

The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral. This is usually at most a slight inconvenience, rather than a serious obstacle. For a given value of t , the integral in equation (47) is a definite integral and can be approximated to any desired degree of accuracy by using readily available numerical integrators. By repeating this process for many values of t and plotting the results, you can obtain a graph of a solution. Alternatively, you can use a numerical approximation method, such as those discussed in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple, Mathematica, MATLAB and Sage readily execute such procedures and produce graphs of solutions of differential equations.

Figure 2.1.4 displays graphs of the solution (47) for several values of c . The particular solution satisfying the initial condition $y(0) = 1$ is shown in black. From the figure it may be plausible to conjecture that all solutions approach a limit as $t \rightarrow \infty$. The limit can also be found analytically (see Problem 22).

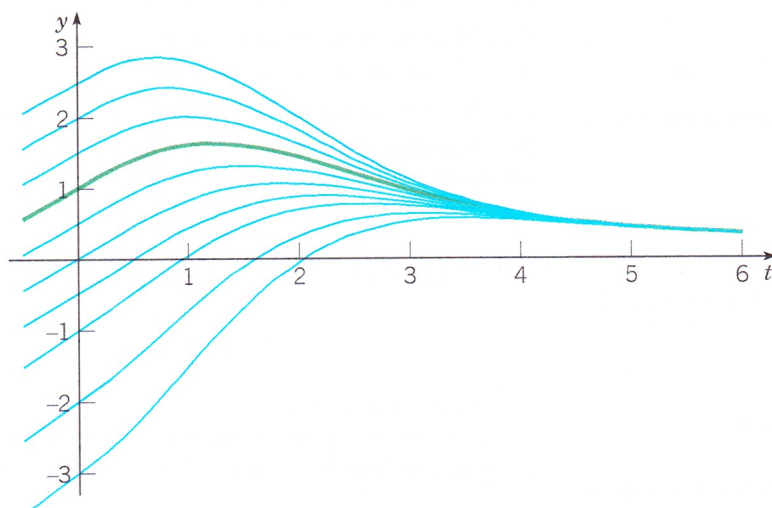


FIGURE 2.1.4 Integral curves of $2y' + ty = 2$; the green curve is the particular solution satisfying the initial condition $y(0) = 1$.

Problems

In each of Problems 1 through 8:

- G a.** Draw a direction field for the given differential equation.
 - b.** Based on an inspection of the direction field, describe how solutions behave for large t .
 - c.** Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \rightarrow \infty$.
1. $y' + 3y = t + e^{-2t}$
 2. $y' - 2y = t^2 e^{2t}$
 3. $y' + y = t e^{-t} + 1$
 4. $y' + \frac{1}{t}y = 3 \cos(2t), \quad t > 0$
 5. $y' - 2y = 3e^t$
 6. $ty' - y = t^2 e^{-t}, \quad t > 0$
 7. $y' + y = 5 \sin(2t)$
 8. $2y' + y = 3t^2$

In each of Problems 9 through 12, find the solution of the given initial value problem.

9. $y' - y = 2te^{2t}, \quad y(0) = 1$
10. $y' + 2y = te^{-2t}, \quad y(1) = 0$
11. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0$
12. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

In each of Problems 13 and 14:

- G a.** Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a ? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0 .
 - b.** Solve the initial value problem and find the critical value a_0 exactly.
 - c.** Describe the behavior of the solution corresponding to the initial value a_0 .
13. $y' - \frac{1}{2}y = 2 \cos t, \quad y(0) = a$
 14. $3y' - 2y = e^{-\pi t/2}, \quad y(0) = a$

In each of Problems 15 and 16:

G a. Draw a direction field for the given differential equation. How do solutions appear to behave as $t \rightarrow 0$? Does the behavior depend on the choice of the initial value a ? Let a_0 be the critical value of a , that is, the initial value such that the solutions for $a < a_0$ and the solutions for $a > a_0$ have different behaviors as $t \rightarrow \infty$. Estimate the value of a_0 .

b. Solve the initial value problem and find the critical value a_0 exactly.

c. Describe the behavior of the solution corresponding to the initial value a_0 .

15. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$

16. $(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$

G 17. Consider the initial value problem

$$y' + \frac{1}{2}y = 2\cos t, \quad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

N 18. Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t -axis.

19. Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

a. Find the solution of this initial value problem and describe its behavior for large t .

N b. Determine the value of t for which the solution first intersects the line $y = 12$.

20. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \rightarrow \infty$.

21. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \quad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \rightarrow \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \rightarrow \infty$?

22. Show that all solutions of $2y' + ty = 2$ [equation (41) of the text] approach a limit as $t \rightarrow \infty$, and find the limiting value.

Hint: Consider the general solution, equation (47). Show that the first

term in the solution (47) is indeterminate with form $0 \cdot \infty$. Then, use l'Hôpital's rule to compute the limit as $t \rightarrow \infty$.

23. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \rightarrow 0$ as $t \rightarrow \infty$.

Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 24 through 27, construct a first-order linear differential equation whose solutions have the required behavior as $t \rightarrow \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

24. All solutions have the limit 3 as $t \rightarrow \infty$.

25. All solutions are asymptotic to the line $y = 3 - t$ as $t \rightarrow \infty$.

26. All solutions are asymptotic to the line $y = 2t - 5$ as $t \rightarrow \infty$.

27. All solutions approach the curve $y = 4 - t^2$ as $t \rightarrow \infty$.

28. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \quad (48)$$

a. If $g(t) = 0$ for all t , show that the solution is

$$y = A \exp\left(-\int p(t) dt\right), \quad (49)$$

where A is a constant.

b. If $g(t)$ is not everywhere zero, assume that the solution of equation (48) is of the form

$$y = A(t) \exp\left(-\int p(t) dt\right), \quad (50)$$

where A is now a function of t . By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t) \exp\left(\int p(t) dt\right). \quad (51)$$

c. Find $A(t)$ from equation (51). Then substitute for $A(t)$ in equation (50) and determine y . Verify that the solution obtained in this manner agrees with that of equation (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second-order linear equations.

In each of Problems 29 and 30, use the method of Problem 28 to solve the given differential equation.

29. $y' - 2y = t^2 e^{2t}$

30. $y' + \frac{1}{t}y = \cos(2t), \quad t > 0$

Problems

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = \frac{x^2}{y}$
2. $y' + y^2 \sin x = 0$
3. $y' = \cos^2(x) \cos^2(2y)$
4. $xy' = (1 - y^2)^{1/2}$
5. $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$
6. $\frac{dy}{dx} = \frac{x^2}{1 + y^2}$
7. $\frac{dy}{dx} = \frac{y}{x}$
8. $\frac{dy}{dx} = \frac{-x}{y}$

In each of Problems 9 through 16:

- a. Find the solution of the given initial value problem in explicit form.
 - G** b. Plot the graph of the solution.
 - c. Determine (at least approximately) the interval in which the solution is defined.
9. $y' = (1 - 2x)y^2$, $y(0) = -1/6$
 10. $y' = (1 - 2x)/y$, $y(1) = -2$
 11. $x dx + ye^{-x} dy = 0$, $y(0) = 1$
 12. $dr/d\theta = r^2/\theta$, $r(1) = 2$
 13. $y' = xy^3(1 + x^2)^{-1/2}$, $y(0) = 1$
 14. $y' = 2x/(1 + 2y)$, $y(2) = 0$
 15. $y' = (3x^2 - e^x)/(2y - 5)$, $y(0) = 1$
 16. $\sin(2x) dx + \cos(3y) dy = 0$, $y(\pi/2) = \pi/3$

Some of the results requested in Problems 17 through 22 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

- G** 17. Solve the initial value problem

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and determine the interval in which the solution is valid.

Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

- G** 18. Solve the initial value problem

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.

Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

- G** 19. Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

- G** 20. Solve the initial value problem

$$y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0$$

and determine where the solution attains its maximum value.

- G** 21. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{3}, \quad y(0) = y_0.$$

- a. Determine how the behavior of the solution as t increases depends on the initial value y_0 .
- b. Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

- G** 22. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{1 + t}, \quad y(0) = y_0 > 0.$$

- a. Determine how the solution behaves as $t \rightarrow \infty$.
- b. If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
- c. Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

23. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c , and d are constants.

24. Use separation of variables to solve the differential equation

$$\frac{dQ}{dt} = r(a + bQ), \quad Q(0) = Q_0,$$

where a, b, r , and Q_0 are constants. Determine how the solution behaves as $t \rightarrow \infty$.

Homogeneous Equations. If the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.¹ Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 25 illustrates how to solve first-order homogeneous equations.

¹The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

N 25. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \quad (29)$$

a. Show that equation (29) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \quad (30)$$

thus equation (29) is homogeneous.

b. Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x , v , and dv/dx .

c. Replace y and dy/dx in equation (30) by the expressions from part **b** that involve v and dv/dx . Show that the resulting differential equation is

$$v + x \frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x \frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \quad (31)$$

Observe that equation (31) is separable.

d. Solve equation (31), obtaining v implicitly in terms of x .

e. Find the solution of equation (29) by replacing v by y/x in the solution in part **d**.

f. Draw a direction field and some integral curves for equation (29). Recall that the right-hand side of equation (29) actually depends only on the ratio y/x . This means that integral curves have the same slope at all points on any given straight line

through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 25 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 26 through 31:

a. Show that the given equation is homogeneous.

b. Solve the differential equation.

G c. Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

26. $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

27. $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$

28. $\frac{dy}{dx} = \frac{4y - 3x}{2x - y}$

29. $\frac{dy}{dx} = -\frac{4x + 3y}{2x + y}$

30. $\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy}$

31. $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$

2.3 Modeling with First-Order Differential Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Step 1: Construction of the Model. In this step the physical situation is translated into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of

To determine the maximum altitude A_{\max} that the body reaches, we set $v = 0$ and $x = A_{\max}$ in equation (30) and then solve for A_{\max} , obtaining

$$A_{\max} = \frac{v_0^2 R}{2gR - v_0^2}. \quad (31)$$

Solving equation (31) for v_0 , we find the initial velocity required to lift the body to the altitude A_{\max} , namely,

$$v_0 = \sqrt{2gR \frac{A_{\max}}{R + A_{\max}}}. \quad (32)$$

The escape velocity v_e is then found by letting $A_{\max} \rightarrow \infty$. Consequently,

$$v_e = \sqrt{2gR}. \quad (33)$$

The numerical value of v_e is approximately 6.9 mi/s, or 11.1 km/s.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.


Problems

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t . Also find the limiting amount of salt in the tank as $t \rightarrow \infty$.

3. A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4} \left(1 + \frac{1}{2} \sin t\right)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

a. Find the amount of salt in the tank at any time.

 b. Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.

c. The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

4. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t . Torricelli's² principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h .

a. Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

b. By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h) \frac{dh}{dt} = -\alpha a \sqrt{2gh}, \quad (34)$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a . The value of α for water is about 0.6.

c. Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m, and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

5. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

a. Find the time T required for the original sum to double in value as a function of r .

b. Determine T if $r = 7\%$.

c. Find the return rate that must be achieved if the initial investment is to double in 8 years.

6. A young person with no initial capital invests k dollars per year at an annual rate of return r . Assume that investments are made continuously and that the return is compounded continuously.

a. Determine the sum $S(t)$ accumulated at any time t .

b. If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

c. If $k = \$2000/\text{year}$, determine the return rate r that must be obtained to have \$1 million available in 40 years.

²Evangelista Torricelli (1608–1647), successor to Galileo as court mathematician in Florence, published this result in 1644. In addition to this work in fluid dynamics, he is also known for constructing the first mercury barometer and for making important contributions to geometry.

7. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k , determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

N 8. A recent college graduate borrows \$150,000 at an interest rate of 6% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800 + 10t$, where t is the number of months since the loan was made.

a. Assuming that this payment schedule can be maintained, when will the loan be fully paid?

b. Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

9. An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby.³ This is a means of determining the age of certain wood and plant remains, and hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years),⁴ measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

a. Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14.

b. Find an expression for $Q(t)$ at any time t , if $Q(0) = Q_0$.

c. Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

N 10. Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$\frac{dy}{dt} = (0.5 + \sin t) \frac{y}{5}.$$

a. If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.

b. Suppose that the growth rate is replaced by its average value $1/10$. Determine the doubling time τ in this case.

c. Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ ?

d. Plot the solutions obtained in parts **a**, **b**, and **c** on a single set of axes.

³Willard F. Libby (1908–1980) was born in rural Colorado and received his education at the University of California at Berkeley. He developed the method of radiocarbon dating beginning in 1947 while he was at the University of Chicago. For this work he was awarded the Nobel Prize in Chemistry in 1960.

⁴McGraw-Hill *Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

N 11. Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \quad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$, and k represents the rate of predation.

G a. Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between $1/2$ and 1 .

b. Estimate the critical initial population y_c below which the population will become extinct.

c. Choose other values of k and find the corresponding y_c for each one.

G d. Use the data you have found in parts **b** and **c** to plot y_c versus k .

12. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F , determine when the coffee reaches a temperature of 150°F .

13. Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann⁵ law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \quad (35)$$

where $u(t)$ is the absolute temperature of the body at time t , T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T , then solutions of equation (35) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \quad (36)$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2.0 \times 10^{-12} \text{ K}^{-3}/\text{s}$.

a. Determine the temperature of the body at any time by solving equation (36).

G b. Plot the graph of u versus t .

N c. Find the time τ at which $u(\tau) = 600$ —that is, twice the ambient temperature. Up to this time the error in using equation (36) to approximate the solutions of equation (35) is no more than 1%.

N 14. Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, u satisfies the differential equation

$$\frac{du}{dt} = -k(u - T(t)), \quad (37)$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos(\omega t)$.

⁵Jozef Stefan (1835–1893), professor of physics at Vienna, stated the radiation law on empirical grounds in 1879. His student Ludwig Boltzmann (1844–1906) derived it theoretically from the principles of thermodynamics in 1884. Boltzmann is best known for his pioneering work in statistical mechanics.

a. Solve equation (37) and express $u(t)$ in terms of t, k, T_0, T_1 , and ω . Observe that part of your solution approaches zero as t becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

G b. Suppose that t is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60^\circ\text{F}$, $T_1 = 15^\circ\text{F}$, and $k = 0.2/\text{h}$. Draw graphs of $S(t)$ and $T(t)$ versus t on the same axes. From your graph estimate the amplitude R of the oscillatory part of $S(t)$. Also estimate the time lag τ between corresponding maxima of $T(t)$ and $S(t)$.

c. Let k, T_0, T_1 , and ω now be unspecified. Write the oscillatory part of $S(t)$ in the form $R \cos(\omega(t - \tau))$. Use trigonometric identities to find expressions for R and τ . Let T_1 and ω have the values given in part b, and plot graphs of R and τ versus k .

15. Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r , and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P . Note that the given assumptions neglect a number of factors that may, in some cases, be important—for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are deposited unevenly throughout the lake but (usually) at isolated points around its periphery. The results below must be interpreted in light of the neglect of such factors as these.

a. If at time $t = 0$ the concentration of pollutant is c_0 , find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \rightarrow \infty$?

b. If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval T that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

c. Table 2.3.2 contains data⁶ for several of the Great Lakes. Using these data, determine from part b the time T that is needed to reduce the contamination of each of these lakes to 10% of the original value.

TABLE 2.3.2 Volume and Flow Data for the Great Lakes

Lake	$10^3 \times V$ (km ³)	r (km ³ /year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

N 16. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

a. Find the maximum height above the ground that the ball reaches.

b. Assuming that the ball misses the building on the way down, find the time that it hits the ground.

G c. Plot the graphs of velocity and position versus time.

N 17. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.

b. Find the time that the ball hits the ground.

G c. Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 16.

N 18. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.

b. Find the time that the ball hits the ground.

G c. Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 16 and 17.

19. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.

a. Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.

b. Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g} \left(1 - \frac{1}{2} \frac{kv_0}{mg} + \frac{1}{3} \left(\frac{kv_0}{mg} \right)^2 - \cdots \right),$$

$$x_m = \frac{v_0^2}{2g} \left(1 - \frac{2}{3} \frac{kv_0}{mg} + \frac{1}{2} \left(\frac{kv_0}{mg} \right)^2 - \cdots \right).$$

c. Show that the quantity kv_0/mg is dimensionless.

20. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

a. Find the velocity $v(t)$ of the body at any time.

b. Use the result of part a to calculate the limit of $v(t)$ as $k \rightarrow 0$ —that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?

c. Use the result of part a to calculate the limit of $v(t)$ as $m \rightarrow 0$ —that is, as the mass approaches zero.

21. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R , a buoyant force B , and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a , the resistive force is given by Stokes's law, $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.⁷

⁶ This problem is based on R. H. Rainey, "Natural Displacement of Pollution from the Great Lakes," *Science* 155 (1967), pp. 1242–1243; the information in the table was taken from that source.

⁷ Sir George Gabriel Stokes (1819–1903) was born in Ireland but spent most of his life at Cambridge University, first as a student and later as a professor. Stokes was one of the foremost applied mathematicians of the nineteenth century, best known for his work in fluid dynamics and the wave theory of light. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series.

- a. Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ .
- b. In 1910 R. A. Millikan⁸ studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e . Assume that E has been adjusted so the droplet is held stationary ($v = 0$) and that w and B are as given above. Find an expression for e . Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

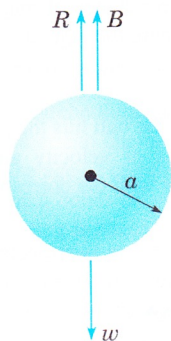


FIGURE 2.3.5 A body falling in a dense fluid (see Problem 21).

22. Let $v(t)$ and $w(t)$ be the horizontal and vertical components, respectively, of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$\frac{dv}{dt} = 0, \quad \frac{dw}{dt} = -g.$$

- a. Show that

$$v = u \cos A, \quad w = -gt + u \sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.

- b. Let $x(t)$ and $y(t)$ be the horizontal and vertical coordinates, respectively, of the ball at time t . If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t .

G c. Let $g = 32$ ft/s², $u = 125$ ft/s, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A ; that is, plot $x(t)$ and $y(t)$ parametrically.

- d. Suppose the outfield wall is at a distance L and has height H . Find a relation between u and A that must be satisfied if the ball is to clear the wall.

e. Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part (d), find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/s.

- f. For $L = 350$ and $H = 10$, find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

- N 23.** A more realistic model (than that in Problem 22) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$\frac{dv}{dt} = -rv, \quad \frac{dw}{dt} = -g - rw,$$

where r is the coefficient of resistance.

⁸Robert A. Millikan (1868–1953) was educated at Oberlin College and Columbia University. Later he was a professor at the University of Chicago and California Institute of Technology. His determination of the charge on an electron was published in 1910. For this work, and for other studies of the photoelectric effect, he was awarded the Nobel Prize for Physics in 1923.

- a. Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A .
- b. Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.
- G c.** Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A . How do the trajectories differ from those in Problem 22 with $r = 0$?
- d. Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 22f.

24. Brachistochrone Problem. One of the famous problems in the history of mathematics is the brachistochrone⁹ problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q , the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hôpital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown in Figure 2.3.6. The lower point Q has coordinates (x_0, y_0) . It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2, \quad (38)$$

where k^2 is a certain positive constant to be determined later.

- a. Solve equation (38) for y' . Why is it necessary to choose the positive square root?
- b. Introduce the new variable t by the relation

$$y = k^2 \sin^2 t. \quad (39)$$

Show that the equation found in part a then takes the form

$$2k^2 \sin^2 t \, dt = dx. \quad (40)$$

- c. Letting $\theta = 2t$, show that the solution of equation (40) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin \theta)/2, \quad y = k^2(1 - \cos \theta)/2. \quad (41)$$

Equations (41) are parametric equations of the solution of equation (38) that passes through $(0, 0)$. The graph of equations (41) is called a **cycloid**.

- d. If we make a proper choice of the constant k , then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$.

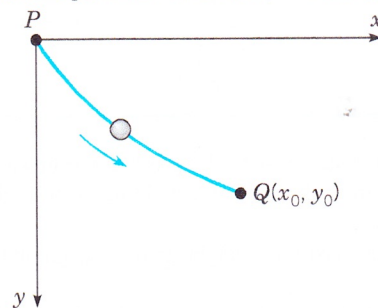


FIGURE 2.3.6 The brachistochrone (see Problem 24).

⁹The word “brachistochrone” comes from the Greek words *brachistos*, meaning shortest, and *chronos*, meaning time.

An introduction to numerical methods for first-order equations is given in Section 2.7, and a systematic discussion of numerical methods appears in Chapter 8. However, it is not necessary to study the numerical algorithms themselves in order to use effectively one of the many software packages that generate and plot numerical approximations to solutions of initial value problems.

Summary. The linear equation $y' + p(t)y = g(t)$ has several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. There is an expression for the solution, namely, equation (7) or equation (8). Moreover, although it involves two integrations, the expression is an explicit one for the solution $y = \phi(t)$ rather than an equation that defines ϕ implicitly.
3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t , then the solution also exists and is differentiable for all t .

None of these statements are true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as on the differential equation.

Problems

In each of Problems 1 through 4, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t - 3)y' + (\ln t)y = 2t$, $y(1) = 2$
2. $y' + (\tan t)y = \sin t$, $y(\pi) = 0$
3. $(4 - t^2)y' + 2ty = 3t^2$, $y(-3) = 1$
4. $(\ln t)y' + y = \cot t$, $y(2) = 3$

In each of Problems 5 through 8, state where in the ty -plane the hypotheses of Theorem 2.4.2 are satisfied.

5. $y' = (1 - t^2 - y^2)^{1/2}$
6. $y' = \frac{\ln |ty|}{1 - t^2 + y^2}$
7. $y' = (t^2 + y^2)^{3/2}$
8. $y' = \frac{1 + t^2}{3y - y^2}$

In each of Problems 9 through 12, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0 .

9. $y' = -4t/y$, $y(0) = y_0$
10. $y' = 2ty^2$, $y(0) = y_0$
11. $y' + y^3 = 0$, $y(0) = y_0$
12. $y' = \frac{t^2}{y(1 + t^3)}$, $y(0) = y_0$

In each of Problems 13 through 16, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases and how their behavior depends on the initial value y_0 when $t = 0$.

13. $y' = ty(3 - y)$
14. $y' = y(3 - ty)$
15. $y' = -y(3 - ty)$
16. $y' = t - 1 - y^2$

17. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.

- a. Is there a solution that passes through the point $(1, 1)$? If so, find it.
- b. Is there a solution that passes through the point $(2, 1)$? If so, find it.
- c. Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.

18. a. Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

Where are these solutions valid?

- b. Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
- c. Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part a for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.
19. a. Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c .
- b. Show that $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$ for $t > 0$, but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part b is nonlinear, while that of part a is linear.
20. Show that if $y = \phi(t)$ is a solution of $y' + p(t)y = 0$, then $y = c\phi(t)$ is also a solution for any value of the constant c .
21. Let $y = y_1(t)$ be a solution of

$$y' + p(t)y = 0, \quad (27)$$

and let $y = y_2(t)$ be a solution of

$$y' + p(t)y = g(t). \quad (28)$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of equation (28).

22. a. Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \quad (29)$$

where c is an arbitrary constant.

- b. Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \quad (30)$$

corresponding to $g(t) = 0$.

- c. Show that y_2 is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher-order linear equations have a pattern similar to equation (29).

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 23 and 25 deal with equations of this type.

23. a. Solve Bernoulli's equation when $n = 0$; when $n = 1$.
- b. Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was formulated by Leibniz in 1696.

In each of Problems 24 through 25, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 23b.

24. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.
25. $y' = \epsilon y - \sigma y^3$, $\epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0 ; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0 .

26. Solve the initial value problem

$$y' + 2y = g(t), \quad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

27. Solve the initial value problem

$$y' + p(t)y = 0, \quad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases}$$

2.5 Autonomous Differential Equations and Population Dynamics

An important class of first-order equations consists of those in which the independent variable does not appear explicitly. Such equations are called **autonomous** and have the form

$$dy/dt = f(y). \quad (1)$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Sections 1.1 and 1.2 we considered the special case of equation (1) in which $f(y) = ay + b$.

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, but the main purpose of this section is to show how geometric methods can be used to obtain important qualitative information directly from the differential equation without solving the equation. Of

Problems

Problems 1 through 4 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y , determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the ty -plane.

- 1. $dy/dt = ay + by^2$, $a > 0$, $b > 0$, $-\infty < y_0 < \infty$
- 2. $dy/dt = y(y-1)(y-2)$, $y_0 \geq 0$
- 3. $dy/dt = e^y - 1$, $-\infty < y_0 < \infty$
- 4. $dy/dt = e^{-y} - 1$, $-\infty < y_0 < \infty$

5. Semistable Equilibrium Solutions. Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side depart from it (see Figure 2.5.9). In this case the equilibrium solution is said to be **semistable**.

a. Consider the equation

$$dy/dt = k(1-y)^2, \quad (19)$$

where k is a positive constant. Show that $y = 1$ is the only critical point, with the corresponding equilibrium solution $\phi(t) = 1$.

b. Sketch $f(y)$ versus y . Show that y is increasing as a function of t for $y < 1$ and also for $y > 1$. The phase line has upward-pointing arrows both below and above $y = 1$. Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore, $\phi(t) = 1$ is semistable.

c. Solve equation (19) subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part b.

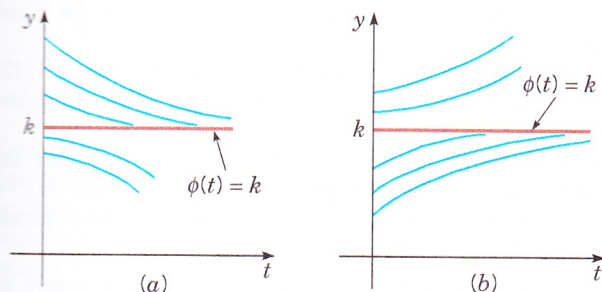


FIGURE 2.5.9 In both cases the equilibrium solution $\phi(t) = k$ is semistable. (a) $dy/dt \leq 0$; (b) $dy/dt \geq 0$.

Problems 6 through 9 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y , determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable (see Problem 5). Draw the phase line, and sketch several graphs of solutions in the ty -plane.

- 6. $dy/dt = y^2(y^2 - 1)$, $-\infty < y_0 < \infty$
- 7. $dy/dt = y(1 - y^2)$, $-\infty < y_0 < \infty$
- 8. $dy/dt = y^2(4 - y^2)$, $-\infty < y_0 < \infty$
- 9. $dy/dt = y^2(1 - y)^2$, $-\infty < y_0 < \infty$

10. Complete the derivation of the explicit formula for the solution (11) of the logistic model by solving equation (10) for y .

11. In Example 1, complete the manipulations needed to arrive at equation (13). That is, solve the solution (11) for t .

12. Complete the derivation of the location of the vertical asymptote in the solution (15) when $y_0 > T$. That is, derive formula (16) by finding the value of t when the denominator of the right-hand side of equation (15) is zero.

13. Complete the derivation of formula (18) for the locations of the inflection points of the solution of the logistic growth model with a threshold (17). *Hint:* Follow the steps outlined on p. 66.

14. Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point—that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.

15. Suppose that a certain population obeys the logistic equation $dy/dt = ry(1 - (y/K))$.

a. If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

b. If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \rightarrow \infty$ as $\alpha \rightarrow 0$ or as $\beta \rightarrow 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.

16. Another equation that has been used to model population growth is the Gompertz¹⁵ equation

$$\frac{dy}{dt} = ry \ln\left(\frac{K}{y}\right),$$

where r and K are positive constants.

a. Sketch the graph of $f(y)$ versus y , find the critical points, and determine whether each is asymptotically stable or unstable.

b. For $0 \leq y \leq K$, determine where the graph of y versus t is concave up and where it is concave down.

c. For each y in $0 < y \leq K$, show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.

17. a. Solve the Gompertz equation

$$\frac{dy}{dt} = ry \ln\left(\frac{K}{y}\right),$$

subject to the initial condition $y(0) = y_0$.

Hint: You may wish to let $u = \ln(y/K)$.

b. For the data given in Example 1 in the text ($r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$), use the Gompertz model to find the predicted value of $y(2)$.

c. For the same data as in part b, use the Gompertz model to find the time τ at which $y(\tau) = 0.75K$.

¹⁵Benjamin Gompertz (1779–1865) was an English actuary. He developed his model for population growth, published in 1825, in the course of constructing mortality tables for his insurance company.

18. A pond forms as water collects in a conical depression of radius a and depth h . Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

- a.** Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha \pi (3a/\pi h)^{2/3} V^{2/3},$$

where α is the coefficient of evaporation.

- b.** Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?
c. Find a condition that must be satisfied if the pond is not to overflow.

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y.$$

Although it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. Problems 19 and 20 explore some of the questions involved in formulating a rational strategy for managing the fishery.¹⁶

19. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y : the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey , where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y - Ey. \quad (20)$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

- a.** Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.
b. Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.
c. A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2 . Find Y as a function of the effort E ; the graph of this function is known as the yield-effort curve.
d. Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m .

20. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y - h. \quad (21)$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

- a.** If $h < rK/4$, show that equation (21) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.
b. Show that y_1 is unstable and y_2 is asymptotically stable.
c. From a plot of $f(y)$ versus y , show that if the initial population $y_0 > y_1$, then $y \rightarrow y_2$ as $t \rightarrow \infty$, but that if

$y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

- d.** If $h > rK/4$, show that y decreases to zero as t increases, regardless of the value of y_0 .

e. If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 5). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 19d. The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.¹⁷ Problems 21 through 23 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

21. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y . Since $x = 1 - y$, we obtain the initial value problem

$$\frac{dy}{dt} = \alpha y(1 - y), \quad y(0) = y_0, \quad (22)$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

- a.** Find the equilibrium points for the differential equation (22) and determine whether each is asymptotically stable, semistable, or unstable.
b. Solve the initial value problem 22 and verify that the conclusions you reached in part a are correct. Show that $y(t) \rightarrow 1$ as $t \rightarrow \infty$, which means that ultimately the disease spreads through the entire population.

22. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β , so

$$\frac{dy}{dt} = -\beta y. \quad (23)$$

Suppose also that the disease spreads at a rate proportional to the product of x and y ; thus

$$\frac{dx}{dt} = -\alpha xy. \quad (24)$$

- a.** Determine y at any time t by solving equation (23) subject to the initial condition $y(0) = y_0$.
b. Use the result of part a to find x at any time t by solving equation (24) subject to the initial condition $x(0) = x_0$.
c. Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \rightarrow \infty$.

¹⁶An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

¹⁷A standard source is the book by Bailey listed in the references. The models in Problems 21, 22, and 23 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

23. Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt , the rate at which the number of susceptibles declines, is given by

$$\frac{dx}{dt} = -(\beta + \mu(t))x. \quad (25)$$

The first term on the right-hand side of equation (25) is the rate at which susceptibles contract smallpox, and the second term is the rate at which they die from all other causes. Also

$$\frac{dn}{dt} = -\nu\beta x - \mu(t)n, \quad (26)$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right-hand side are the death rates due to smallpox and to all other causes, respectively.

a. Let $z = x/n$, and show that z satisfies the initial value problem

$$\frac{dz}{dt} = -\beta z(1 - \nu z), \quad z(0) = 1. \quad (27)$$

Observe that the initial value problem (27) does not depend on $\mu(t)$.

b. Find $z(t)$ by solving equation (27).

c. Bernoulli estimated that $\nu = \beta = 1/8$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: On the basis of the model just described and the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years, 7 months. He therefore supported the inoculation program.

Bifurcation Points. For an equation of the form

$$\frac{dy}{dt} = f(a, y), \quad (28)$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a . As a steadily increases or decreases, it often happens that at a certain value of a , called a **bifurcation point**, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 24 through 26 describe three types of bifurcations that can occur in simple equations of the form (28).

24. Consider the equation

$$\frac{dy}{dt} = a - y^2. \quad (29)$$

a. Find all of the critical points for equation (29). Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$.

b. Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.

c. In each case sketch several solutions of equation (29) in the ty -plane.

Note: If we plot the location of the critical points as a function of a in the ay -plane, we obtain Figure 2.5.10. This is called the **bifurcation diagram** for equation (29). The bifurcation at $a = 0$ is called a **saddle-node bifurcation**. This name is more natural in the context of second-order systems, which are discussed in Chapter 9.

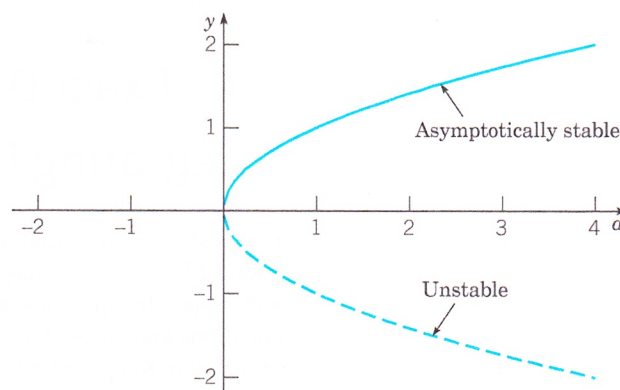


FIGURE 2.5.10 Bifurcation diagram for $y' = a - y^2$.

25. Consider the equation

$$\frac{dy}{dt} = ay - y^3 = y(a - y^2). \quad (30)$$

a. Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

b. In each case sketch several solutions of equation (30) in the ty -plane.

c. Draw the bifurcation diagram for equation (30)—that is, plot the location of the critical points versus a .

Note: For equation (30) the bifurcation point at $a = 0$ is called a **pitchfork bifurcation**. Your diagram may suggest why this name is appropriate.

26. Consider the equation

$$\frac{dy}{dt} = ay - y^2 = y(a - y). \quad (31)$$

a. Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

b. In each case sketch several solutions of equation (31) in the ty -plane.

c. Draw the bifurcation diagram for equation (31).

Note: Observe that for equation (31) there are the same number of critical points for $a < 0$ and $a > 0$ but that their stability has changed. For $a < 0$ the equilibrium solution $y = 0$ is asymptotically stable and $y = a$ is unstable, while for $a > 0$ the situation is reversed. Thus there has been an **exchange of stability** as a passes through the bifurcation point $a = 0$. This type of bifurcation is called a **transcritical bifurcation**.

27. Chemical Reactions. A second-order chemical reaction involves the interaction (collision) of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X ; this is denoted by $P + Q \rightarrow X$. Suppose that p and q , where $p \neq q$, are the initial concentrations of P and Q , respectively, and let $x(t)$ be the concentration of X at time t . Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t , and the rate at which the reaction occurs is given by the equation

$$\frac{dx}{dt} = \alpha(p - x)(q - x), \quad (32)$$

where α is a positive constant.

a. If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \rightarrow \infty$ without solving the differential equation. Then solve the initial value problem and find $x(t)$ for any t .

b. If the substances P and Q are the same, then $p = q$ and equation (32) is replaced by

$$\frac{dx}{dt} = \alpha(p - x)^2. \quad (33)$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \rightarrow \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t .

2.6 Exact Differential Equations and Integrating Factors

For first-order differential equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact differential equations for which there is also a well-defined method of solution. Keep in mind, however, that the first-order differential equations that can be solved by elementary integration methods are rather special; most first-order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \quad (1)$$

Solution:

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \quad 2xy = \frac{\partial \psi}{\partial y}. \quad (2)$$

Therefore, the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (3)$$

Assuming that y is a function of x , we can use the chain rule to write the left-hand side of equation (3) as $d\psi(x, y)/dx$. Then equation (3) has the form

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \quad (4)$$

Integrating equation (4) we obtain

$$\psi(x, y) = x^2 + xy^2 = c, \quad (5)$$

where c is an arbitrary constant. The level curves of $\psi(x, y)$ are the integral curves of equation (1). Solutions of equation (1) are defined implicitly by equation (5).

In solving equation (1) the key step was the recognition that there is a function ψ that satisfies equations (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \quad (6)$$

Integrating the first of equations (32) with respect to x , we obtain

$$\psi(x, y) = x^3 y + \frac{1}{2} x^2 y^2 + h(y).$$

Substituting this expression for $\psi(x, y)$ in the second of equations (32), we find that

$$x^3 + x^2 y + h'(y) = x^3 + x^2 y,$$

so $h'(y) = 0$ and $h(y)$ is a constant. Thus the solutions of equation (31), and hence of equation (19), are given implicitly by

$$x^3 y + \frac{1}{2} x^2 y^2 = c. \quad (33)$$

Solutions may also be found in explicit form since equation (33) is quadratic in y .

You may also verify that a second integrating factor for equation (19) is

$$\mu(x, y) = \frac{1}{xy(2x + y)}$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 22).

Problems

Determine whether each of the equations in Problems 1 through 8 is exact. If it is exact, find the solution.

- $(2x + 3) + (2y - 2)y' = 0$
- $(2x + 4y) + (2x - 2y)y' = 0$
- $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
- $\frac{dy}{dx} = -\frac{ax + by}{bx + cy}$
- $\frac{dy}{dx} = -\frac{ax - by}{bx - cy}$
- $(ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + 2x) + (xe^{xy} \cos(2x) - 3)y' = 0$
- $(y/x + 6x) + (\ln x - 2)y' = 0, \quad x > 0$
- $\frac{x}{(x^2 + y^2)^{3/2}} + \frac{y}{(x^2 + y^2)^{3/2}} \frac{dy}{dx} = 0$

In each of Problems 9 and 10, solve the given initial value problem and determine at least approximately where the solution is valid.

- $(2x - y) + (2y - x)y' = 0, \quad y(1) = 3$
- $(9x^2 + y - 1) - (4y - x)y' = 0, \quad y(1) = 0$

In each of Problems 11 and 12, find the value of b for which the given equation is exact, and then solve it using that value of b .

- $(xy^2 + bx^2 y) + (x + y)x^2 y' = 0$
- $(ye^{2xy} + x) + bxe^{2xy} y' = 0$
- Assume that equation (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x, y)$ is

$$\psi(x, y) = \int_{x_0}^x M(s, y_0) ds + \int_{y_0}^y N(x, t) dt,$$

where (x_0, y_0) is a point in R .

- Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

In each of Problems 15 and 16, show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

- $x^2 y^3 + x(1 + y^2)y' = 0, \quad \mu(x, y) = 1/(xy^3)$
- $(x + 2) \sin y + (x \cos y)y' = 0, \quad \mu(x, y) = xe^x$
- Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y) dy.$$

In each of Problems 18 through 21, find an integrating factor and solve the given equation.

- $(3x^2 y + 2xy + y^3) + (x^2 + y^2)y' = 0$
- $y' = e^{2x} + y - 1$
- $1 + (x/y - \sin y)y' = 0$
- $y + (2xy - e^{-2y})y' = 0$
- Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x, y) = (xy(2x + y))^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

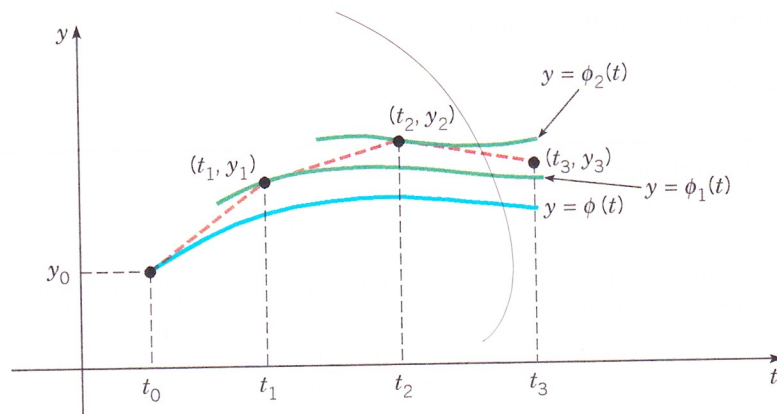


FIGURE 2.7.4 The Euler method.

In Example 2 the general solution of the differential equation is

$$y = 14 - 4t + ce^{-t/2} \quad (17)$$

and the solution of the initial value problem (9) corresponds to $c = -13$. The family of solutions (17) is a converging family since the term involving the arbitrary constant c approaches zero as $t \rightarrow \infty$. It does not matter very much which solutions we are approximating by tangent lines in the implementation of Euler's method, since all the solutions are getting closer and closer to each other as t increases.

On the other hand, in Example 3 the general solution of the differential equation is

$$y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}, \quad (18)$$

and, because the term involving the arbitrary constant c grows without bound as $t \rightarrow \infty$, this is a diverging family. Note that solutions corresponding to two nearby values of c become arbitrarily far apart as t increases. In Example 3 we are trying approximate the solution for $c = 11/4$, but in the use of Euler's method we are actually at each step following another solution that separates from the desired one faster and faster as t increases. This explains why the errors in Example 3 are so much larger than those in Example 2.

In using a numerical procedure such as the Euler method, you must always keep in mind the question of whether the results are accurate enough to be useful. In the preceding examples, the accuracy of the numerical results could be determined directly by a comparison with the solution obtained analytically. Of course, usually the analytical solution is not available if a numerical procedure is to be employed, so what we usually need are bounds for, or at least estimates of, the error that do not require a knowledge of the exact solution. You should also keep in mind that the best that we can expect, or hope for, from a numerical approximation is that it reflects the behavior of the actual solution. Thus a member of a diverging family of solutions will always be harder to approximate than a member of a converging family.

If you wish to read more about numerical approximations to solutions of initial value problems, you may go directly to Chapter 8 at this point. There, we present some information on the analysis of errors and also discuss several algorithms that are computationally much more efficient than the Euler method.

Problems

Note about Variations of Computed Results. Most of the problems in this section call for fairly extensive numerical computations. To handle these problems you need suitable computing hardware and software. Keep in mind that numerical results may vary somewhat, depending on how your program is constructed and on how your computer executes arithmetic steps, rounds off, and so forth. Minor variations in the last decimal place may be due to such causes and do not necessarily indicate that something is amiss. Answers in the back

of the book are recorded to six digits in most cases, although more digits were retained in the intermediate calculations.

In each of Problems 1 through 4:

- N a.** Find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4 using the Euler method with $h = 0.1$.
- N b.** Repeat part (a) with $h = 0.05$. Compare the results with those found in **a**.

- C** c. Repeat part a with $h = 0.025$. Compare the results with those found in a and b.
- D** d. Find the solution $y = \phi(t)$ of the given problem and evaluate $\phi(t)$ at $t = 0.1, 0.2, 0.3$, and 0.4 . Compare these values with the results of a, b, and c.

1. $y' = 3 + t - y, \quad y(0) = 1$
2. $y' = 2y - 1, \quad y(0) = 1$
3. $y' = 0.5 - t + 2y, \quad y(0) = 1$
4. $y' = 3 \cos t - 2y, \quad y(0) = 0$

In each of Problems 5 through 8, draw a direction field for the given differential equation and state whether you think that the solutions are converging or diverging.

5. $y' = 5 - 3\sqrt{y}$
6. $y' = y(3 - ty)$
7. $y' = -ty + 0.1y^3$
8. $y' = t^2 + y^2$

In each of Problems 9 and 10, use Euler's method to find approximate values of the solution of the given initial value problem at $t = 0.5, 1, 1.5, 2, 2.5$, and 3 : (a) With $h = 0.1$, (b) With $h = 0.05$, (c) With $h = 0.025$, (d) With $h = 0.01$.

9. $y' = 5 - 3\sqrt{y}, \quad y(0) = 2$
10. $y' = y(3 - ty), \quad y(0) = 0.5$

11. Consider the initial value problem

$$y' = \frac{3t^2}{3y^2 - 4}, \quad y(1) = 0.$$

- A** a. Use Euler's method with $h = 0.1$ to obtain approximate values of the solution at $t = 1.2, 1.4, 1.6$, and 1.8 .
- B** b. Repeat part a with $h = 0.05$.
- C** c. Compare the results of parts a and b. Note that they are reasonably close for $t = 1.2, 1.4$, and 1.6 but are quite different for $t = 1.8$. Also note (from the differential equation) that the line tangent to the solution is parallel to the y -axis when $y = \pm 2/\sqrt{3} \cong \pm 1.155$. Explain how this might cause such a difference in the calculated values.

D 12. Consider the initial value problem

$$y' = t^2 + y^2, \quad y(0) = 1.$$

Use Euler's method with $h = 0.1, 0.05, 0.025$, and 0.01 to explore the solution of this problem for $0 \leq t \leq 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field in Problem 8?

13. Consider the initial value problem

$$y' = -ty + 0.1y^3, \quad y(0) = \alpha,$$

where α is a given number.

- G** a. Draw a direction field for the differential equation (or reexamine the one from Problem 7). Observe that there is a critical value of α in the interval $2 \leq \alpha \leq 3$ that separates converging solutions from diverging ones. Call this critical value α_0 .

N b. Use Euler's method with $h = 0.01$ to estimate α_0 . Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

14. Consider the initial value problem

$$y' = y^2 - t^2, \quad y(0) = \alpha,$$

where α is a given number.

- G** a. Draw a direction field for the differential equation. Note that there is a critical value of α in the interval $0 \leq \alpha \leq 1$ that separates converging solutions from diverging ones. Call this critical value α_0 .

N b. Use Euler's method with $h = 0.01$ to estimate α_0 . Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

15. Convergence of Euler's Method. It can be shown that under suitable conditions on f , the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \quad y(t_0) = y_0.$$

- A** a. Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

N b. Using the Euler formula, show that

$$y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \dots$$

- C** c. Noting that $y_1 = (1 + h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \quad (19)$$

for each positive integer n .

- D** d. Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n . Note also that $h \rightarrow 0$ as $n \rightarrow \infty$. By substituting for h in equation (19) and letting $n \rightarrow \infty$, show that $y_n \rightarrow \phi(t)$ as $n \rightarrow \infty$.

Hint: $\lim_{n \rightarrow \infty} (1 + a/n)^n = e^a$.

In each of Problems 16 and 17, use the technique discussed in Problem 15 to show that the approximation obtained by the Euler method converges to the exact solution at any fixed point as $h \rightarrow 0$.

16. $y' = y, \quad y(0) = 1$

17. $y' = 2y - 1, \quad y(0) = 1$ *Hint:* $y_1 = (1 + 2h)/2 + 1/2$

2.8 The Existence and Uniqueness Theorem

In this section we discuss the proof of Theorem 2.4.2, the fundamental existence and uniqueness theorem for first-order initial value problems. Recall that this theorem states that under certain conditions on $f(t, y)$, the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0 \quad (1)$$

has a unique solution in some interval containing the point t_0 .

4. Are there other solutions of the integral equation (3) besides $y = \phi(t)$?

To show the uniqueness of the solution $y = \phi(t)$, we can proceed much as in the example. First, assume the existence of another solution $y = \psi(t)$. It is then possible to show (see Problem 18) that the difference $\phi(t) - \psi(t)$ satisfies the inequality

$$|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)| ds \quad (30)$$

for $0 \leq t \leq h$ and a suitable positive number A . From this point the argument is identical to that given in the example, and we conclude that there is no solution of the initial value problem (2) other than the one generated by the method of successive approximations.

Problems

In each of Problems 1 and 2, transform the given initial value problem into an equivalent problem with the initial point at the origin.

1. $dy/dt = t^2 + y^2, \quad y(1) = 2$
2. $dy/dt = 1 - y^3, \quad y(-1) = 3$

In each of Problems 3 through 4, let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximations.

- a. Determine $\phi_n(t)$ for an arbitrary value of n .
 - G** b. Plot $\phi_n(t)$ for $n = 1, \dots, 4$. Observe whether the iterates appear to be converging.
 - c. Express $\lim_{n \rightarrow \infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.
 - G** d. Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \dots, 4$. For each of $\phi_1(t), \dots, \phi_4(t)$, estimate the interval in which it is a reasonably good approximation to the actual solution.
- N** 3. $y' = 2(y + 1), \quad y(0) = 0$
 - N** 4. $y' = -y/2 + t, \quad y(0) = 0$

In each of Problems 5 and 6, let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

- a. Determine $\phi_n(t)$ for an arbitrary value of n .
 - G** b. Plot $\phi_n(t)$ for $n = 1, \dots, 4$. Observe whether the iterates appear to be converging.
 - c. Show that the sequence $\{\phi_n(t)\}$ converges.
5. $y' = ty + 1, \quad y(0) = 0$
 6. $y' = t^2y - t, \quad y(0) = 0$

In each of Problems 7 and 8, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

- a. Calculate $\phi_1(t), \dots, \phi_3(t)$.
 - G** b. Plot $\phi_1(t), \dots, \phi_3(t)$. Observe whether the iterates appear to be converging.
7. $y' = t^2 + y^2, \quad y(0) = 0$
 8. $y' = 1 - y^3, \quad y(0) = 0$

In each of Problems 9 and 10, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

- a. Calculate $\phi_1(t), \dots, \phi_4(t)$, or (if necessary) Taylor approximations to these iterates. Keep terms up to order six.
 - G** b. Plot the functions you found in part a and observe whether they appear to be converging.
9. $y' = -\sin y + 1, \quad y(0) = 0$
 10. $y' = \frac{3t^2 + 4t + 2}{2(y - 1)}, \quad y(0) = 0$

11. Let $\phi_n(x) = x^n$ for $0 \leq x \leq 1$ and show that

$$\lim_{n \rightarrow \infty} \phi_n(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

This example shows that a sequence of continuous functions may converge to a limit function that is discontinuous.

12. Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}$, $0 \leq x \leq 1$.

- a. Show that $\lim_{n \rightarrow \infty} \phi_n(x) = 0$ for $0 \leq x \leq 1$; hence

$$\int_0^1 \lim_{n \rightarrow \infty} \phi_n(x) dx = 0.$$

- b. Show that $\int_0^1 2nxe^{-nx^2} dx = 1 - e^{-n}$; hence

$$\lim_{n \rightarrow \infty} \int_0^1 \phi_n(x) dx = 1.$$

Thus, in this example,

$$\lim_{n \rightarrow \infty} \int_a^b \phi_n(x) dx \neq \int_a^b \lim_{n \rightarrow \infty} \phi_n(x) dx,$$

even though $\lim_{n \rightarrow \infty} \phi_n(x)$ exists and is continuous.

13. a. Verify that $\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$ is a solution of the integral equation (9).

- b. Verify that $\phi(t)$ is also a solution of the initial value problem (6).

- c. Use the fact that $\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$ to evaluate $\phi(t)$ in terms of elementary functions.

- d. Solve initial value problem (6) as a separable equation.

- e. Solve initial value problem (6) as a first order linear equation.

In Problems 14 through 17, we indicate how to prove that the sequence $\{\phi_n(t)\}$, defined by equations (4) through (7), converges.

- a. Verify that $\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}$ is a solution of the integral equation (9).

- b. Verify that $\phi(t)$ is also a solution of the initial value problem (6).

- c. Use the fact that $\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$ to evaluate $\phi(t)$ in terms of elementary functions.

- d. Solve initial value problem (6) as a separable equation.

- e. Solve initial value problem (6) as a first order linear equation.

14. If $\partial f / \partial y$ is continuous in the rectangle D , show that there is a positive constant K such that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|, \quad (31)$$

where (t, y_1) and (t, y_2) are any two points in D having the same t coordinate. This inequality is known as a Lipschitz²² condition.

Hint: Hold t fixed and use the mean value theorem on f as a function of y only. Choose K to be the maximum value of $|\partial f / \partial y|$ in D .

15. If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 14 to show that

$$|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \leq K|\phi_n(t) - \phi_{n-1}(t)|.$$

16. a. Show that if $|t| \leq h$, then

$$|\phi_1(t)| \leq M|t|,$$

where M is chosen so that $|f(t, y)| \leq M$ for (t, y) in D .

b. Use the results of Problem 15 and part a of Problem 16 to show that

$$|\phi_2(t) - \phi_1(t)| \leq \frac{MK|t|^2}{2}.$$

c. Show, by mathematical induction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \leq \frac{MK^{n-1}|t|^n}{n!} \leq \frac{MK^{n-1}h^n}{n!}.$$

17. Note that

$$\phi_n(t) = \phi_1(t) + (\phi_2(t) - \phi_1(t)) + \cdots + (\phi_n(t) - \phi_{n-1}(t)).$$

²²The German mathematician Rudolf Lipschitz (1832–1903), professor at the University of Bonn for many years, worked in several areas of mathematics. The inequality (i) can replace the hypothesis that $\partial f / \partial y$ is continuous in Theorem 2.8.1; this results in a slightly stronger theorem.

a. Show that

$$|\phi_n(t)| \leq |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

b. Use the results of Problem 16 to show that

$$|\phi_n(t)| \leq \frac{M}{K} \left(Kh + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!} \right).$$

c. Show that the sum in part b converges as $n \rightarrow \infty$ and, hence, the sum in part a also converges as $n \rightarrow \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

18. In this problem we deal with the question of uniqueness of the solution of the integral equation (3)

$$\phi(t) = \int_0^t f(s, \phi(s)) ds.$$

a. Suppose that ϕ and ψ are two solutions of equation (3). Show that, for $t \geq 0$,

$$\phi(t) - \psi(t) = \int_0^t (f(s, \phi(s)) - f(s, \psi(s))) ds.$$

b. Show that

$$|\phi(t) - \psi(t)| \leq \int_0^t (f(s, \phi(s)) - f(s, \psi(s))) ds.$$

c. Use the result of Problem 14 to show that

$$|\phi(t) - \psi(t)| \leq K \int_0^t |\phi(s) - \psi(s)| ds,$$

where K is an upper bound for $|\partial f / \partial y|$ in D . This is the same as equation (30), and the rest of the proof may be constructed as indicated in the text.

2.9 First-Order Difference Equations

Although a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more natural. For instance, the continuous model of compound interest used in Section 2.3 is only an approximation to the actual discrete process. Similarly, sometimes population growth may be described more accurately by a discrete model than by a continuous model. This is true, for example, of species whose generations do not overlap and that propagate at regular intervals, such as at particular times of the calendar year. Then the population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_n in the preceding year; that is,

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \dots \quad (1)$$

Equation (1) is called a **first-order difference equation**. It is first-order because the value of y_{n+1} depends on the value of y_n but not on earlier values y_{n-1} , y_{n-2} , and so forth. As for differential equations, the difference equation (1) is **linear** if f is a linear function of y_n ; otherwise, it is **nonlinear**. A **solution** of the difference equation (1) is a sequence of numbers y_0, y_1, y_2, \dots that satisfy the equation for each n . In addition to the difference equation itself, there may also be an **initial condition**

$$y_0 = \alpha \quad (2)$$

that prescribes the value of the first term of the solution sequence.

We now assume temporarily that the function f in equation (1) depends only on y_n , but not on n . In this case

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2, \dots \quad (3)$$

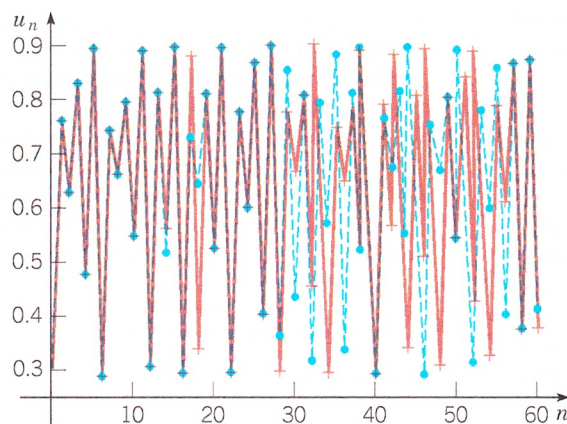


FIGURE 2.9.7 Two solutions of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.65$; $u_0 = 0.3$ and $u_0 = 0.305$.

of his analysis of this equation as a model of the population of certain insect species, May suggested that if the growth rate ρ is too large, then it will be impossible to make effective long-range predictions about these insect populations. The occurrence of chaotic solutions in seemingly simple problems has stimulated an enormous amount of research, but many questions remain unanswered. It is increasingly clear, however, that chaotic solutions are much more common than was suspected at first and that they may be a part of the investigation of a wide range of phenomena.

Problems

In each of Problems 1 through 4, solve the given difference equation in terms of the initial value y_0 . Describe the behavior of the solution as $n \rightarrow \infty$.

1. $y_{n+1} = -0.9y_n$

2. $y_{n+1} = \sqrt{\frac{n+3}{n+1}} y_n$

3. $y_{n+1} = (-1)^{n+1} y_n$

4. $y_{n+1} = 0.5y_n + 6$

5. An investor deposits \$1000 in an account paying interest at a rate of 8%, compounded monthly, and also makes additional deposits of \$25 per month. Find the balance in the account after 3 years.

6. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. What monthly payment rate is required to pay off the loan in 3 years? Compare your result with that of Problem 7 in Section 2.3.

7. A homebuyer takes out a mortgage of \$100,000 with an interest rate of 9%. What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?

8. If the interest rate on a 20-year mortgage is fixed at 10% and if a monthly payment of \$1000 is the maximum that the buyer can afford, what is the maximum mortgage loan that can be made under these conditions?

9. A homebuyer wishes to finance the purchase with a \$95,000 mortgage with a 20-year term. What is the maximum interest rate the buyer can afford if the monthly payment is not to exceed \$900?

The Logistic Difference Equation. Problems 10 through 15 deal with the difference equation (21), $u_{n+1} = \rho u_n(1 - u_n)$.

10. Carry out the details in the linear stability analysis of the equilibrium solution $u_n = (\rho - 1)/\rho$. That is, derive the difference equation (26) in the text for the perturbation v_n .

11. **N a.** For $\rho = 3.2$, plot or calculate the solution of the logistic equation (21) for several initial conditions, say, $u_0 = 0.2, 0.4, 0.6$, and 0.8 . Observe that in each case the solution approaches a steady oscillation between the same two values. This illustrates that the long-term behavior of the solution is independent of the initial conditions.

N b. Make similar calculations and verify that the nature of the solution for large n is independent of the initial condition for other values of ρ , such as 2.6, 2.8, and 3.4.

12. Assume that $\rho > 1$ in equation (21).

G a. Draw a qualitatively correct staircase diagram and thereby show that if $u_0 < 0$, then $u_n \rightarrow -\infty$ as $n \rightarrow \infty$.

G b. In a similar way, determine what happens as $n \rightarrow \infty$ if $u_0 > 1$.

13. The solutions of equation (21) change from convergent sequences to periodic oscillations of period 2 as the parameter ρ passes through the value 3. To see more clearly how this happens, carry out the following calculations.

N a. Plot or calculate the solution for $\rho = 2.9, 2.95$, and 2.99 , respectively, using an initial value u_0 of your choice in the interval $(0, 1)$. In each case estimate how many iterations are required for the solution to get “very close” to the limiting value. Use any convenient interpretation of what “very close” means in the preceding sentence.

N b. Plot or calculate the solution for $\rho = 3.01, 3.05$, and 3.1 , respectively, using the same initial condition as in part a. In each case estimate how many iterations are needed to reach a steady-state oscillation. Also find or estimate the two values in the steady-state oscillation.

N 14. By calculating or plotting the solution of equation (21) for different values of ρ , estimate the value of ρ at which the solution changes from an oscillation of period 2 to one of period 4. In the same way, estimate the value of ρ at which the solution changes from period 4 to period 8.

N 15. Let ρ_k be the value of ρ at which the solution of equation (21) changes from period 2^{k-1} to period 2^k . Thus, as noted in the text, $\rho_1 = 3$, $\rho_2 \cong 3.449$, and $\rho_3 \cong 3.544$.

a. Using these values of ρ_1, ρ_2 , and ρ_3 , or those you found in Problem 14, calculate $(\rho_2 - \rho_1)/(\rho_3 - \rho_2)$.

b. Let $\delta_n = (\rho_n - \rho_{n-1})/(\rho_{n+1} - \rho_n)$. It can be shown that δ_n approaches a limit δ as $n \rightarrow \infty$, where $\delta \cong 4.6692$ is known as the Feigenbaum²⁴ number. Determine the percentage difference between the limiting value δ and δ_2 , as calculated in part a.

c. Assume that $\delta_3 = \delta$ and use this relation to estimate ρ_4 , the value of ρ at which solutions of period 16 appear.

G d. By plotting or calculating solutions near the value of ρ_4 found in part c, try to detect the appearance of a period 16 solution.

e. Observe that

$$\rho_n = \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + \cdots + (\rho_n - \rho_{n-1}).$$

Assuming that

$$\rho_4 - \rho_3 = (\rho_3 - \rho_2)\delta^{-1}, \quad \rho_5 - \rho_4 = (\rho_3 - \rho_2)\delta^{-2},$$

and so forth, express ρ_n as a geometric sum. Then find the limit ρ_n as $n \rightarrow \infty$. This is an estimate of the value of ρ at which the onset of chaos occurs in the solution of the logistic equation (21).

²⁴This result for the logistic difference equation was discovered in August 1975 by Mitchell Feigenbaum (1944–), while he was working at the Los Alamos National Laboratory. Within a few weeks he had established that the same limiting value also appears in a large class of period-doubling difference equations. Feigenbaum, who has a doctorate in physics from M.I.T., is now at Rockefeller University.

Chapter Review Problems

Miscellaneous Problems. One of the difficulties in solving first-order differential equations is that there are several methods of solution, each of which can be used on a certain type of equation. It may take some time to become proficient in matching solution methods with equations. The first 24 of the following problems are presented to give you some practice in identifying the method or methods applicable to a given equation. The remaining problems involve certain types of equations that can be solved by specialized methods.

In each of Problems 1 through 24, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

1. $\frac{dy}{dx} = \frac{x^3 - 2y}{x}$

2. $\frac{dy}{dx} = \frac{1 + \cos x}{2 - \sin y}$

3. $\frac{dy}{dx} = \frac{2x + y}{3 + 3y^2 - x}, \quad y(0) = 0$

4. $\frac{dy}{dx} = 3 - 6x + y - 2xy$

5. $\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}$

6. $x \frac{dy}{dx} + xy = 1 - y, \quad y(1) = 0$

7. $x \frac{dy}{dx} + 2y = \frac{\sin x}{x}, \quad y(2) = 1$

8. $\frac{dy}{dx} = -\frac{2xy + 1}{x^2 + 2y}$

9. $(x^2y + xy - y) + (x^2y - 2x^2) \frac{dy}{dx} = 0$

10. $(x^2 + y) + (x + e^y) \frac{dy}{dx} = 0$

11. $(x + y) + (x + 2y) \frac{dy}{dx} = 0, \quad y(2) = 3$

12. $(e^x + 1) \frac{dy}{dx} = y - ye^x$

13. $\frac{dy}{dx} = \frac{e^{-x} \cos y - e^{2y} \cos x}{-e^{-x} \sin y + 2e^{2y} \sin x}$

14. $\frac{dy}{dx} = e^{2x} + 3y$

15. $\frac{dy}{dx} + 2y = e^{-x^2 - 2x}, \quad y(0) = 3$

16. $\frac{dy}{dx} = \frac{3x^2 - 2y - y^3}{2x + 3xy^2}$

17. $y' = e^{x+y}$

18. $\frac{dy}{dx} + \frac{2y^2 + 6xy - 4}{3x^2 + 4xy + 3y^2} = 0$

19. $t \frac{dy}{dt} + (t + 1)y = e^{t^2}$

20. $xy' = y + xe^{y/x}$

21. $\frac{dy}{dx} = \frac{x}{x^2y + y^3} \quad \text{Hint: Let } u = x^2.$

22. $\frac{dy}{dx} = \frac{x + y}{x - y}$

23. $(3y^2 + 2xy) - (2xy + x^2) \frac{dy}{dx} = 0$

24. $xy' + y - y^2e^{2x} = 0, \quad y(1) = 2$

25. Riccati Equations. The equation

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati²⁵ equation. Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution

$$y = y_1(t) + \frac{1}{v(t)}.$$

Show that $v(t)$ satisfies the first-order linear equation

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.$$

Note that $v(t)$ will contain a single arbitrary constant.

26. Verify that the given function is a particular solution of the given Riccati equation. Then use the method of Problem 25 to solve the following Riccati equations:

a. $y' = 1 + t^2 - 2ty + y^2$; $y_1(t) = t$

b. $y' = -\frac{1}{t^2} - \frac{y}{t} + y^2$; $y_1(t) = \frac{1}{t}$

c. $\frac{dy}{dt} = \frac{2\cos^2 t - \sin^2 t + y^2}{2\cos t}$; $y_1(t) = \sin t$

27. The propagation of a single action in a large population (for example, drivers turning on headlights at sunset) often depends partly on external circumstances (gathering darkness) and partly on a tendency to imitate others who have already performed the action in question. In this case the proportion $y(t)$ of people who have performed the action can be described²⁶ by the equation

$$dy/dt = (1 - y)(x(t) + by), \quad (28)$$

where $x(t)$ measures the external stimulus and b is the imitation coefficient.

a. Observe that equation (28) is a Riccati equation and that $y_1(t) = 1$ is one solution. Use the transformation suggested in Problem 25, and find the linear equation satisfied by $v(t)$.

b. Find $v(t)$ in the case that $x(t) = at$, where a is a constant. Leave your answer in the form of an integral.

²⁵Riccati equations are named for Jacopo Francesco Riccati (1676–1754), a Venetian nobleman, who declined university appointments in Italy, Austria, and Russia to pursue his mathematical studies privately at home. Riccati studied these equations extensively; however, it was Euler (in 1760) who discovered the result stated in this problem.

²⁶See Anatol Rapoport, "Contribution to the Mathematical Theory of Mass Behavior: I. The Propagation of Single Acts," *Bulletin of Mathematical Biophysics* 14 (1952), pp. 159–169, and John Z. Hearon, "Note on the Theory of Mass Behavior," *Bulletin of Mathematical Biophysics* 17 (1955), pp. 7–13.

Some Special Second-Order Differential Equations. Second-order differential equations involve the second derivative of the unknown function and have the general form $y'' = f(t, y, y')$. Usually, such equations cannot be solved by methods designed for first-order equations. However, there are two types of second-order equations that can be transformed into first-order equations by a suitable change of variable. The resulting equation can sometimes be solved by the methods presented in this chapter. Problems 28 through 37 deal with these types of equations.

Equations with the Dependent Variable Missing. For a second-order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first-order differential equation of the form $v' = f(t, v)$. If this equation can be solved for v , then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first-order equation for v , and a second is introduced in the integration for y . In each of Problems 28 through 31, use this substitution to solve the given equation.

28. $t^2y'' + 2ty' - 1 = 0$, $t > 0$

29. $ty'' + y' = 1$, $t > 0$

30. $y'' + t(y')^2 = 0$

31. $2t^2y'' + (y')^3 = 2ty'$, $t > 0$

Equations with the Independent Variable Missing. Consider second-order differential equations of the form $y'' = f(y, y')$, in which the independent variable t does not appear explicitly. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right-hand side of this equation depends on y and v , rather than on t and v , this equation contains too many variables. However, if we think of y as the independent variable, then by the chain rule, $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first-order equation can be solved, we obtain v as a function of y . A relation between y and t results from solving $dy/dt = v(y)$, which is a separable equation. Again, there are two arbitrary constants in the final result. In each of Problems 32 through 35, use this method to solve the given differential equation.

32. $yy'' + (y')^2 = 0$

33. $y'' + y = 0$

34. $yy'' - (y')^3 = 0$

35. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 35 the transformed equation is a Bernoulli equation. See Problem 23 in Section 2.4.

In each of Problems 36 through 37, solve the given initial value problem using the methods of Problems 28 through 35.

36. $y'y'' = 2$, $y(0) = 1$, $y'(0) = 2$

37. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0$, $y(1) = 2$, $y'(1) = -1$

References

The two books mentioned in Section 2.5 are

Bailey, N. T. J., *The Mathematical Theory of Infectious Diseases and Its Applications* (2nd ed.) (New York: Hafner Press, 1975).

Clark, Colin W., *Mathematical Bioeconomics* (2nd ed.) (New York: Wiley-Interscience, 1990).

A good introduction to population dynamics, in general, is

Frauenthal, J. C., *Introduction to Population Modeling* (Boston: Birkhauser, 1980).

A fuller discussion of the proof of the fundamental existence and uniqueness theorem can be found in many more advanced books on differential equations. Two that are reasonably accessible to elementary readers are