whose general solution is $y = c_1 e^{2t} + c_2 e^{-3t}$.

G 14. Find the solution of the initial value problem

$$y'' - y = 0, \quad y(0) = \frac{5}{4}, \quad y'(0) = -\frac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

15. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

16. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.

In each of Problems 17 and 18, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

17. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$
18. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$
19. Consider the initial value problem (see Example 5)

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

   a. Solve the initial value problem.
   b. Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
   c. Determine the smallest value of β for which $y_m \ge 4$.
   d. Determine the behavior of t_m and y_m as $\beta \to \infty$.

20. Consider the equation $ay'' + by' + cy = d$, where a, b, c, and d are constants.
   a. Find all equilibrium, or constant, solutions of this differential equation.
   b. Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

21. Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:
   a. real, different, and negative.
   b. real with opposite signs.
   c. real, different, and positive.

In each case, determine the behavior of the solution as $t \to \infty$.

Next, we let $W(t) = W[y_1, y_2](t)$ and observe that

$$W' = y_1 y_2'' - y_1'' y_2. \tag{26}$$

Then we can write equation (25) in the form

$$W' + p(t)W = 0. \tag{27}$$

Equation (27) can be solved immediately since it is both a first-order linear differential equation (Section 2.1) and a separable differential equation (Section 2.2). Thus

$$W(t) = c\exp\left(-\int p(t)\,dt\right), \tag{28}$$

where c is a constant.

The value of c depends on which pair of solutions of equation (22) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t. This completes the proof of Theorem 3.2.7.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

EXAMPLE 7

In Example 5 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0. \tag{29}$$

Verify that the Wronskian of y_1 and y_2 is given by Abel's formula (23).

Solution:

From the example just cited we know that $W[y_1, y_2](t) = -\frac{3}{2}t^{-3/2}$. To use equation (23), we must write the differential equation (29) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = \dfrac{3}{2t}$. Hence

$$W[y_1, y_2](t) = c\exp\left(-\int \frac{3}{2t}\,dt\right) = c\exp\left(-\frac{3}{2}\ln t\right)$$
$$= c\,t^{-3/2}. \tag{30}$$

Equation (30) gives the Wronskian of any pair of solutions of equation (29). For the particular solutions given in this example, we must choose $c = -\dfrac{3}{2}$.

Summary. We can summarize the discussion in this section as follows: to find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions, and the general solution is

$$y = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a given point in $\alpha < t < \beta$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

Problems

In each of Problems 1 through 5, find the Wronskian of the given pair of functions.

1. $e^{2t}, \; e^{-3t/2}$
2. $\cos t, \; \sin t$
3. $e^{-2t}, \; te^{-2t}$
4. $e^t \sin t, \; e^t \cos t$
5. $\cos^2\theta, \; 1 + \cos(2\theta)$

In each of Problems 6 through 9, determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

6. $ty'' + 3y = t, \quad y(1) = 1, \; y'(1) = 2$
7. $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3) = 0, \; y'(3) = -1$
8. $y'' + (\cos t)y' + 3(\ln|t|)y = 0, \quad y(2) = 3, \; y'(2) = 1$
9. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \; y'(3) = 2$
10. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.
11. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.
12. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.
13. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.
14. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.
15. If the Wronskian of f and g is $t\cos t - \sin t$, and if $u = f + 3g$, $v = f - g$, find the Wronskian of u and v.
16. Assume that y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where a_1, a_2, b_1, and b_2 are any constants. Show that
$$W[y_3, y_4] = (a_1b_2 - a_2b_1)W[y_1, y_2].$$
Are y_3 and y_4 also a fundamental set of solutions? Why or why not?

In each of Problems 17 and 18, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

17. $y'' + y' - 2y = 0, \quad t_0 = 0$
18. $y'' + 4y' + 3y = 0, \quad t_0 = 1$

In each of Problems 19 through 21, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

19. $y'' + 4y = 0; \quad y_1(t) = \cos(2t), \; y_2(t) = \sin(2t)$
20. $y'' - 2y' + y = 0; \quad y_1(t) = e^t, \; y_2(t) = te^t$
21. $x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0;$
$y_1(x) = x, \; y_2(x) = xe^x$
22. Consider the equation $y'' - y' - 2y = 0$.
    a. Show that $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions.
    b. Let $y_3(t) = -2e^{2t}$, $y_4(t) = y_1(t) + 2y_2(t)$, and $y_5(t) = 2y_1(t) - 2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?
    c. Determine whether each of the following pairs forms a fundamental set of solutions: $\{y_1(t), y_3(t)\}$; $\{y_2(t), y_3(t)\}$; $\{y_1(t), y_4(t)\}$; $\{y_4(t), y_5(t)\}$.

In each of Problems 23 through 25, find the Wronskian of two solutions of the given differential equation without solving the equation.

23. $t^2y'' - t(t+2)y' + (t+2)y = 0$
24. $(\cos t)y'' + (\sin t)y' - ty = 0$
25. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad$ Legendre's equation
26. Show that if p is differentiable and $p(t) > 0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']' + q(t)y = 0$ is $W(t) = c/p(t)$, where c is a constant.
27. If the differential equation $ty'' + 2y' + te^ty = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W[y_1, y_2](1) = 2$, find the value of $W[y_1, y_2](5)$.
28. If the Wronskian of any two solutions of $y'' + p(t)y' + q(t)y = 0$ is constant, what does this imply about the coefficients p and q?

In Problems 29 and 30, assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I.

29. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.
30. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.
31. **Exact Equations.** The equation
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
is said to be exact if it can be written in the form
$$(P(x)y')' + (f(x)y)' = 0,$$
where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first-order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is
$$P''(x) - Q'(x) + R(x) = 0.$$
It can be shown that this is also a sufficient condition.

In each of Problems 32 through 34, use the result of Problem 31 to determine whether the given equation is exact. If it is, then solve the equation.

32. $y'' + xy' + y = 0$
33. $xy'' - (\cos x)y' + (\sin x)y = 0, \quad x > 0$
34. $x^2y'' + xy' - y = 0, \quad x > 0$

35. The Adjoint Equation. If a second-order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

can be written in the form

$$(\mu(x)P(x)y')' + (f(x)y)' = 0.$$

By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the **adjoint** of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second-order equation.

In each of Problems 36 and 37, use the result of Problem 35 to find the adjoint of the given differential equation.

36. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation

37. $y'' - xy = 0$, Airy's equation

38. A second-order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be **self-adjoint** if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 36 and 37 is self-adjoint.

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the second-order linear differential equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of equation (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of equation (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp((\lambda - i\mu)t). \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from equation (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in equations (5), we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to discover how this extension of the exponential function should be defined. Here we use a method based on infinite series; an alternative is outlined in Problem 20.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty}\frac{t^n}{n!}, \quad -\infty < t < \infty. \tag{7}$$

If we now assume that we can substitute it for t in equation (7), then we have

$$e^{it} = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}. \tag{8}$$

To simplify this series, we write $(it)^n = i^nt^n$ and make use of the facts that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. When n is even, there is an integer k with $n = 2k$; in this case $i^n = i^{2k} = (-1)^k$. And when n is odd, $n = 2k + 1$, so $i^n = i^{2k+1} = i(-1)^k$. This suggests separating the terms in the right-hand side of (8) into its real and imaginary parts. The result is[5]

$$e^{it} = \sum_{k=0}^{\infty}\frac{(-1)^kt^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}\frac{(-1)^kt^{2k+1}}{(2k+1)!}. \tag{9}$$

The first series in equation (9) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i\sin t. \tag{10}$$

Equation (10) is known as **Euler's formula** and is an extremely important mathematical relationship.

Although our derivation of equation (10) is based on the unverified assumption that the series (7) can be used for complex as well as real values of the independent variable, our intention is to use this derivation only to make equation (10) seem plausible. We now put matters on a firm foundation by adopting equation (10) as the *definition* of e^{it}. In other words, whenever we write e^{it}, we mean the expression on the right-hand side of equation (10).

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in equation (10) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i\sin t. \tag{11}$$

Further, if t is replaced by μt in equation (10), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t). \tag{12}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp((\lambda + i\mu)t)$ to satisfy

$$e^{(\lambda+i\mu)t} = e^{\lambda t}e^{i\mu t}. \tag{13}$$

Then, substituting for $e^{i\mu t}$ from equation (12), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\ &= e^{\lambda t}\cos(\mu t) + ie^{\lambda t}\sin(\mu t). \end{aligned} \tag{14}$$

We now take equation (14) as the definition of $\exp[(\lambda + i\mu)t]$. The value of the exponential function with a complex exponent is a complex number whose real and imaginary parts are given by the terms on the right-hand side of equation (14). Observe that the real and imaginary parts of $\exp((\lambda + i\mu)t)$ are expressed entirely in terms of elementary real-valued functions. For example, the quantity in equation (6) has the value

$$e^{-3+6i} = e^{-3}\cos 6 + ie^{-3}\sin 6 \cong 0.0478041 - 0.0139113i.$$

With the definitions (10) and (14), it is straightforward to show that the usual laws of exponents are valid for the complex exponential function. You can also use equation (14) to verify that the differentiation formula

$$\frac{d}{dt}(e^{rt}) = re^{rt} \tag{15}$$

holds for complex values of r.

[5] Recall from calculus that the reordering of terms in the right-hand side of equation (9) is allowed because the series converges absolutely for all $-\infty < t < \infty$.

Solution:

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = \frac{1}{4} \pm 3i$. Thus the general solution of the differential equation is

$$y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t). \tag{27}$$

To apply the first initial condition, we set $t = 0$ in equation (27); this gives

$$y(0) = c_1 = -2.$$

For the second initial condition, we must differentiate equation (27) before substituting $t = 0$. In this way we find that

$$y'(0) = \frac{1}{4}c_1 + 3c_2 = 1,$$

from which we determine that $c_2 = \frac{1}{2}$. Using these values of c_1 and c_2 in the general solution (27), we obtain

$$y = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t) \tag{28}$$

as the solution of the initial value problem (26). The graph of this solution is shown in Figure 3.3.2.

In this case we observe that the solution is a growing oscillation. Again the trigonometric factors in equation (28) determine the oscillatory part of the solution (again with period $2\pi/3$), while the exponential factor (with a positive exponent this time) causes the magnitude of the oscillation to increase with time.

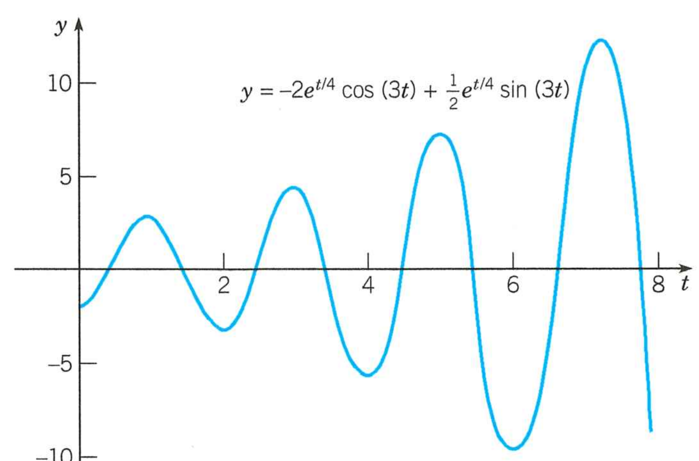

FIGURE 3.3.2 Solution of the initial value problem (26): $16y'' - 8y' + 145y = 0,\ y(0) = -2,\ y'(0) = 1.$

EXAMPLE 3

Find the general solution of

$$y'' + 9y = 0. \tag{29}$$

Solution:

The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$; thus $\lambda = 0$ and $\mu = 3$. The general solution is

$$y = c_1\cos(3t) + c_2\sin(3t). \tag{30}$$

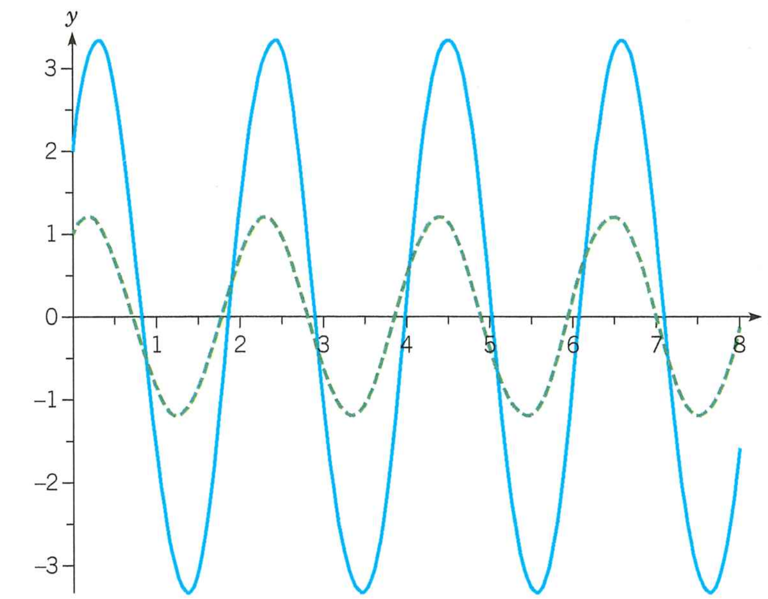

FIGURE 3.3.3 Solutions of equation (29): $y'' + 9y = 0$, with two sets of initial conditions: $y(0) = 1, y'(0) = 2$ (dashed, green) and $y(0) = 2, y'(0) = 8$ (solid, blue). Both solutions have the same period, but different amplitudes and phase shifts.

Note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution. Figure 3.3.3 shows the graph of two solutions of equation (28) with different initial conditions. In each case the solution is a pure oscillation with period $2\pi/3$ but whose amplitude and phase shift are determined by the initial conditions. Since there is no exponential factor in the solution (30), the amplitude of each oscillation remains constant in time.

Problems

In each of Problems 1 through 4, use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(2 - 3i)$
2. $e^{i\pi}$
3. $e^{2-(\pi/2)i}$
4. 2^{1-i}

In each of Problems 5 through 11, find the general solution of the given differential equation.

5. $y'' - 2y' + 2y = 0$
6. $y'' - 2y' + 6y = 0$
7. $y'' + 2y' + 2y = 0$
8. $y'' + 6y' + 13y = 0$
9. $y'' + 2y' + 1.25y = 0$
10. $9y'' + 9y' - 4y = 0$
11. $y'' + 4y' + 6.25y = 0$

In each of Problems 12 through 15, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

G 12. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

G 13. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

G 14. $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$

G 15. $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

N 16. Consider the initial value problem

$$3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

a. Find the solution $u(t)$ of this problem.
b. For $t > 0$, find the first time at which $|u(t)| = 10$.

N 17. Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \quad u(0) = 2, \quad u'(0) = 1.$$

a. Find the solution $u(t)$ of this problem.
b. Find the smallest T such that $|u(t)| \le 0.1$ for all $t > T$.

N 18. Consider the initial value problem

$$y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \ge 0.$$

a. Find the solution $y(t)$ of this problem.
b. Find α such that $y = 0$ when $t = 1$.
c. Find, as a function of α, the smallest positive value of t for which $y = 0$.
d. Determine the limit of the expression found in part c as $\alpha \to \infty$.

19. Show that $W\left[e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\right] = \mu e^{2\lambda t}$.

20. In this problem we outline a different derivation of Euler's formula.

a. Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.

b. Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{31}$$

for some constants c_1 and c_2. Why is this so?

c. Set $t = 0$ in equation (31) to show that $c_1 = 1$.

d. Assuming that equation (15) is true, differentiate equation (31) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in equation (31) to arrive at Euler's formula.

21. Using Euler's formula, show that

$$\frac{e^{it} + e^{-it}}{2} = \cos t, \qquad \frac{e^{it} - e^{-it}}{2i} = \sin t.$$

22. If e^{rt} is given by equation (14), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.

23. Consider the differential equation

$$ay'' + by' + cy = 0,$$

where $b^2 - 4ac < 0$ and the characteristic equation has complex roots $\lambda \pm i\mu$. Substitute the functions

$$u(t) = e^{\lambda t}\cos(\mu t) \quad \text{and} \quad v(t) = e^{\lambda t}\sin(\mu t)$$

for y in the differential equation and thereby confirm that they are solutions.

24. If the functions y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.
Hint: Suppose that t_1 and t_2 are two zeros of y_1 between which there are no zeros of y_2. Apply Rolle's theorem to y_1/y_2 to reach a contradiction.

Change of Variables. Sometimes a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \tag{32}$$

can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 25 through 36. In particular, in Problem 25 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 26 through 31 are examples of this type of equation. Problem 32 determines conditions under which the more general equation (32) can be transformed into a differential equation with constant coefficients. Problems 33 through 36 give specific applications of this procedure.

25. Euler Equations. An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \quad t > 0, \tag{33}$$

where α and β are real constants, is called an **Euler equation.**

a. Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

b. Use the results of part a to transform equation (33) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{34}$$

Observe that differential equation (34) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of equation (34), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of equation (33).

In each of Problems 26 through 31, use the method of Problem 25 to solve the given equation for $t > 0$.

26. $t^2y'' + ty' + y = 0$

27. $t^2y'' + 4ty' + 2y = 0$

28. $t^2y'' - 4ty' - 6y = 0$

29. $t^2y'' - 4ty' + 6y = 0$

30. $t^2y'' + 3ty' - 3y = 0$

31. $t^2y'' + 7ty' + 10y = 0$

32. In this problem we determine conditions on p and q that enable equation (32) to be transformed into an equation with constant coefficients by a change of the independent variable. Let $x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

a. Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

b. Show that the differential equation (32) becomes

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{35}$$

c. In order for equation (35) to have constant coefficients, the coefficients of d^2y/dx^2, dy/dx, and y must all be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence, after integrating with respect to t,

$$x = u(t) = \int (q(t))^{1/2}\,dt. \tag{36}$$

d. With x chosen as in part c, show that the coefficient of dy/dx in equation (35) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2(q(t))^{3/2}} \tag{37}$$

is a constant. Thus equation (32) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant.

e. How must the analysis and results in d be modified if $q(t) < 0$?

In each of Problems 33 through 36, try to transform the given equation into one with constant coefficients by the method of Problem 32. If this is possible, find the general solution of the given equation.

33. $y'' + ty' + e^{-t^2}y = 0, \quad -\infty < t < \infty$

34. $y'' + 3ty' + t^2y = 0, \quad -\infty < t < \infty$

35. $ty'' + (t^2 - 1)y' + t^3y = 0, \quad 0 < t < \infty$

36. $y'' + ty' - e^{-t^2}y = 0$

3.4 Repeated Roots; Reduction of Order

In Sections 3.1 and 3.3 we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional between the other two and occurs when the discriminant $b^2 - 4ac$ is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -\frac{b}{2a}. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/(2a)} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE 1

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{5}$$

Solution:

The characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore, one solution of equation (5) is $y_1(t) = e^{-2t}$. To find the general solution of equation (5), we need a second solution that is not a constant multiple of y_1. This second solution can be found in several ways (see Problems 15 through 17); here we use a method originated by d'Alembert[7] in the eighteenth century. Recall that since $y_1(t)$ is a solution of equation (1), so is $cy_1(t)$ for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is also a solution of equation (1).

To carry out this program, we substitute $y = v(t)y_1(t)$ in equation (5) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-2t}, \tag{6}$$

we differentiate once to find

$$y' = v'(t)e^{-2t} - 2v(t)e^{-2t} \tag{7}$$

and a second differentiation yields

$$y'' = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}. \tag{8}$$

By substituting the expressions in equations (6), (7), and (8) in equation (5) and collecting terms, we obtain

$$(v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t))e^{-2t} = 0,$$

[7]Jean d'Alembert (1717–1783), a French mathematician, was a contemporary of Euler and Daniel Bernoulli and is known primarily for his work in mechanics and differential equations. D'Alembert's principle in mechanics and d'Alembert's paradox in hydrodynamics are named for him, and the wave equation first appeared in his paper on vibrating strings in 1747. In his later years he devoted himself primarily to philosophy and to his duties as science editor of Diderot's *Encyclopédie*.

Note that the coefficient of v is zero, as it should be; this provides a useful check on our algebraic calculations.

If we let $w = v'$, then the second-order linear differential equation (34) reduces to the separable first-order differential equation

$$2tw' - w = 0.$$

Separating the variables and solving for $w(t)$, we find that

$$w(t) = v'(t) = ct^{1/2};$$

then, one final integration yields

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = v(t)t^{-1} = \frac{2}{3}ct^{1/2} + kt^{-1}, \tag{35}$$

where c and k are arbitrary constants. The second term on the right-hand side of equation (35) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2(t) = t^{1/2}$. You can verify that the Wronskian of y_1 and y_2 is

$$W[y_1, y_2](t) = \frac{3}{2}t^{-3/2} \neq 0 \text{ for } t > 0. \tag{36}$$

Consequently, y_1 and y_2 form a fundamental set of solutions of equation (33) for $t > 0$.

Problems

In each of Problems 1 through 8, find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $y'' - 2y' + 10y = 0$
5. $y'' - 6y' + 9y = 0$
6. $4y'' + 17y' + 4y = 0$
7. $16y'' + 24y' + 9y = 0$
8. $2y'' + 2y' + y = 0$

In each of Problems 9 through 11, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

9. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
10. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
11. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$
12. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b, and then determine the critical value of b that separates solutions that remain positive for all $t > 0$ from those that eventually become negative.

N 13. Consider the initial value problem

$$4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

a. Solve the initial value problem and plot the solution.
b. Determine the coordinates (t_M, y_M) of the maximum point.
c. Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
d. Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

14. Consider the equation $ay'' + by' + cy = 0$. If the roots of the corresponding characteristic equation are real, show that a solution to the differential equation either is everywhere zero or else can take on the value zero at most once.

Problems 15 through 17 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

15. a. Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$ so that one solution of the equation is e^{-at}.
b. Use Abel's formula [equation (23) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.
c. Let $y_1(t) = e^{-at}$ and use the result of part b to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

16. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1t)$ and $\exp(r_2t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that

$$\phi(t; r_1, r_2) = \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1}$$

is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed, and use l'Hôpital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

17. a. If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L\left[e^{rt}\right] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2e^{rt}. \tag{37}$$

Since the right-hand side of equation (37) is zero when $r = r_1$, it follows that $\exp(r_1t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.
b. Differentiate equation (37) with respect to r, and interchange differentiation with respect to r and with respect to t, thus showing that

$$\begin{aligned}\frac{\partial}{\partial r}L\left[e^{rt}\right] &= L\left[\frac{\partial}{\partial r}e^{rt}\right] = L\left[te^{rt}\right]\\ &= ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1).\end{aligned} \tag{38}$$

Since the right-hand side of equation (36) is zero when $r = r_1$, conclude that $t\exp(r_1t)$ is also a solution of $L[y] = 0$.

In each of Problems 18 through 22, use the method of reduction of order to find a second solution of the given differential equation.

18. $t^2y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$
19. $t^2y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$
20. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
21. $xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin(x^2)$
22. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$
23. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution, and then find the general solution in the form of an integral.

24. The method of Problem 15 can be extended to second-order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W[y_1, y_2]/y_1^2$, where $W[y_1, y_2]$ is the Wronskian of y_1 and y_2. Then use Abel's formula (equation (23) of Section 3.2) to determine y_2.

In each of Problems 25 through 27, use the method of Problem 24 to find a second independent solution of the given equation.

25. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
26. $ty'' - y' + 4t^3y = 0, \quad t > 0; \quad y_1(t) = \sin(t^2)$
27. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$

Behavior of Solutions as $t \to \infty$. Problems 28 through 30 are concerned with the behavior of solutions as $t \to \infty$.

28. If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.
29. a. If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 28 is no longer true, but that all solutions are bounded as $t \to \infty$.
b. If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 28 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y_0'$.
30. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t\sin t)y = 0$$

for any value of the constant k. If $0 < k < 2$, show that $1 - k\cos t\sin t > 0$ and $k\sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable-coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 28. Thus we observe a not unusual situation in the study of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. In each of Problems 31 through 34, use the substitution introduced in Problem 25 in Section 3.3 to solve the given differential equation.

31. $t^2y'' - 3ty' + 4y = 0, \quad t > 0$
32. $t^2y'' + 2ty' + 0.25y = 0, \quad t > 0$
33. $t^2y'' + 3ty' + y = 0, \quad t > 0$
34. $4t^2y'' - 8ty' + 9y = 0, \quad t > 0$

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now turn our attention to the nonhomogeneous second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{1}$$

where p, q, and g are given (continuous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{2}$$

To obtain a particular solution, we assume that

$$Y(t) = A_0t^n + A_1t^{n-1} + \cdots + A_{n-2}t^2 + A_{n-1}t + A_n. \qquad (29)$$

Substituting in equation (28), we obtain

$$a\left(n(n-1)A_0t^{n-2} + \cdots + 2A_{n-2}\right) + b(nA_0t^{n-1} + \cdots + A_{n-1})$$
$$+ c(A_0t^n + A_1t^{n-1} + \cdots + A_n) = a_0t^n + \cdots + a_n. \qquad (30)$$

Equating the coefficients of like powers of t, beginning with t^n, leads to the following sequence of equations:

$$cA_0 = a_0,$$
$$cA_1 + nbA_0 = a_1,$$
$$\vdots$$
$$cA_n + bA_{n-1} + 2aA_{n-2} = a_n.$$

Provided that $c \neq 0$, the solution of the first equation is $A_0 = a_0/c$, and the remaining equations determine $A_1, \ldots, A_n$ successively.

If $c = 0$ but $b \neq 0$, then the polynomial on the left-hand side of equation (30) is of degree $n-1$, and we cannot satisfy equation (30). To be sure that $aY''(t) + bY'(t)$ is a polynomial of degree n, we must choose $Y(t)$ to be a polynomial of degree $n+1$. Hence we assume that

$$Y(t) = t(A_0t^n + \cdots + A_n).$$

Substituting this guess into equation (28), with $c = 0$, and simplifying yields

$$aY'' + bY' = bA_0(n+1)t^n + (aA_0(n+1)n + bA_1n)t^{n-1} + \cdots$$
$$= a_0t^n + a_1t^{n-1} + \cdots + a_n.$$

There is no constant term in this expression for $Y(t)$, but there is no need to include such a term since a constant is a solution of the homogeneous equation when $c = 0$. Since $b \neq 0$, we find $A_0 = a_0/(b(n+1))$, and the other coefficients $A_1, \ldots, A_n$ can be determined similarly.

If both c and b are zero, then the characteristic equation is $ar^2 = 0$ and $r = 0$ is a repeated root. Thus $y_1 = e^{0t} = 1$ and $y_2 = te^{0t} = t$ form a fundamental set of solutions of the corresponding homogeneous equation. This leads us to assume that

$$Y(t) = t^2(A_0t^n + \cdots + A_n).$$

The term $aY''(t)$ gives rise to a term of degree n, and we can proceed as before. Again the constant and linear terms in $Y(t)$ are omitted since, in this case, they are both solutions of the homogeneous equation.

Case 2: $g(t) = e^{\alpha t}P_n(t)$. The problem of determining a particular solution of

$$ay'' + by' + cy = e^{\alpha t}P_n(t) \qquad (31)$$

can be reduced to the preceding case by a substitution. Let

$$Y(t) = e^{\alpha t}u(t);$$

then

$$Y'(t) = e^{\alpha t}(u'(t) + \alpha u(t))$$

and

$$Y''(t) = e^{\alpha t}(u''(t) + 2\alpha u'(t) + \alpha^2u(t)).$$

Substituting for y, y', and y'' in equation (31), canceling the factor $e^{\alpha t}$, and collecting terms, we obtain

$$au''(t) + (2a\alpha + b)u'(t) + (a\alpha^2 + b\alpha + c)u(t) = P_n(t). \qquad (32)$$

The determination of a particular solution of equation (32) is precisely the same problem, except for the names of the constants, as solving equation (28). Therefore, if $a\alpha^2 + b\alpha + c$ is not zero, we assume that $u(t) = A_0t^n + \cdots + A_n$; hence a particular solution of equation (31) is of the form

$$Y(t) = e^{\alpha t}(A_0t^n + A_1t^{n-1} + \cdots + A_n). \qquad (33)$$

On the other hand, if $a\alpha^2 + b\alpha + c$ is zero but $2a\alpha + b$ is not, we must take $u(t)$ to be of the form $t(A_0t^n + \cdots + A_n)$. The corresponding form for $Y(t)$ is t times the expression on the right-hand side of equation (33). Note that if $a\alpha^2 + b\alpha + c$ is zero, then $e^{\alpha t}$ is a solution of the homogeneous equation.

If both $a\alpha^2 + b\alpha + c$ and $2a\alpha + b$ are zero (and this implies that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation), then the correct form for $u(t)$ is $t^2(A_0t^n + \cdots + A_n)$. Hence $Y(t)$ is t^2 times the expression on the right-hand side of equation (33).

Case 3: $g(t) = e^{\alpha t}P_n(t)\cos(\beta t)$ or $e^{\alpha t}P_n(t)\sin(\beta t)$. These two cases are similar, so we consider only the latter. We can reduce this problem to the preceding one by noting that, as a consequence of Euler's formula, $\sin(\beta t) = (e^{i\beta t} - e^{-i\beta t})/(2i)$. Hence $g(t)$ is of the form

$$g(t) = P_n(t)\frac{e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}}{2i},$$

and we should choose

$$Y(t) = e^{(\alpha+i\beta)t}(A_0t^n + \cdots + A_n) + e^{(\alpha-i\beta)t}(B_0t^n + \cdots + B_n),$$

or, equivalently,

$$Y(t) = e^{\alpha t}(A_0t^n + \cdots + A_n)\cos(\beta t) + e^{\alpha t}(B_0t^n + \cdots + B_n)\sin(\beta t).$$

Usually, the latter form is preferred because it does not involve the use of complex-valued coefficients. If $\alpha \pm i\beta$ satisfy the characteristic equation corresponding to the homogeneous equation, we must, of course, multiply each of the polynomials by t to increase their degrees by 1.

If the nonhomogeneous function involves both $\cos(\beta t)$ and $\sin(\beta t)$, it is usually convenient to treat these terms together, since each one individually may give rise to the same form for a particular solution. For example, if $g(t) = t\sin t + 2\cos t$, the form for $Y(t)$ would be

$$Y(t) = (A_0t + A_1)\sin t + (B_0t + B_1)\cos t,$$

provided that $\sin t$ and $\cos t$ are not solutions of the homogeneous equation.

Problems

In each of Problems 1 through 10, find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - y' - 2y = -2t + 4t^2$
3. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$
4. $y'' - 2y' - 3y = -3te^{-t}$
5. $y'' + 2y' = 3 + 4\sin(2t)$
6. $y'' + 2y' + y = 2e^{-t}$
7. $y'' + y = 3\sin(2t) + t\cos(2t)$
8. $u'' + \omega_0^2u = \cos(\omega t), \quad \omega^2 \neq \omega_0^2$
9. $u'' + \omega_0^2u = \cos(\omega_0 t)$
10. $y'' + y' + 4y = 2\sinh t \quad$ *Hint:* $\sinh t = (e^t - e^{-t})/2$

In each of Problems 11 through 15, find the solution of the given initial value problem.

11. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
12. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
13. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
14. $y'' + 4y = 3\sin(2t), \quad y(0) = 2, \quad y'(0) = -1$
15. $y'' + 2y' + 5y = 4e^{-t}\cos(2t), \quad y(0) = 1, \quad y'(0) = 0$

In each of Problems 16 through 21:

a. Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

N b. Use a computer algebra system to find a particular solution of the given equation.

16. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t)$
17. $y'' - 5y' + 6y = e^t\cos(2t) + e^{2t}(3t+4)\sin t$
18. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$
19. $y'' + 4y = t^2\sin(2t) + (6t+7)\cos(2t)$
20. $y'' + 3y' + 2y = e^t(t^2+1)\sin(2t) + 3e^{-t}\cos t + 4e^t$
21. $y'' + 2y' + 5y = 3te^{-t}\cos(2t) - 2te^{-2t}\cos t$
22. Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \qquad (34)$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.4), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

a. Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into equation (34) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.
b. Let $w(t) = v'(t)$ and show that $w(t)$ satisfies $w' - 5w = 2$. Solve this equation for $w(t)$.
c. Integrate $w(t)$ to find $v(t)$ and then show that

$$Y(t) = -\frac{2}{5}te^{-t} + \frac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right-hand side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

23. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t),$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

N **24.** In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

Behavior of Solutions as $t \to \infty$. In Problems 25 and 26, we continue the discussion started with Problems 28 through 30 of Section 3.4. Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{35}$$

where a, b, and c are positive.

25. If $Y_1(t)$ and $Y_2(t)$ are solutions of equation (35), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?

26. If $g(t) = d$, a constant, show that every solution of equation (35) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?

27. In this problem we indicate an alternative procedure[8] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{36}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

a. Verify that equation (36) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

b. Let $u = (D - r_2)y$. Then show that the solution of equation (36) can be found by solving the following two first-order equations:

$$(D - r_1)u = g(t), \quad (D - r_2)y = u(t).$$

In each of Problems 28 through 30, use the method of Problem 27 to solve the given differential equation.

28. $y'' - 3y' - 4y = 3e^{2t}$ (see Example 1)

29. $y'' + 2y' + y = 2e^{-t}$ (see Problem 6)

30. $y'' + 2y' = 3 + 4\sin(2t)$ (see Problem 5)

[8] R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201. Also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second-Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal* 8 (1977), pp. 132–141, for a more general discussion of factoring operators.

3.6 Variation of Parameters

In this section we describe a second method of finding a particular solution of a nonhomogeneous equation. This method, **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second-order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires us to evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find the general solution of

$$y'' + 4y = 8\tan t \quad -\pi/2 < t < \pi/2. \tag{1}$$

Solution:

Observe that this problem is not a good candidate for the method of undetermined coefficients, as described in Section 3.5, because the nonhomogeneous term $g(t) = 8\tan t$ involves a quotient (rather than a sum or a product) of $\sin t$ and $\cos t$. Therefore, the method of undetermined coefficients cannot be applied; we need a different approach.

Observe also that the homogeneous equation corresponding to equation (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of equation (2) is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t). \tag{3}$$

The basic idea in the method of variation of parameters is similar to the method of reduction of order introduced at the end of Section 3.4. In the general solution (3), replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, respectively, and then determine these functions so that the resulting expression

$$y = u_1(t)\cos(2t) + u_2(t)\sin(2t) \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2, we need to substitute for y from equation (4) in differential equation (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and u_2. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.[9]

Returning now to equation (4), we differentiate it and rearrange the terms, thereby obtaining

$$y' = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t). \tag{5}$$

Keeping in mind the possibility of choosing a second condition on u_1 and u_2, let us require the sum of the last two terms on the right-hand side of equation (5) to be zero; that is, we require that

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0. \tag{6}$$

It then follows from equation (5) that

$$y' = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t). \tag{7}$$

Although the ultimate effect of the condition (6) is not yet clear, the removal of the terms involving u_1' and u_2' has simplified the expression for y'. Further, by differentiating equation (7), we obtain

$$y'' = -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t). \tag{8}$$

Then, substituting for y and y'' in equation (1) from equations (4) and (8), respectively, we find that

$$\begin{aligned} y'' + 4y = &-4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) \\ &+ 4u_1(t)\cos(2t) + 4u_2(t)\sin(2t) = 8\tan t. \end{aligned}$$

Hence u_1 and u_2 must satisfy

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t. \tag{9}$$

Summarizing our results to this point, we want to choose u_1 and u_2 so as to satisfy equations (6) and (9). These equations can be viewed as a pair of linear *algebraic* equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$. Equations (6) and (9) can be solved in various ways. For example, solving equation (6) for $u_2'(t)$, we have

$$u_2'(t) = -u_1'(t)\frac{\cos(2t)}{\sin(2t)}. \tag{10}$$

Then, substituting for $u_2'(t)$ in equation (9) and simplifying, we obtain

$$u_1'(t) = -\frac{8\tan t\sin(2t)}{2} = -8\sin^2 t. \tag{11}$$

[9] An alternate, and more mathematically appealing, derivation of the second condition can be found in Problems 17 to 19 in Section 7.9.

By examining the expression (30) and reviewing the process by which we derived it, we can see that there may be two major difficulties in carrying out the method of variation of parameters. As we have mentioned earlier, one is the determination of functions $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions of the homogeneous equation (29) when the coefficients in that equation are not constants. The other possible difficulty lies in the evaluation of the integrals appearing in equation (30). This depends entirely on the nature of the functions y_1, y_2, and g. In using equation (30), be sure that the differential equation is exactly in the form (28); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that equation (30) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions. (See Problems 18 to 22.)

Problems

In each of Problems 1 through 3, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$
2. $y'' - y' - 2y = 2e^{-t}$
3. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 4 through 9, find the general solution of the given differential equation. In Problems 9, g is an arbitrary continuous function.

4. $y'' + y = \tan t, \quad 0 < t < \pi/2$
5. $y'' + 9y = 9\sec^2(3t), \quad 0 < t < \pi/6$
6. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$
7. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$
8. $y'' - 2y' + y = e^t/(1+t^2)$
9. $y'' - 5y' + 6y = g(t)$

In each of Problems 10 through 15, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 14 and 15, g is an arbitrary continuous function.

10. $t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$
11. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0;$ $y_1(t) = t, \quad y_2(t) = te^t$
12. $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1+t, \quad y_2(t) = e^t$
13. $x^2y'' - 3xy' + 4y = x^2\ln x, \quad x > 0; \quad y_1(x) = x^2,$ $y_2(x) = x^2\ln x$
14. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin x, \quad x > 0;$ $y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
15. $x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0;$ $y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$
16. By choosing the lower limit of integration in equation (30) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)\,ds.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0.$$

17. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{32}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \qquad u(t_0) = y_0, \quad u'(t_0) = y_0', \tag{33}$$

$$L[v] = g(t), \qquad v(t_0) = 0, \quad v'(t_0) = 0, \tag{34}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known. And, as shown in Problem 16, the function v is given by equation (30).

18. a. Use the result of Problem 16 to show that the solution of the initial value problem

$$y'' + y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0 \tag{35}$$

is

$$y = \int_{t_0}^{t} \sin(t-s)\,g(s)\,ds. \tag{36}$$

b. Use the result of Problem 17 to find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'.$$

19. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where $L[y] = (D-a)(D-b)y$ for real numbers a and b with $a \neq b$. Note that $L[y] = y'' - (a+b)y' + aby$.

20. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where $L[y] = (D - (\lambda + i\mu))(D - (\lambda - i\mu))y$; that is, $L[y] = y'' - 2\lambda y' + (\lambda^2 + \mu^2)y$. Note that the roots of the characteristic equation are $\lambda \pm i\mu$.

21. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where $L[y] = (D-a)^2y$, that is, $L[y] = y'' - 2ay' + a^2y$, and a is any real number.

22. By combining the results of Problems 19 through 21, show that the solution of the initial value problem

$$L[y] = (D^2 + bD + c)y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where b and c are constants, can be written in the form

$$y = \phi(t) = \int_{t_0}^{t} K(t-s)g(s)\,ds, \tag{37}$$

where the function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. When we think of $g(t)$ as the input to the problem and of $\phi(t)$ as the output, it follows from equation (37) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in equation (37) is called the **convolution** of K and g, and K is referred to as the **kernel**.

23. The method of reduction of order (Section 3.4) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{38}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies equation (38) if v is a solution of

$$y_1(t)v'' + \big(2y_1'(t) + p(t)y_1(t)\big)v' = g(t). \tag{39}$$

Equation (39) is a first-order linear differential equation for v'. By solving equation (39) for v', integrating the result to find v, and then multiplying by $y_1(t)$, you can find the general solution of equation (38). This method simultaneously finds both the second homogeneous solution and a particular solution.

In each of Problems 24 through 26, use the method outlined in Problem 23 to solve the given differential equation.

24. $t^2y'' - 2ty' + 2y = 4t^2, \quad t > 0; \quad y_1(t) = t$
25. $t^2y'' + 7ty' + 5y = t, \quad t > 0; \quad y_1(t) = t^{-1}$
26. $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t$ (see Problem 12)

3.7 Mechanical and Electrical Vibrations

One of the reasons why second-order linear differential equations with constant coefficients are worth studying is that they serve as mathematical models of many important physical processes. Two important areas of application are the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an initial value problem of the form

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

This illustrates a fundamental relationship between mathematics and physics: *many physical problems may have mathematically equivalent models.* Thus, once we know how to solve the initial value problem (1), it is only necessary to make appropriate interpretations of the constants a, b, and c, and of the functions y and g, to obtain solutions of different physical problems.

We will study the motion of a mass on a spring in detail because understanding the behavior of this simple system is the first step in the investigation of more complex vibrating systems. Further, the principles involved are common to many problems.

Consider a mass m hanging at rest on the end of a vertical spring of original length l, as shown in Figure 3.7.1. The mass causes an elongation L of the spring in the downward (positive) direction. In this static situation there are two forces acting at the point where the mass is attached to the spring; see Figure 3.7.2. The gravitational force, or weight of the mass, acts downward and has magnitude $w = mg$, where g is the acceleration due to gravity. There is also a force F_s, due to the spring, that acts upward. If we assume that the elongation L of the spring is small, the spring force is very nearly proportional to L; this is known as **Hooke's**[10] **law**. Thus we write $F_s = -kL$, where the constant of proportionality k is called the

[10] Robert Hooke (1635–1703) was an English scientist with wide-ranging interests. His most important book, *Micrographia*, was published in 1665 and described a variety of microscopical observations. Hooke first published his law of elastic behavior in 1676 as *ceiiinosssttuv*; in 1678 he gave the interpretation *ut tensio sic vis*, which means, roughly, "as the force so is the displacement."

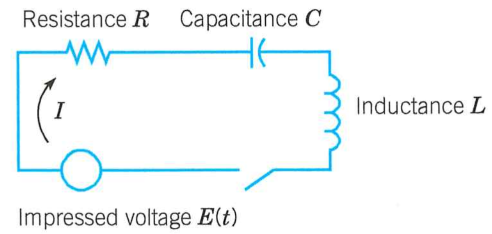

FIGURE 3.7.9 A simple electric circuit.

Electric Circuits. A second example of the occurrence of second-order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit shown in Figure 3.7.9. The current I, measured in amperes (A), is a function of time t. The resistance R in ohms (Ω), the capacitance C in farads (F), and the inductance L in henrys (H) are all positive and are assumed to be known constants. The impressed voltage E in volts (V) is a given function of time. Another physical quantity that enters the discussion is the total charge Q in coulombs (C) on the capacitor at time t. The relation between charge Q and current I is

$$I = \frac{dQ}{dt}. \tag{31}$$

The flow of current in the circuit is governed by Kirchhoff's[11] second law: *In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.*

According to the elementary laws of electricity, we know that

The voltage drop across the resistor is RI.

The voltage drop across the capacitor is $\dfrac{Q}{C}$.

The voltage drop across the inductor is $L\dfrac{dI}{dt}$.

Hence, by Kirchhoff's law,

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{32}$$

The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:

$$1 \text{ volt} = 1 \text{ ohm} \cdot 1 \text{ ampere} = 1 \text{ coulomb}/1 \text{ farad} = 1 \text{ henry} \cdot 1 \text{ ampere}/1 \text{ second}.$$

Substituting for I from equation (31), we obtain the differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{33}$$

for the charge Q. The initial conditions are

$$Q(t_0) = Q_0, \quad Q'(t_0) = I(t_0) = I_0. \tag{34}$$

Thus to know the charge at any time it is sufficient to know the charge on the capacitor and the current in the circuit at some initial time t_0.

Alternatively, we can obtain a differential equation for the current I by differentiating equation (33) with respect to t, and then substituting for dQ/dt from equation (31). The result is

$$LI'' + RI' + \frac{1}{C}I = E'(t), \tag{35}$$

with the initial conditions

$$I(t_0) = I_0, \quad I'(t_0) = I_0'. \tag{36}$$

[11]Gustav Kirchhoff (1824–1887) was a German physicist and professor at Breslau, Heidelberg, and Berlin. He formulated the basic laws of electric circuits about 1845 while still a student at Albertus University in his native Königsberg. In 1857 he discovered that an electric current in a resistanceless wire travels at the speed of light. He is also famous for fundamental work in electromagnetic absorption and emission and was one of the founders of spectroscopy.

From equation (32) it follows that

$$I_0' = \frac{E(t_0) - RI_0 - \dfrac{Q_0}{C}}{L}. \tag{37}$$

Hence I_0' is also determined by the initial charge and current, which are physically measurable quantities.

The most important conclusion from this discussion is that the flow of current in the circuit is described by an initial value problem of precisely the same form as the one that describes the motion of a spring-mass system. This is a good example of the unifying role of mathematics: once you know how to solve second-order linear equations with constant coefficients, you can interpret the results in terms of mechanical vibrations, electric circuits, or any other physical situation that leads to the same problem.

Problems

In each of Problems 1 and 2, determine ω_0, R, and δ so as to write the given expression in the form $u = R\cos(\omega_0 t - \delta)$.

1. $u = 3\cos(2t) + 4\sin(2t)$

2. $u = -2\cos(\pi t) - 3\sin(\pi t)$

3. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

4. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

G **5.** A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyn·s/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Plot u versus t. Determine the quasi-frequency and the quasi-period. Determine the ratio of the quasi-period to the period of the corresponding undamped motion. Also find the time τ such that $|u(t)| < 0.05$ cm for all $t > \tau$.

6. A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasi-frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

7. A series circuit has a capacitor of 10^{-5} F, a resistor of $3 \times 10^2\ \Omega$, and an inductor of 0.2 H. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

8. A vibrating system satisfies the equation $u'' + \gamma u' + u = 0$. Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

9. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass, and g is the acceleration due to gravity.

10. Show that the solution of the initial value problem

$$mu'' + \gamma u' + ku = 0, \quad u(t_0) = u_0, \quad u'(t_0) = u_0'$$

can be expressed as the sum $u = v + w$, where v satisfies the initial conditions $v(t_0) = u_0$, $v'(t_0) = 0$, w satisfies the initial conditions $w(t_0) = 0$, $w'(t_0) = u_0'$, and both v and w satisfy the same differential equation as u. This is another instance of superposing solutions of simpler problems to obtain the solution of a more general problem.

11. **a.** Show that $A\cos(\omega_0 t) + B\sin(\omega_0 t)$ can be written in the form $r\sin(\omega_0 t - \theta)$. Determine r and θ in terms of A and B.
b. If $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$, determine the relationship among R, r, δ, and θ.

12. If a series circuit has a capacitor of $C = 0.8 \times 10^{-6}$ F and an inductor of $L = 0.2$ H, find the resistance R so that the circuit is critically damped.

13. Assume that the system described by the differential equation $mu'' + \gamma u' + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
Hint: Determine all possible values of t for which $u = 0$.

14. Assume that the system described by the differential equation $mu'' + \gamma u' + ku = 0$ is critically damped and that the initial conditions are $u(0) = u_0$, $u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$ but that u is never zero. If u_0 is positive, determine a condition on v_0 that will ensure that the mass passes through its equilibrium position after it is released.

15. **Logarithmic Decrement.** **a.** For the damped oscillation described by equation (26), show that the time between successive maxima is $T_d = 2\pi/\mu$.
b. Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/(2m))$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the **logarithmic decrement** and is denoted by Δ.
c. Show that $\Delta = \pi\gamma/(m\mu)$. Since m, μ, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

16. Referring to Problem 15, find the logarithmic decrement of the system in Problem 5.

17. The position of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \quad u(0) = 2, \quad u'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.

18. Consider the initial value problem

$$mu'' + \gamma u' + ku = 0, \quad u(0) = u_0, \quad u'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.

a. Solve the initial value problem.
b. Write the solution in the form $u(t) = Re^{-\gamma t/(2m)}\cos(\mu t - \delta)$. Determine R in terms of m, γ, k, u_0, and v_0.
c. Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

19. A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion and determine the period of the motion.
Hint: Use Archimedes'[12] principle: an object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

20. The position of a certain undamped spring-mass system satisfies the initial value problem

$$u'' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2.$$

a. Find the solution of this initial value problem.
G **b.** Plot u versus t and u' versus t on the same axes.
G **c.** Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. This plot is known as a **phase plot**, and the uu'-plane is called the **phase plane**. Observe that a closed curve in the phase plane corresponds to a periodic solution $u(t)$. What is the direction of motion on the phase plot as t increases?

21. The position of a certain spring-mass system satisfies the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 0, \quad u(0) = 0, \quad u'(0) = 2.$$

a. Find the solution of this initial value problem.
G **b.** Plot u versus t and u' versus t on the same axes.
G **c.** Plot u' versus u in the phase plane (see Problem 20). Identify several corresponding points on the curves in parts b and c. What is the direction of motion on the phase plot as t increases?

22. In the absence of damping, the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \quad u(0) = a, \quad u'(0) = b.$$

a. Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so initially the total energy in the system is $(ka^2 + mb^2)/2$.
b. Solve the given initial value problem.
c. Using the solution in part b, determine the total energy in the system at any time t. Your result should confirm the principle of conservation of energy for this system.

23. Suppose that a mass m slides without friction on a horizontal surface. The mass is attached to a spring with spring constant k, as shown in Figure 3.7.10, and is also subject to viscous air resistance with coefficient γ. Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies equation (21). How does the derivation of the equation of motion in this case differ from the derivation given in the text?

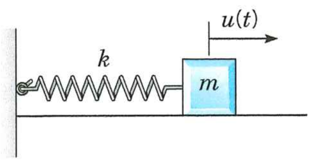

FIGURE 3.7.10 A spring-mass system.

24. In the spring-mass system of Problem 23, suppose that the spring force is not given by Hooke's law but instead satisfies the relation

$$F_s = -(ku + \epsilon u^3),$$

where $k > 0$ and ϵ is small but may be of either sign. The spring is called a hardening spring if $\epsilon > 0$ and a softening spring if $\epsilon < 0$. Why are these terms appropriate?

a. Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies the differential equation

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0.$$

Suppose that the initial conditions are

$$u(0) = 0, \quad u'(0) = 1.$$

In the remainder of this problem, assume that $m = 1$, $k = 1$, and $\gamma = 0$.
b. Find $u(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.
G **c.** Let $\epsilon = 0.1$. Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.
G **d.** Repeat part c for $\epsilon = 0.2$ and $\epsilon = 0.3$.
G **e.** Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.
G **f.** Repeat parts c, d, and e for negative values of ϵ.

[12] Archimedes (287–212 BCE) was the foremost of the ancient Greek mathematicians. He lived in Syracuse on the island of Sicily. His most notable discoveries were in geometry, but he also made important contributions to hydrostatics and other branches of mechanics. His method of exhaustion is a precursor of the integral calculus developed by Newton and Leibniz almost two millennia later. He died at the hands of a Roman soldier during the Second Punic War.

3.8 Forced Periodic Vibrations

We will now investigate the situation in which a periodic external force is applied to a spring-mass system. The behavior of this simple system models that of many oscillatory systems with an external force due, for example, to a motor attached to the system. We will first consider the case in which damping is present and will look later at the idealized special case in which there is assumed to be no damping.

Forced Vibrations with Damping. The algebraic calculations can be fairly complicated in this kind of problem, so we will begin with a relatively simple example.

EXAMPLE 1

Suppose that the motion of a certain spring-mass system satisfies the differential equation

$$u'' + u' + \frac{5}{4}u = 3\cos t \tag{1}$$

and the initial conditions

$$u(0) = 2, \quad u'(0) = 3. \tag{2}$$

Find the solution of this initial value problem and describe the behavior of the solution for large t.

Solution:

The homogeneous equation corresponding to equation (1) has the characteristic equation $r^2 + r + \frac{5}{4} = 0$ with roots $r = -\frac{1}{2} \pm i$. Thus a general solution $u_c(t)$ of this homogeneous equation is

$$u_c(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t. \tag{3}$$

A particular solution of equation (1) has the form $U(t) = A\cos t + B\sin t$, where A and B are found by substituting $U(t)$ for u in equation (1). We have $U'(t) = -A\sin t + B\cos t$ and $U''(t) = -A\cos t - B\sin t$. Thus, from equation (1) we obtain

$$\left(\frac{1}{4}A + B\right)\cos t + \left(-A + \frac{1}{4}B\right)\sin t = 3\cos t.$$

Consequently, A and B must satisfy the equations

$$\frac{1}{4}A + B = 3, \quad -A + \frac{1}{4}B = 0,$$

with the result that $A = \frac{12}{17}$ and $B = \frac{48}{17}$. Therefore, the particular solution is

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t, \tag{4}$$

and the general solution of equation (1) is

$$u = u_c(t) + U(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t. \tag{5}$$

The remaining constants c_1 and c_2 are determined by the initial conditions (2). From equation (5), and its first derivative, we have

$$u(0) = c_1 + \frac{12}{17} = 2, \quad u'(0) = -\frac{1}{2}c_1 + c_2 + \frac{48}{17} = 3,$$

so $c_1 = \frac{22}{17}$ and $c_2 = \frac{14}{17}$. Thus we finally arrive at the solution of the given initial value problem (1), (2), namely,

$$u = \frac{22}{17}e^{-t/2}\cos t + \frac{14}{17}e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t. \tag{6}$$

The graph of the solution (6) is shown by the green curve in Figure 3.8.1.

A graph of this solution is shown in Figure 3.8.7. The amplitude variation has a slow frequency of 0.1 and a corresponding slow period of $2\pi/0.1 = 20\pi$. Note that a half-period of 10π corresponds to a single cycle of increasing and then decreasing amplitude. The displacement of the spring-mass system oscillates with a relatively fast frequency of 0.9, which is only slightly less than the natural frequency ω_0.

Now imagine that the forcing frequency ω is increased, say, to $\omega = 0.9$. Then the slow frequency is halved to 0.05, and the corresponding slow half-period is doubled to 20π. The multiplier 2.7778 also increases substantially, to 5.263. However, the fast frequency is only marginally increased, to 0.95. Can you visualize what happens as ω takes on values closer and closer to the natural frequency $\omega_0 = 1$?

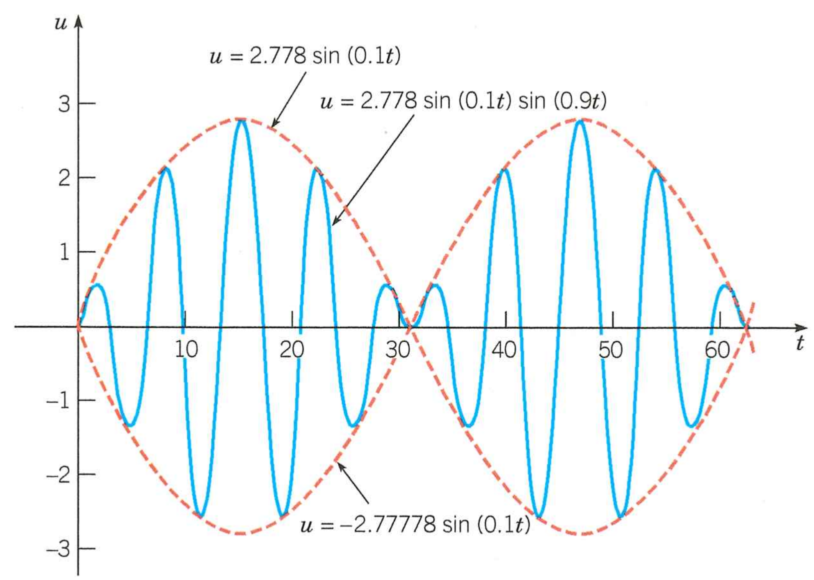

FIGURE 3.8.7 A beat; the solution (solid blue) of equation (22): $u'' + u = \frac{1}{2}\cos(0.8t)$, $u(0) = 0, u'(0) = 0$ is $u = 2.778\sin(0.1t)\sin(0.9t)$. The dashed red curve is the external force $F(t) = \frac{1}{2}\cos(0.8t)$.

Now let us return to equation (17) and consider the case of resonance, where $\omega = \omega_0$; that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $F_0\cos(\omega t)$ is a solution of the homogeneous equation. In this case the solution of equation (17) is

$$u = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t). \tag{24}$$

Consider the following example.

EXAMPLE 4

Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos t, \quad u(0) = 0, \quad u'(0) = 0, \tag{25}$$

and plot the graph of the solution.

Solution:

The general solution of the differential equation is

$$u = c_1\cos t + c_2\sin t + \frac{t}{4}\sin t,$$

and the initial conditions require that $c_1 = c_2 = 0$. Thus the solution of the given initial value problem is

$$u = \frac{t}{4}\sin t. \tag{26}$$

The graph of the solution is shown in Figure 3.8.8.

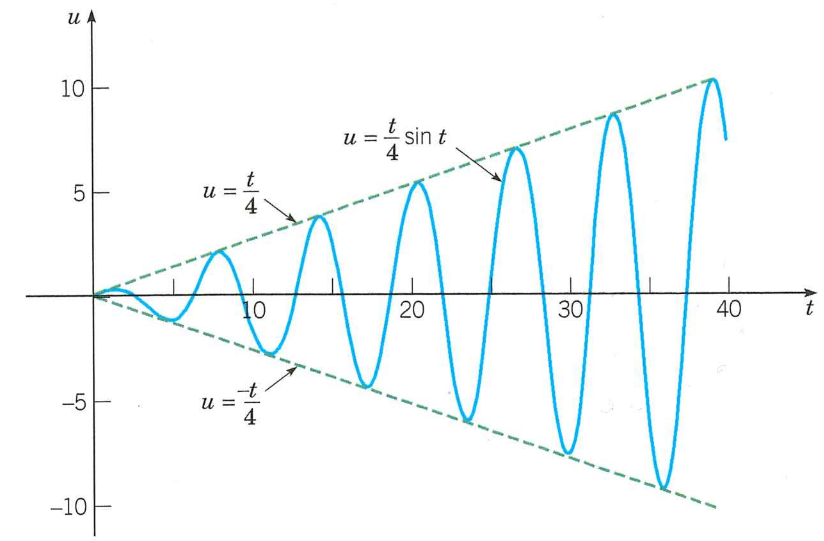

FIGURE 3.8.8 Resonance; the solution (solid blue) of equation (25): $u'' + u = \frac{1}{2}\cos t$, $u(0) = 0$, $u'(0) = 0$ is $u = \frac{t}{4}\sin t$.

Because of the term $t\sin(\omega_0 t)$, the solution (24) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of c_1 and c_2, and Figure 3.8.8 bears this out. Of course, in reality, unbounded oscillations do not occur, because the spring cannot stretch infinitely far. Moreover, as soon as u becomes large, the mathematical model on which equation (17) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that u be small. As we have seen, if damping is included in the model, the predicted motion remains bounded; however, the response to the input function $F_0\cos(\omega t)$ may be quite large if the damping is small and ω is close to ω_0.

Problems

In each of Problems 1 through 3, write the given expression as a product of two trigonometric functions of different frequencies.

1. $\sin(7t) - \sin(6t)$
2. $\cos(\pi t) + \cos(2\pi t)$
3. $\sin(3t) + \sin(4t)$
4. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.
5. **a.** Find the solution of the initial value problem in Problem 4.
   b. Identify the transient and steady-state parts of the solution.
   G **c.** Plot the graph of the steady-state solution.
   N **d.** If the given external force is replaced by a force of $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.
6. N A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8\sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.
7. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of $\frac{1}{4}$ lb·s/ft and is acted on by an external force of $4\cos(2t)$ lb.
   a. Determine the steady-state response of this system.
   b. If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.

8. A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady-state response. Express your answer in the form $R\cos(\omega t - \delta)$.

9. In this problem we ask you to supply some of the details in the analysis of a forced damped oscillator.

a. Derive equations (10), (11), and (12) for the steady-state solution of equation (8).

b. Derive the expression in equation (13) for Rk/F_0.

c. Show that $\omega^2_{\max}$ and $R_{\max}$ are given by equations (14) and (15), respectively.

d. Verify that Rk/F_0, ω/ω_0, and $\Gamma = \gamma^2/(mk)$ are all dimensionless quantities.

10. Find the velocity of the steady-state response given by equation (10). Then show that the velocity is maximum when $\omega = \omega_0$.

11. Find the solution of the initial value problem

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

N **12.** A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of $5 \times 10^3\ \Omega$, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12 V battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

N **13.** Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

a. Find the solution $u(t)$ for $\omega \neq 1$.

G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

14. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 1, \quad u'(0) = 1.$$

a. Find the solution for $\omega \neq 1$.

G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 13; that is, describe the effect of the nonzero initial conditions.

G **15.** For the initial value problem in Problem 13, plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. (Recall that such a plot is called a phase plot.) Use a t interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 16 through 18 deal with the initial value problem

$$u'' + \frac{1}{8}u' + 4u = F(t), \quad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

G **a.** Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.

G **b.** Draw the phase plot of the solution; that is, plot u' versus u.

16. $F(t) = 3\cos(t/4)$

17. $F(t) = 3\cos(2t)$

18. $F(t) = 3\cos(6t)$

G **19.** A spring-mass system with a hardening spring (Problem 24 of Section 3.7) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$u'' + u + \frac{1}{5}u^3 = \cos\omega t, \quad u(0) = 0, \quad u'(0) = 0.$$

a. Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?

b. Plot the solution for several values of ω between 1/2 and 2. Describe how the solution changes as ω increases.

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CHAPTER 4

Higher-Order Linear Differential Equations

The theoretical structure and methods of solution developed in the preceding chapter for second-order linear equations extend directly to linear equations of third and higher order. In this chapter we briefly review this generalization, taking particular note of those instances where new phenomena may appear, because of the greater variety of situations that can occur for equations of higher order.

4.1 General Theory of n^{th} Order Linear Differential Equations

An n^{th} order linear differential equation is an equation of the form

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t). \tag{1}$$

We assume that the functions $P_0, \ldots, P_n$, and G are continuous real-valued functions on some interval $I\colon \alpha < t < \beta$, and that P_0 is nowhere zero in this interval. Then, dividing equation (1) by $P_0(t)$, we obtain

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t). \tag{2}$$

The linear differential operator L of order n defined by equation (2) is similar to the second-order operator introduced in Chapter 3. The mathematical theory associated with equation (2) is completely analogous to that for the second-order linear equation; for this reason we simply state the results for the n^{th} order problem. The proofs of most of the results are also similar to those for the second-order equation and are usually left as exercises.

Since equation (2) involves the n^{th} derivative of y with respect to t, it will, so to speak, require n integrations to solve equation (2). Each of these integrations introduces an arbitrary constant. Hence we expect that to obtain a unique solution it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}, \tag{3}$$

where t_0 may be any point in the interval I and $y_0, y_0', \ldots, y_0^{(n-1)}$ are any prescribed real constants. The following theorem, which is similar to Theorem 3.2.1, guarantees that the initial value problem (2), (3) has a solution and that it is unique.

Theorem 4.1.1

If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3), where t_0 is any point in I. This solution exists throughout the interval I.