CHAPTER 1

Introduction

In this first chapter we provide a foundation for your study of differential equations in several different ways. First, we use two problems to illustrate some of the basic ideas that we will return to, and elaborate upon, frequently throughout the remainder of the book. Later, to provide organizational structure for the book, we indicate several ways of classifying differential equations.

The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. On the other hand, it is important to recognize that differential equations remains a dynamic field of inquiry today, with many interesting open questions. We outline some of the major trends in the historical development of the subject and mention a few of the outstanding mathematicians who have contributed to it. Additional biographical information about some of these contributors will be highlighted at appropriate times in later chapters.

1.1 Some Basic Mathematical Models; Direction Fields

Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it), you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1 | A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Problems

In each of Problems 1 through 4, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

G **1.** $y' = 3 - 2y$

G **2.** $y' = 2y - 3$

G **3.** $y' = -1 - 2y$

G **4.** $y' = 1 + 2y$

In each of Problems 5 and 6, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

5. All solutions approach $y = 2/3$.

6. All other solutions diverge from $y = 2$.

In each of Problems 7 through 10, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

G **7.** $y' = y(4 - y)$

G **8.** $y' = -y(5 - y)$

G **9.** $y' = y^2$

G **10.** $y' = y(y - 2)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.1.5 through 1.1.10. In each of Problems 11 through 16, identify the differential equation that corresponds to the given direction field.

a. $y' = 2y - 1$
b. $y' = 2 + y$
c. $y' = y - 2$
d. $y' = y(y + 3)$
e. $y' = y(y - 3)$
f. $y' = 1 + 2y$
g. $y' = -2 - y$
h. $y' = y(3 - y)$
i. $y' = 1 - 2y$
j. $y' = 2 - y$

11. The direction field of Figure 1.1.5.

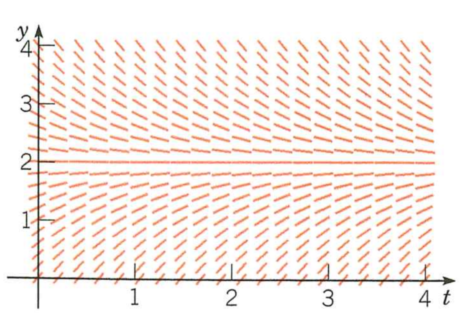

FIGURE 1.1.5 Problem 11.

12. The direction field of Figure 1.1.6.

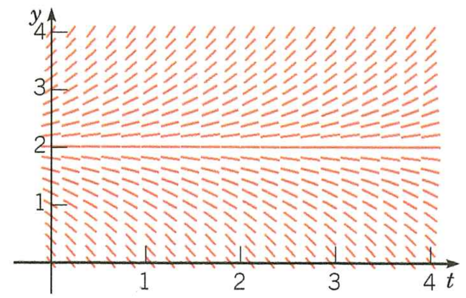

FIGURE 1.1.6 Problem 12.

13. The direction field of Figure 1.1.7.

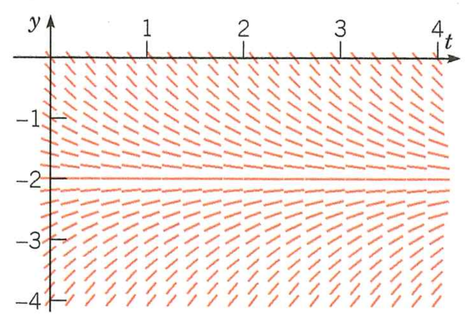

FIGURE 1.1.7 Problem 13.

14. The direction field of Figure 1.1.8.

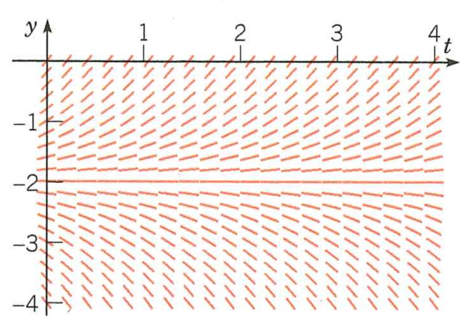

FIGURE 1.1.8 Problem 14.

15. The direction field of Figure 1.1.9.

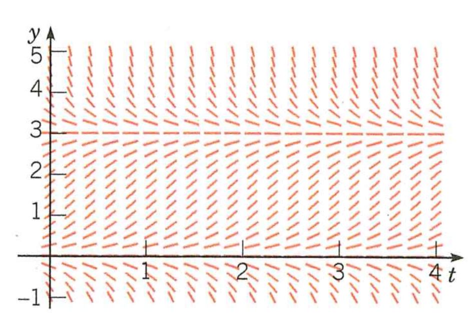

FIGURE 1.1.9 Problem 15.

16. The direction field of Figure 1.1.10.

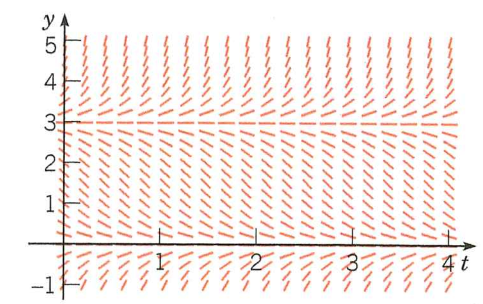

FIGURE 1.1.10 Problem 16.

17. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 grams of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

a. Write a differential equation for the amount of chemical in the pond at any time.
b. How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?
c. Write a differential equation for the concentration of the chemical in the pond at time t. *Hint:* The concentration is $c = a/v = a(t)/10^6$.

18. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

19. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is 0.05 $(\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

20. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4/h.

a. Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.
b. How much of the drug is present in the bloodstream after a long time?

N **21.** For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

a. Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.
b. Determine the limiting velocity after a long time.
c. If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.
N **d.** Using the data in part c, draw a direction field and compare it with Figure 1.1.3.

In each of Problems 22 through 25, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right-hand sides of these equations depend on t as well as y; therefore, their solutions can exhibit more complicated behavior than those in the text.

G **22.** $y' = -2 + t - y$

G **23.** $y' = e^{-t} + y$

G **24.** $y' = 3\sin t + 1 + y$

G **25.** $y' = -\dfrac{2t + y}{2y}$

[2] See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

1.2 Solutions of Some Differential Equations

In the preceding section we derived the differential equations

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k. \tag{2}$$

Equation (1) models a falling object, and equation (2) models a population of field mice preyed on by owls. Both of these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of equations (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

Further Remarks on Mathematical Modeling. Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object, the underlying physical principle (Newton's laws of motion) is well established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient γ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly—for example, by measuring the time of fall from a given height and then calculating the value of γ that predicts this observed time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate r and the predation rate k depends on observations of actual populations, which may be subject to considerable variation. The assumption that r and k are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the field mouse population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

If the differences between actual observations and a mathematical model's predictions are too great, then you need to consider refining the model, making more careful observations, or perhaps both. There is almost always a tradeoff between accuracy and simplicity. Both are desirable, but a gain in one usually involves a loss in the other. However, even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also give satisfactory results under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

Historical Background, Part II: Euler, Lagrange, and Laplace. The greatest mathematician of the eighteenth century, Leonhard Euler (1707–1783), grew up near Basel, Switzerland and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. For the remainder of his life he was associated with the St. Petersburg Academy (1727–1741 and 1766–1783) and the Berlin Academy (1741–1766). Losing sight in his right eye in 1738, and in his left eye in 1766, did not stop Euler from being one of the most prolific mathematicians of all time. In addition to publishing more than 500 books and papers during his life, an additional 400 have appeared posthumously.

Of particular interest here is Euler's formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Among other things, Euler identified the condition for exactness of first-order differential equations (Section 2.6) in 1734–1735, developed the theory of integrating factors (Section 2.6) in the same paper, and gave the general solution of homogeneous linear differential equations with constant coefficients (Sections 3.1, 3.3, 3.4, and 4.2) in 1743. He extended the latter results to nonhomogeneous differential equations in 1750–1751. Beginning about 1750, Euler made frequent use of power series (Chapter 5) in solving differential equations. He also proposed a numerical procedure (Sections 2.7 and 8.1) in 1768–1769, made important contributions in partial differential equations, and gave the first systematic treatment of the calculus of variations.

Joseph-Louis Lagrange (1736–1813) became professor of mathematics in his native Turin, Italy, at the age of 19. He succeeded Euler in the chair of mathematics at the Berlin Academy in 1766 and moved on to the Paris Academy in 1787. He is most famous for his monumental work *Mécanique analytique*, published in 1788, an elegant and comprehensive treatise of Newtonian mechanics. With respect to elementary differential equations, Lagrange showed in 1762–1765 that the general solution of a homogeneous nth order linear differential equation is a linear combination of n independent solutions (Sections 3.2 and 4.1). Later, in 1774–1775, he offered a complete development of the method of variation of parameters (Sections 3.6 and 4.4). Lagrange is also known for fundamental work in partial differential equations and the calculus of variations.

Pierre-Simon de Laplace (1749–1827) lived in Normandy, France, as a boy but arrived in Paris in 1768 and quickly made his mark in scientific circles, winning election to the Académie des Sciences in 1773. He was preeminent in the field of celestial mechanics; his greatest work, *Traité de mécanique céleste*, was published in five volumes between 1799 and 1825. Laplace's equation is fundamental in many branches of mathematical physics, and Laplace studied it extensively in connection with gravitational attraction. The Laplace transform (Chapter 6) is also named for him, although its usefulness in solving differential equations was not recognized until much later.

By the end of the eighteenth century many elementary methods of solving ordinary differential equations had been discovered. In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on power series expansions (see Chapter 5). These methods find their natural setting in the complex plane. Consequently, they benefitted from, and to some extent stimulated, the more or less simultaneous development of the theory of complex analytic functions. Partial differential equations also began to be studied intensively, as their crucial role in mathematical physics became clear. In this connection a number of functions, arising as solutions of certain ordinary differential equations, occurred repeatedly and were studied exhaustively. Known collectively as higher transcendental functions, many of them are associated with the names of mathematicians, including Bessel (Section 5.7), Legendre (Section 5.3), Hermite (Section 5.2), Chebyshev (Section 5.3), Hankel, and many others.

Problems

N **1.** Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

a. $dy/dt = -y + 5, \quad y(0) = y_0$
b. $dy/dt = -2y + 5, \quad y(0) = y_0$
c. $dy/dt = -2y + 10, \quad y(0) = y_0$

G **2.** Follow the instructions for Problem 1 for the following initial-value problems:

a. $dy/dt = y - 5, \quad y(0) = y_0$
G **b.** $dy/dt = 2y - 5, \quad y(0) = y_0$
c. $dy/dt = 2y - 10, \quad y(0) = y_0$

3. Consider the differential equation

$$dy/dt = -ay + b,$$

where both a and b are positive numbers.

a. Find the general solution of the differential equation.
G **b.** Sketch the solution for several different initial conditions.
c. Describe how the solutions change under each of the following conditions:

i. a increases.
ii. b increases.
iii. Both a and b increase, but the ratio b/a remains the same.

4. Consider the differential equation $dy/dt = ay - b$.

a. Find the equilibrium solution y_e.
b. Let $Y(t) = y - y_e$; thus $Y(t)$ is the deviation from the equilibrium solution. Find the differential equation satisfied by $Y(t)$.

5. Undetermined Coefficients. Here is an alternative way to solve the equation

$$\frac{dy}{dt} = ay - b. \tag{31}$$

a. Solve the simpler equation

$$\frac{dy}{dt} = ay. \tag{32}$$

Call the solution $y_1(t)$.

b. Observe that the only difference between equations (31) and (32) is the constant $-b$ in equation (31). Therefore, it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k such that $y = y_1(t) + k$ is a solution of equation (31).

c. Compare your solution from part b with the solution given in the text in equation (17).

Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.5 in connection with second-order equations.

6. Use the method of Problem 5 to solve the equation

$$\frac{dy}{dt} = -ay + b.$$

7. The field mouse population in Example 1 satisfies the differential equation

$$\frac{dy}{dt} = \frac{p}{2} - 450.$$

a. Find the time at which the population becomes extinct if $p(0) = 850$.
b. Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
N **c.** Find the initial population p_0 if the population is to become extinct in 1 year.

8. The falling object in Example 2 satisfies the initial value problem

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0.$$

a. Find the time that must elapse for the object to reach 98% of its limiting velocity.
b. How far does the object fall in the time found in part a?

9. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

a. If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$\frac{dv}{dt} = \frac{1}{245}(49^2 - v^2).$$

Also see Problem 21 of Section 1.1.

b. If $v(0) = 0$, find an expression for $v(t)$ at any time.

G c. Plot your solution from part b and the solution (26) from Example 2 on the same axes.

d. Based on your plots in part c, compare the effect of a quadratic drag force with that of a linear drag force.

e. Find the distance $x(t)$ that the object falls in time t.

N f. Find the time T it takes the object to fall 300 m.

10. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

a. If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

b. Find an expression for the amount of thorium-234 present at any time t.

c. Find the time required for the thorium-234 to decay to one-half its original amount.

11. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

12. According to Newton's law of cooling (see Problem 19 of Section 1.1), the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T),$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

a. Find the temperature of the object at any time.

b. Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

13. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[5]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

G a. If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.

b. Find the limiting value Q_L that $Q(t)$ approaches after a long time.

G c. Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

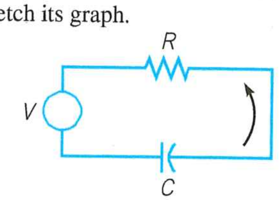

FIGURE 1.2.3 The electric circuit of Problem 13.

N 14. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 17 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

a. Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

b. Solve the problem in part a for $Q(t)$. How much chemical is in the pond after 1 year?

c. At the end of 1 year the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

d. Solve the initial value problem in part c. How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

e. How long does it take for $Q(t)$ to be reduced to 10 g?

G f. Plot $Q(t)$ versus t for 3 years.

[5] This equation results from Kirchhoff's laws, which are discussed in Section 3.7.

1.3 Classification of Differential Equations

The main purposes of this book are to discuss some of the properties of solutions of differential equations and to present some of the methods that have proved effective in finding solutions or, in some cases, in approximating them. To provide a framework for our presentation, we describe here several useful ways of classifying differential equations. Mastery of this vocabulary is essential to selecting appropriate solution methods and to describing properties of solutions of differential equations that you encounter later in this book—and in the real world.

Ordinary and Partial Differential Equations. One important classification is based on whether the unknown function depends on a single independent variable or on several

independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation**. In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation**.

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{1}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L; this equation is derived in Section 3.7. Typical examples of partial differential equations are the heat conduction equation

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{2}$$

and the wave equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{3}$$

Here, α^2 and a^2 are certain physical constants. Note that in both equations (2) and (3) the dependent variable u depends on the two independent variables x and t. The heat conduction equation describes the conduction of heat in a solid body, and the wave equation arises in a variety of problems involving wave motion in solids or fluids.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one differential equation is sufficient. However, if there are two or more unknown functions, then a system of differential equations is required. For example, the Lotka-Volterra, or predator-prey, equations are important in ecological modeling. They have the form

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= -cy + \gamma xy, \end{aligned} \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The positive constants a, α, c, and γ are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka-Volterra equations are examined in Section 9.5. In some areas of application it is not unusual to encounter very large systems containing hundreds, or even many thousands, of differential equations.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first-order equations, whereas equation (1) is a second-order equation. Equations (2) and (3) are also second-order partial differential equations. More generally, the equation

$$F\left(t, u(t), u'(t), \ldots, u^{(n)}(t)\right) = 0 \tag{5}$$

is an ordinary differential equation of the n^{th} order. Equation (5) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(t), u''(t), \ldots, u^{(n)}(t)$. Thus equation (5) is written as

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0. \tag{6}$$

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{7}$$

is a third-order differential equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

previously computed results. (Lorenz restarted the computation with three-digit approximate solutions, not the six-digit approximations that were stored in the computer.) In 1976 the Australian mathematician Sir Robert M. May (1938–) introduced and analyzed the logistic map, showing that there are special values of the problem's parameter where the solutions undergo drastic changes. The common trait that small changes in the problem produce large changes in the solution is one of the defining characteristics of chaos. May's logistic map is discussed in more detail in Section 2.9. Other classical examples of what we now recognize as "chaos" include the work by French mathematician Henri Poincaré (1854–1912) on planetary motion and the studies of turbulent fluid flow by Soviet mathematician Andrey Nikolaevich Kolmogorov (1903–1987), American mathematician Mitchell Feigenbaum (1944–), and many others. In addition to these and other classical examples of chaos, new examples continue to be found.

Solitons and chaos are just two of many examples where computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Other unexpected phenomena (Section 9.8), such as strange attractors (David Ruelle, Belgium, 1935–) and fractals (Benoit Mandelbrot, Poland, 1924–2010), have been discovered, are being intensively studied, and are leading to important new insights in a variety of applications. Although it is an old subject about which much is known, the study of differential equations in the twenty-first century remains a fertile source of fascinating and important unsolved problems.

Problems

In each of Problems 1 through 4, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$
2. $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$
3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$
4. $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin t$

In each of Problems 5 through 10, verify that each given function is a solution of the differential equation.

5. $y'' - y = 0;\quad y_1(t) = e^t,\ \ y_2(t) = \cosh t$
6. $y'' + 2y' - 3y = 0;\quad y_1(t) = e^{-3t},\ \ y_2(t) = e^t$
7. $ty' - y = t^2;\quad y = 3t + t^2$
8. $y'''' + 4y''' + 3y = t;\quad y_1(t) = t/3,\ \ y_2(t) = e^{-t} + t/3$
9. $t^2y'' + 5ty' + 4y = 0,\ \ t > 0;\quad y_1(t) = t^{-2},\ \ y_2(t) = t^{-2}\ln t$
10. $y' - 2ty = 1;\quad y = e^{t^2}\displaystyle\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 11 through 13, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

11. $y' + 2y = 0$
12. $y'' + y' - 6y = 0$
13. $y''' - 3y'' + 2y' = 0$

In each of Problems 14 and 15, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

14. $t^2y'' + 4ty' + 2y = 0$
15. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 16 through 18, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

16. $u_{xx} + u_{yy} + u_{zz} = 0$
17. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
18. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 19 through 21, verify that each given function is a solution of the given partial differential equation.

19. $u_{xx} + u_{yy} = 0;\quad u_1(x,y) = \cos x \cosh y,$ $u_2(x,y) = \ln(x^2+y^2)$
20. $\alpha^2 u_{xx} = u_t;\quad u_1(x,t) = e^{-\alpha^2 t}\sin x,$ $u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x,\ \ \lambda$ a real constant
21. $a^2u_{xx} = u_{tt};\quad u_1(x,t) = \sin(\lambda x)\sin(\lambda a t),$ $u_2(x,t) = \sin(x - at),\ \ \lambda$ a real constant

22. Follow the steps indicated here to derive the equation of motion of a pendulum, equation (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.
    a. Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.
    b. Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.
    c. Simplify the result from part b to obtain equation (12) in the text.

23. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.
    a. Show that the kinetic energy T of the pendulum in motion is
    $$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$
    b. Show that the potential energy V of the pendulum, relative to its rest position, is
    $$V = mgL(1-\cos\theta).$$
    c. By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate dE/dt, set it equal to zero, and show that the resulting equation reduces to equation (12).

24. A third derivation of the pendulum equation depends on the principle of angular momentum: The rate of change of angular momentum about any point is equal to the net external moment about the same point.
    a. Show that the angular momentum M, or moment of momentum, about the point of support is given by $M = mL^2 d\theta/dt$.
    b. Set dM/dt equal to the moment of the gravitational force, and show that the resulting equation reduces to equation (12). Note that positive moments are counterclockwise.

References

Computer software for differential equations changes too fast for particulars to be given in a book such as this. A Google search for Maple, Mathematica, Sage, or MATLAB is a good way to begin if you need information about one of these computer algebra and numerical systems.

There are many instructional books on computer algebra systems, such as the following:

Cheung, C.-K., Keough, G. E., Gross, R. H., and Landraitis, C., *Getting Started with Mathematica* (3rd ed.) (New York: Wiley, 2009).

Meade, D. B., May, M., Cheung, C.-K., and Keough, G. E., *Getting Started with Maple* (3rd ed.) (New York: Wiley, 2009).

For further reading in the history of mathematics, see books such as those listed below:

Boyer, C. B., and Merzbach, U. C., *A History of Mathematics* (2nd ed.) (New York: Wiley, 1989).

Kline, M., *Mathematical Thought from Ancient to Modern Times* (3 vols.) (New York: Oxford University Press, 1990).

A useful historical appendix on the early development of differential equations appears in

Ince, E. L., *Ordinary Differential Equations* (London: Longmans, Green, 1927; New York: Dover, 1956).

Encyclopedic sources of information about the lives and achievements of mathematicians of the past are

Gillespie, C. C., ed., *Dictionary of Scientific Biography* (15 vols.) (New York: Scribner's, 1971).

Koertge, N., ed., *New Dictionary of Scientific Biography* (8 vols.) (New York: Scribner's, 2007).

Koertge, N., ed., *Complete Dictionary of Scientific Biography* (New York: Scribner's, 2007 [e-book]).

Much historical information can be found on the Internet. One excellent site is the MacTutor History of Mathematics archive

http://www-history.mcs.st-and.ac.uk/history/

created by O'Connor, J. J., and Robertson, E. F., Department of Mathematics and Statistics, University of St. Andrews, Scotland.

CHAPTER 2

First-Order Differential Equations

This chapter deals with differential equations of first order

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our objective is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first-order equations.

The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first-order differential equations, introduce the idea of approximating a solution by numerical computation, and discuss some theoretical questions related to the existence and uniqueness of solutions. The final section includes an example of chaotic solutions in the context of first-order difference equations, which have some important points of similarity with differential equations and are simpler to investigate.

2.1 Linear Differential Equations; Method of Integrating Factors

If the function f in equation (1) depends linearly on the dependent variable y, then equation (1) is a first-order linear differential equation. In Sections 1.1 and 1.2 we discussed a restricted type of first-order linear differential equation in which the coefficients are constants. A typical example is

$$\frac{dy}{dt} = -ay + b, \tag{2}$$

where a and b are given constants. Recall that an equation of this form describes the motion of an object falling in the atmosphere.

Now we want to consider the most general first-order linear differential equation, which is obtained by replacing the coefficients a and b in equation (2) by arbitrary functions of t. We will usually write the general **first-order linear differential equation** in the standard form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{3}$$

where p and g are given functions of the independent variable t. Sometimes it is more convenient to write the equation in the form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t), \tag{4}$$

where P, Q, and G are given. Of course, as long as $P(t) \neq 0$, you can convert equation (4) to equation (3) by dividing both sides of equation (4) by $P(t)$.

In some cases it is possible to solve a first-order linear differential equation immediately by integrating the equation, as in the next example.

EXAMPLE 1

Solve the differential equation

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t. \tag{5}$$

Solution:

The left-hand side of equation (5) is a linear combination of dy/dt and y, a combination that also appears in the rule from calculus for differentiating a product. In fact,

$$(4+t^2)\frac{dy}{dt} + 2ty = \frac{d}{dt}\big((4+t^2)y\big);$$

it follows that equation (5) can be rewritten as

$$\frac{d}{dt}\big((4+t^2)y\big) = 4t. \tag{6}$$

Thus, even though y is unknown, we can integrate both sides of equation (6) with respect to t, thereby obtaining

$$(4+t^2)y = 2t^2 + c, \tag{7}$$

where c is an arbitrary constant of integration. Solving for y, we find that

$$y = \frac{2t^2}{4+t^2} + \frac{c}{4+t^2}. \tag{8}$$

This is the general solution of equation (5).

Unfortunately, most first-order linear differential equations cannot be solved as illustrated in Example 1 because their left-hand sides are not the derivative of the product of y and some other function. However, Leibniz discovered that if the differential equation is multiplied by a certain function $\mu(t)$, then the equation is converted into one that is immediately integrable by using the product rule for derivatives, just as in Example 1. The function $\mu(t)$ is called an **integrating factor** and our main task in this section is to determine how to find it for a given equation. We will show how this method works first for an example and then for the general first-order linear differential equation in the standard form (3).

EXAMPLE 2

Find the general solution of the differential equation

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}. \tag{9}$$

Draw some representative integral curves; that is, plot solutions corresponding to several values of the arbitrary constant c. Also find the particular solution whose graph contains the point $(0, 1)$.

Solution:

The first step is to multiply equation (9) by a function $\mu(t)$, as yet undetermined; thus

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}. \tag{10}$$

The question now is whether we can choose $\mu(t)$ so that the left-hand side of equation (10) is the derivative of the product $\mu(t)y$. For any differentiable function $\mu(t)$ we have

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y. \tag{11}$$

Thus the left-hand side of equation (10) and the right-hand side of equation (11) are identical, provided that we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t). \tag{12}$$

As in Example 3, this is another instance where there is a critical initial value, namely, $y_0 = 1$, that separates solutions that behave in one way from others that behave quite differently.

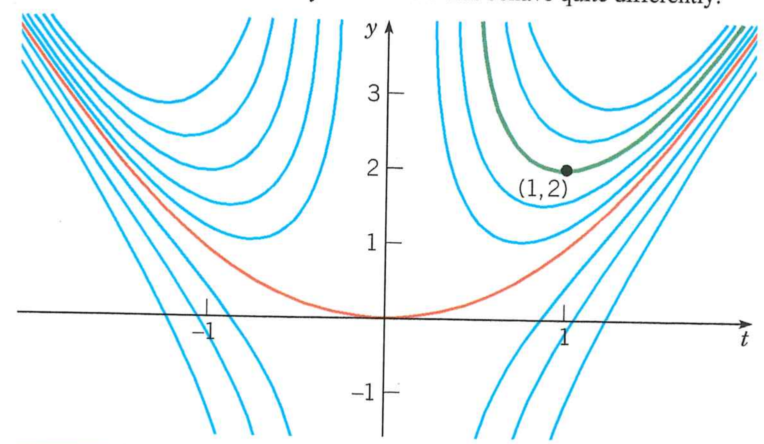

FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the green curve is the particular solution with $y(1) = 2$. The red curve is the particular solution with $y(1) = 1$.

EXAMPLE 5

Solve the initial value problem

$$2y' + ty = 2, \tag{41}$$

$$y(0) = 1. \tag{42}$$

Solution:

To convert the differential equation (41) to the standard form (3), we must divide equation (41) by 2, obtaining

$$y' + \frac{t}{2}y = 1. \tag{43}$$

Thus $p(t) = t/2$, and the integrating factor is $\mu(t) = \exp(t^2/4)$. Then multiply equation (43) by $\mu(t)$, so that

$$e^{t^2/4}y' + \frac{t}{2}e^{t^2/4}y = e^{t^2/4}. \tag{44}$$

The left-hand side of equation (44) is the derivative of $e^{t^2/4}y$, so by integrating both sides of equation (44), we obtain

$$e^{t^2/4}y = \int e^{t^2/4}\,dt + c. \tag{45}$$

The integral on the right-hand side of equation (45) cannot be evaluated in terms of the usual elementary functions, so we leave the integral unevaluated. By choosing the lower limit of integration as the initial point $t = 0$, we can replace equation (45) by

$$e^{t^2/4}y = \int_0^t e^{s^2/4}\,ds + c, \tag{46}$$

where c is an arbitrary constant. It then follows that the general solution y of equation (41) is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}. \tag{47}$$

To determine the particular solution that satisfies the initial condition (42), set $t = 0$ and $y = 1$ in equation (47):

$$\begin{aligned} 1 &= e^0 \int_0^0 e^{-s^2/4}\,ds + ce^0 \\ &= 0 + c, \end{aligned}$$

so $c = 1$.

The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral. This is usually at most a slight inconvenience, rather than a serious obstacle. For a given value of t, the integral in equation (47) is a definite integral and can be approximated to any desired degree of accuracy by using readily available numerical integrators. By repeating this process for many values of t and plotting the results, you can obtain a graph of a solution. Alternatively, you can use a numerical approximation method, such as those discussed in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple, Mathematica, MATLAB and Sage readily execute such procedures and produce graphs of solutions of differential equations.

Figure 2.1.4 displays graphs of the solution (47) for several values of c. The particular solution satisfying the initial condition $y(0) = 1$ is shown in black. From the figure it may be plausible to conjecture that all solutions approach a limit as $t \to \infty$. The limit can also be found analytically (see Problem 22).

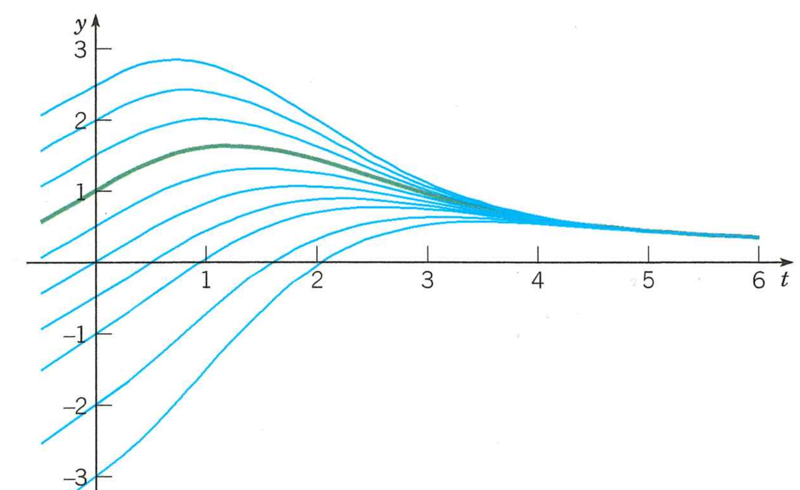

FIGURE 2.1.4 Integral curves of $2y' + ty = 2$; the green curve is the particular solution satisfying the initial condition $y(0) = 1$.

Problems

In each of Problems 1 through 8:

a. Draw a direction field for the given differential equation.
b. Based on an inspection of the direction field, describe how solutions behave for large t.
c. Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$
2. $y' - 2y = t^2e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + \frac{1}{t}y = 3\cos(2t), \quad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' - y = t^2e^{-t}, \quad t > 0$
7. $y' + y = 5\sin(2t)$
8. $2y' + y = 3t^2$

In each of Problems 9 through 12, find the solution of the given initial value problem.

9. $y' - y = 2te^{2t}, \quad y(0) = 1$
10. $y' + 2y = te^{-2t}, \quad y(1) = 0$
11. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0$
12. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

In each of Problems 13 and 14:

a. Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
b. Solve the initial value problem and find the critical value a_0 exactly.
c. Describe the behavior of the solution corresponding to the initial value a_0.

13. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$
14. $3y' - 2y = e^{-\pi t/2}, \quad y(0) = a$

In each of Problems 15 and 16:

G **a.** Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the critical value of a, that is, the initial value such that the solutions for $a < a_0$ and the solutions for $a > a_0$ have different behaviors as $t \to \infty$. Estimate the value of a_0.

b. Solve the initial value problem and find the critical value a_0 exactly.

c. Describe the behavior of the solution corresponding to the initial value a_0.

15. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$

16. $(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$

G **17.** Consider the initial value problem

$$y' + \frac{1}{2}y = 2\cos t, \quad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

N **18.** Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

19. Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

a. Find the solution of this initial value problem and describe its behavior for large t.

N **b.** Determine the value of t for which the solution first intersects the line $y = 12$.

20. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

21. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \quad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

22. Show that all solutions of $2y' + ty = 2$ [equation (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, equation (47). Show that the first term in the solution (47) is indeterminate with form $0 \cdot \infty$. Then, use l'Hôpital's rule to compute the limit as $t \to \infty$.

23. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 24 through 27, construct a first-order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

24. All solutions have the limit 3 as $t \to \infty$.

25. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.

26. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.

27. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.

28. Variation of Parameters. Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{48}$$

a. If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left(-\int p(t)\,dt\right), \tag{49}$$

where A is a constant.

b. If $g(t)$ is not everywhere zero, assume that the solution of equation (48) is of the form

$$y = A(t)\exp\left(-\int p(t)\,dt\right), \tag{50}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left(\int p(t)\,dt\right). \tag{51}$$

c. Find $A(t)$ from equation (51). Then substitute for $A(t)$ in equation (50) and determine y. Verify that the solution obtained in this manner agrees with that of equation (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second-order linear equations.

In each of Problems 29 and 30, use the method of Problem 28 to solve the given differential equation.

29. $y' - 2y = t^2e^{2t}$

30. $y' + \frac{1}{t}y = \cos(2t), \quad t > 0$

2.2 Separable Differential Equations

In Section 1.2 we used a process of direct integration to solve first-order linear differential equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of nonlinear differential equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first-order differential equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear differential equations were considered in the preceding section, but if equation (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first-order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite equation (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. When M is a function of x only and N is a function of y only, then equation (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the **differential form**

$$M(x)\,dx + N(y)\,dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be placed on opposite sides of the equation. The differential form (5) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by an example and then discuss it in general for equation (4).

EXAMPLE 1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

Solution:

If we write equation (6) as

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Recall from calculus that if y is a function of x, then by the chain rule,

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\frac{dy}{dx} = f'(y)\frac{dy}{dx}.$$

Problems

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = \dfrac{x^2}{y}$
2. $y' + y^2 \sin x = 0$
3. $y' = \cos^2(x)\cos^2(2y)$
4. $xy' = (1 - y^2)^{1/2}$
5. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
6. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$
7. $\dfrac{dy}{dx} = \dfrac{y}{x}$
8. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

In each of Problems 9 through 16:

a. Find the solution of the given initial value problem in explicit form.
b. Plot the graph of the solution.
c. Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
14. $y' = 2x/(1 + 2y), \quad y(2) = 0$
15. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
16. $\sin(2x)\,dx + \cos(3y)\,dy = 0, \quad y(\pi/2) = \pi/3$

Some of the results requested in Problems 17 through 22 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

17. Solve the initial value problem

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

18. Solve the initial value problem

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

19. Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

20. Solve the initial value problem

$$y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0$$

and determine where the solution attains its maximum value.

21. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{3}, \quad y(0) = y_0.$$

a. Determine how the behavior of the solution as t increases depends on the initial value y_0.
b. Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

22. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{1 + t}, \quad y(0) = y_0 > 0.$$

a. Determine how the solution behaves as $t \to \infty$.
b. If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.
c. Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

23. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

24. Use separation of variables to solve the differential equation

$$\frac{dQ}{dt} = r(a + bQ), \quad Q(0) = Q_0,$$

where a, b, r, and Q_0 are constants. Determine how the solution behaves as $t \to \infty$

Homogeneous Equations. If the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 25 illustrates how to solve first-order homogeneous equations.

[1]The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

25. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{29}$$

a. Show that equation (29) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{30}$$

thus equation (29) is homogeneous.
b. Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.
c. Replace y and dy/dx in equation (30) by the expressions from part b that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \tag{31}$$

Observe that equation (31) is separable.
d. Solve equation (31), obtaining v implicitly in terms of x.
e. Find the solution of equation (29) by replacing v by y/x in the solution in part d.
f. Draw a direction field and some integral curves for equation (29). Recall that the right-hand side of equation (29) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 25 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 26 through 31:

a. Show that the given equation is homogeneous.
b. Solve the differential equation.
c. Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

26. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$
27. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$
28. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$
29. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$
30. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$
31. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First-Order Differential Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Step 1: Construction of the Model. In this step the physical situation is translated into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of

EXAMPLE 4 | Escape Velocity

A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with distance, find an expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude A_{max} above the surface of the earth, and find the least initial velocity for which the body will not return to the earth; the latter is the **escape velocity**.

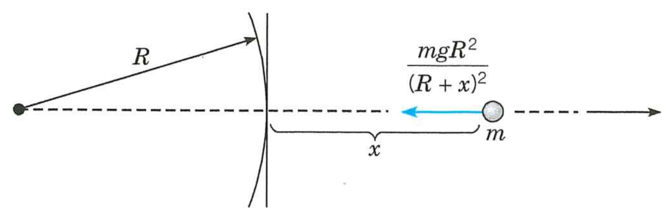

FIGURE 2.3.4 A body in the earth's gravitational field is pulled towards the center of the earth.

Solution:

Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface; see Figure 2.3.4. The figure is drawn horizontally to remind you that gravity is directed toward the center of the earth, which is not necessarily downward from a perspective away from the earth's surface. The gravitational force acting on the body (that is, its weight) is inversely proportional to the square of the distance from the center of the earth and is given by $w(x) = -k/(x+R)^2$, where k is a constant, R is the radius of the earth, and the minus sign signifies that $w(x)$ is directed in the negative x direction. We know that on the earth's surface $w(0)$ is given by $-mg$, where g is the acceleration due to gravity at sea level. Therefore, $k = mgR^2$ and

$$w(x) = -\frac{mgR^2}{(R+x)^2}. \tag{25}$$

Since there are no other forces acting on the body, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \tag{26}$$

and the initial condition is

$$v(0) = v_0. \tag{27}$$

Unfortunately, equation (26) involves too many variables since it depends on t, x, and v. To remedy this situation, we can eliminate t from equation (26) by thinking of x, rather than t, as the independent variable. Then we can express dv/dt in terms of dv/dx by using the chain rule; hence

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and equation (26) is replaced by

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}. \tag{28}$$

Equation (28) is separable but not linear, so by separating the variables and integrating, we obtain

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c. \tag{29}$$

Since $x = 0$ when $t = 0$, the initial condition (27) at $t = 0$ can be replaced by the condition that $v = v_0$ when $x = 0$. Hence $c = (v_0^2/2) - gR$ and

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}. \tag{30}$$

Note that equation (30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign must be chosen if it is falling back to earth.

To determine the maximum altitude A_{max} that the body reaches, we set $v = 0$ and $x = A_{max}$ in equation (30) and then solve for A_{max}, obtaining

$$A_{max} = \frac{v_0^2 R}{2gR - v_0^2}. \tag{31}$$

Solving equation (31) for v_0, we find the initial velocity required to lift the body to the altitude A_{max}, namely,

$$v_0 = \sqrt{2gR\frac{A_{max}}{R + A_{max}}}. \tag{32}$$

The escape velocity v_e is then found by letting $A_{max} \to \infty$. Consequently,

$$v_e = \sqrt{2gR}. \tag{33}$$

The numerical value of v_e is approximately 6.9 mi/s, or 11.1 km/s.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.

Problems

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}\left(1 + \frac{1}{2}\sin t\right)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

a. Find the amount of salt in the tank at any time.

G **b.** Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.

c. The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

4. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's[2] principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h.

a. Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

b. By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}, \tag{34}$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. The value of α for water is about 0.6.

c. Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m, and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

5. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

a. Find the time T required for the original sum to double in value as a function of r.

b. Determine T if $r = 7\%$.

c. Find the return rate that must be achieved if the initial investment is to double in 8 years.

6. A young person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

a. Determine the sum $S(t)$ accumulated at any time t.

b. If $r = 7.5\%$, determine k so that \$1 million will be available for retirement in 40 years.

c. If $k = \$2000$/year, determine the return rate r that must be obtained to have \$1 million available in 40 years.

[2] Evangelista Torricelli (1608–1647), successor to Galileo as court mathematician in Florence, published this result in 1644. In addition to this work in fluid dynamics, he is also known for constructing the first mercury barometer and for making important contributions to geometry.

7. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

N 8. A recent college graduate borrows \$150,000 at an interest rate of 6% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800 + 10t$, where t is the number of months since the loan was made.

a. Assuming that this payment schedule can be maintained, when will the loan be fully paid?

b. Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

9. An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby.[3] This is a means of determining the age of certain wood and plant remains, and hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years),[4] measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the *proportion* of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

a. Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for carbon-14.

b. Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

c. Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

N 10. Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

$$\frac{dy}{dt} = (0.5 + \sin t)\frac{y}{5}.$$

a. If $y(0) = 1$, find (or estimate) the time τ at which the population has doubled. Choose other initial conditions and determine whether the doubling time τ depends on the initial population.

b. Suppose that the growth rate is replaced by its average value $1/10$. Determine the doubling time τ in this case.

c. Suppose that the term $\sin t$ in the differential equation is replaced by $\sin 2\pi t$; that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time τ?

d. Plot the solutions obtained in parts a, b, and c on a single set of axes.

[3] Willard F. Libby (1908–1980) was born in rural Colorado and received his education at the University of California at Berkeley. He developed the method of radiocarbon dating beginning in 1947 while he was at the University of Chicago. For this work he was awarded the Nobel Prize in Chemistry in 1960.

[4] *McGraw-Hill Encyclopedia of Science and Technology* (8th ed.) (New York: McGraw-Hill, 1997), Vol. 5, p. 48.

N 11. Suppose that a certain population satisfies the initial value problem

$$dy/dt = r(t)y - k, \qquad y(0) = y_0,$$

where the growth rate $r(t)$ is given by $r(t) = (1 + \sin t)/5$, and k represents the rate of predation.

G a. Suppose that $k = 1/5$. Plot y versus t for several values of y_0 between $1/2$ and 1.

b. Estimate the critical initial population y_c below which the population will become extinct.

c. Choose other values of k and find the corresponding y_c for each one.

G d. Use the data you have found in parts b and c to plot y_c versus k.

12. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

13. Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann[5] law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4), \tag{35}$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant depending on the physical parameters of the body. However, if u is much larger than T, then solutions of equation (35) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \tag{36}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2.0 \times 10^{-12}$ K^{-3}/s.

a. Determine the temperature of the body at any time by solving equation (36).

G b. Plot the graph of u versus t.

N c. Find the time τ at which $u(\tau) = 600$—that is, twice the ambient temperature. Up to this time the error in using equation (36) to approximate the solutions of equation (35) is no more than 1%.

N 14. Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, u satisfies the differential equation

$$\frac{du}{dt} = -k(u - T(t)), \tag{37}$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1\cos(\omega t)$.

[5] Jozef Stefan (1835–1893), professor of physics at Vienna, stated the radiation law on empirical grounds in 1879. His student Ludwig Boltzmann (1844–1906) derived it theoretically from the principles of thermodynamics in 1884. Boltzmann is best known for his pioneering work in statistical mechanics.

a. Solve equation (37) and express $u(t)$ in terms of t, k, T_0, T_1, and ω. Observe that part of your solution approaches zero as t becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

G b. Suppose that t is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60$°F, $T_1 = 15$°F, and $k = 0.2$/h. Draw graphs of $S(t)$ and $T(t)$ versus t on the same axes. From your graph estimate the amplitude R of the oscillatory part of $S(t)$. Also estimate the time lag τ between corresponding maxima of $T(t)$ and $S(t)$.

c. Let k, T_0, T_1, and ω now be unspecified. Write the oscillatory part of $S(t)$ in the form $R\cos(\omega(t-\tau))$. Use trigonometric identities to find expressions for R and τ. Let T_1 and ω have the values given in part b, and plot graphs of R and τ versus k.

15. Consider a lake of constant volume V containing at time t an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration k of pollutant enters the lake at a rate r, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P. Note that the given assumptions neglect a number of factors that may, in some cases, be important—for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are deposited unevenly throughout the lake but (usually) at isolated points around its periphery. The results below must be interpreted in light of the neglect of such factors as these.

a. If at time $t = 0$ the concentration of pollutant is c_0, find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \to \infty$?

b. If the addition of pollutants to the lake is terminated ($k = 0$ and $P = 0$ for $t > 0$), determine the time interval T that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

c. Table 2.3.2 contains data[6] for several of the Great Lakes. Using these data, determine from part b the time T that is needed to reduce the contamination of each of these lakes to 10% of the original value.

TABLE 2.3.2 Volume and Flow Data for the Great Lakes

Lake	$10^3 \times V$ (km^3)	r (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

N 16. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

a. Find the maximum height above the ground that the ball reaches.

b. Assuming that the ball misses the building on the way down, find the time that it hits the ground.

G c. Plot the graphs of velocity and position versus time.

[6] This problem is based on R. H. Rainey, "Natural Displacement of Pollution from the Great Lakes," *Science 155* (1967), pp. 1242–1243; the information in the table was taken from that source.

N 17. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.

b. Find the time that the ball hits the ground.

G c. Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 16.

N 18. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.

b. Find the time that the ball hits the ground.

G c. Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 16 and 17.

19. A body of constant mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Neglect changes in the gravitational force.

a. Find the maximum height x_m attained by the body and the time t_m at which this maximum height is reached.

b. Show that if $kv_0/mg < 1$, then t_m and x_m can be expressed as

$$t_m = \frac{v_0}{g}\left(1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right),$$

$$x_m = \frac{v_0^2}{2g}\left(1 - \frac{2}{3}\frac{kv_0}{mg} + \frac{1}{2}\left(\frac{kv_0}{mg}\right)^2 - \cdots\right).$$

c. Show that the quantity kv_0/mg is dimensionless.

20. A body of mass m is projected vertically upward with an initial velocity v_0 in a medium offering a resistance $k|v|$, where k is a constant. Assume that the gravitational attraction of the earth is constant.

a. Find the velocity $v(t)$ of the body at any time.

b. Use the result of part a to calculate the limit of $v(t)$ as $k \to 0$—that is, as the resistance approaches zero. Does this result agree with the velocity of a mass m projected upward with an initial velocity v_0 in a vacuum?

c. Use the result of part a to calculate the limit of $v(t)$ as $m \to 0$—that is, as the mass approaches zero.

21. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force R, a buoyant force B, and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law, $R = 6\pi\mu a|v|$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid.[7]

[7] Sir George Gabriel Stokes (1819–1903) was born in Ireland but spent most of his life at Cambridge University, first as a student and later as a professor. Stokes was one of the foremost applied mathematicians of the nineteenth century, best known for his work in fluid dynamics and the wave theory of light. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series.

a. Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.
b. In 1910 R. A. Millikan[8] studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e. Assume that E has been adjusted so the droplet is held stationary ($v = 0$) and that w and B are as given above. Find an expression for e. Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

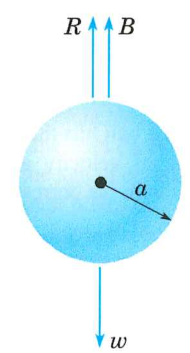

FIGURE 2.3.5 A body falling in a dense fluid (see Problem 21).

22. Let $v(t)$ and $w(t)$ be the horizontal and vertical components, respectively, of the velocity of a batted (or thrown) baseball. In the absence of air resistance, v and w satisfy the equations

$$\frac{dv}{dt} = 0, \qquad \frac{dw}{dt} = -g.$$

a. Show that

$$v = u\cos A, \qquad w = -gt + u\sin A,$$

where u is the initial speed of the ball and A is its initial angle of elevation.
b. Let $x(t)$ and $y(t)$ be the horizontal and vertical coordinates, respectively, of the ball at time t. If $x(0) = 0$ and $y(0) = h$, find $x(t)$ and $y(t)$ at any time t.
G c. Let $g = 32$ ft/s^2, $u = 125$ ft/s, and $h = 3$ ft. Plot the trajectory of the ball for several values of the angle A; that is, plot $x(t)$ and $y(t)$ parametrically.
d. Suppose the outfield wall is at a distance L and has height H. Find a relation between u and A that must be satisfied if the ball is to clear the wall.
e. Suppose that $L = 350$ ft and $H = 10$ ft. Using the relation in part (d), find (or estimate from a plot) the range of values of A that correspond to an initial velocity of $u = 110$ ft/s.
f. For $L = 350$ and $H = 10$, find the minimum initial velocity u and the corresponding optimal angle A for which the ball will clear the wall.

N 23. A more realistic model (than that in Problem 22) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

$$\frac{dv}{dt} = -rv, \qquad \frac{dw}{dt} = -g - rw,$$

where r is the coefficient of resistance.
a. Determine $v(t)$ and $w(t)$ in terms of initial speed u and initial angle of elevation A.
b. Find $x(t)$ and $y(t)$ if $x(0) = 0$ and $y(0) = h$.
G c. Plot the trajectory of the ball for $r = 1/5$, $u = 125$, $h = 3$, and for several values of A. How do the trajectories differ from those in Problem 22 with $r = 0$?
d. Assuming that $r = 1/5$ and $h = 3$, find the minimum initial velocity u and the optimal angle A for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 22f.

24. Brachistochrone Problem. One of the famous problems in the history of mathematics is the brachistochrone[9] problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hôpital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown in Figure 2.3.6. The lower point Q has coordinates (x_0, y_0). It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2, \tag{38}$$

where k^2 is a certain positive constant to be determined later.
a. Solve equation (38) for y'. Why is it necessary to choose the positive square root?
b. Introduce the new variable t by the relation

$$y = k^2 \sin^2 t. \tag{39}$$

Show that the equation found in part a then takes the form

$$2k^2 \sin^2 t\, dt = dx. \tag{40}$$

c. Letting $\theta = 2t$, show that the solution of equation (40) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin\theta)/2, \qquad y = k^2(1 - \cos\theta)/2. \tag{41}$$

Equations (41) are parametric equations of the solution of equation (38) that passes through $(0, 0)$. The graph of equations (41) is called a **cycloid**.
d. If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$.

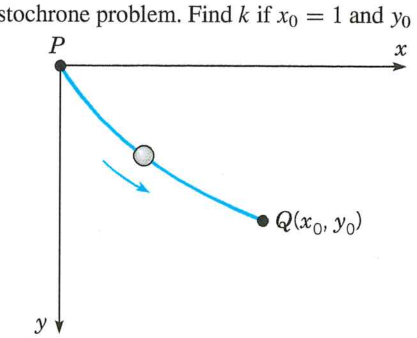

FIGURE 2.3.6 The brachistochrone (see Problem 24).

[8]Robert A. Millikan (1868–1953) was educated at Oberlin College and Columbia University. Later he was a professor at the University of Chicago and California Institute of Technology. His determination of the charge on an electron was published in 1910. For this work, and for other studies of the photoelectric effect, he was awarded the Nobel Prize for Physics in 1923.

[9]The word "brachistochrone" comes from the Greek words *brachistos*, meaning shortest, and *chronos*, meaning time.

2.4 Differences Between Linear and Nonlinear Differential Equations

Up to now, we have been primarily concerned with showing that first-order differential equations can be used to investigate many different kinds of problems in the natural sciences, and with presenting methods of solving such equations if they are either linear or separable. Now it is time to turn our attention to some more general questions about differential equations and to explore in more detail some important ways in which nonlinear equations differ from linear ones.

Existence and Uniqueness of Solutions. So far, we have discussed a number of initial value problems, each of which had a solution and apparently only one solution. That raises the question of whether this is true of all initial value problems for first-order equations. In other words, does every initial value problem have exactly one solution? This may be an important question even for nonmathematicians. If you encounter an initial value problem in the course of investigating some physical problem, you might want to know that it has a solution before spending very much time and effort in trying to find it. Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions. For linear equations, the answers to these questions are given by the following fundamental theorem.

Theorem 2.4.1 | Existence and Uniqueness Theorem for First-Order Linear Equations

If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t) \tag{1}$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0, \tag{2}$$

where y_0 is an arbitrary prescribed initial value.

Observe that Theorem 2.4.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and the *uniqueness* of the solution of the initial value problem (1). In addition, it states that the solution exists throughout any interval I containing the initial point t_0 in which the coefficients p and g are continuous. That is, the solution can be discontinuous or fail to exist only at points where at least one of p and g is discontinuous. Such points can often be identified at a glance.

The proof of this theorem is partly contained in the discussion in Section 2.1 leading to the formula (see equation (32) in Section 2.1)

$$\mu(t)y = \int \mu(t)g(t)\,dt + c, \tag{3}$$

where [equation (30) in Section 2.1]

$$\mu(t) = \exp\int p(t)\,dt. \tag{4}$$

The derivation in Section 2.1 shows that if equation (1) has a solution, then it must be given by equation (3). By looking slightly more closely at that derivation, we can also conclude that the differential equation (1) must indeed have a solution. Since p is continuous for $\alpha < t < \beta$, it follows that on the interval $\alpha < t < \beta$, μ is defined, is a differentiable function, and is is the solution of the initial value problem with the initial condition (24). Observe that the solution (25) becomes unbounded as $t \to 1/y_0$, so the interval of existence of the solution is $-\infty < t < 1/y_0$ if $y_0 > 0$, and is $1/y_0 < t < \infty$ if $y_0 < 0$. This example illustrates another feature of initial value problems for nonlinear equations: the singularities of the solution may depend in an essential way on the initial conditions as well as on the differential equation.

General Solution. Another way in which linear and nonlinear equations differ concerns the concept of a general solution. For a first-order linear differential equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant. For instance, for the differential equation $y' = y^2$ in Example 4, the expression in equation (22) contains an arbitrary constant but does not include all solutions of the differential equation. To show this, observe that the function $y = 0$ for all t is certainly a solution of the differential equation, but it cannot be obtained from equation (22) by assigning a value to c. In this example we might anticipate that something of this sort might happen, because to rewrite the original differential equation in the form (21), we must require that y is not zero. However, the existence of "additional" solutions is not uncommon for nonlinear equations; a less obvious example is given in Problem 18. Thus we will use the term "general solution" only when discussing linear equations.

Implicit Solutions. Recall again that for an initial value problem for a first-order linear differential equation, equation (8) provides an explicit formula for the solution $y = \phi(t)$. As long as the necessary antiderivatives can be found, the value of the solution at any point can be determined merely by substituting the appropriate value of t into the equation. The situation for nonlinear equations is much less satisfactory. Usually, the best that we can hope for is to find an equation

$$F(t, y) = 0 \tag{26}$$

involving t and y that is satisfied by the solution $y = \phi(t)$. Even this can be done only for differential equations of certain particular types, of which separable equations are the most important. The equation (26) is called an integral, or first integral, of the differential equation, and (as we have already noted) its graph is an integral curve, or perhaps a family of integral curves. Equation (26), assuming it can be found, defines the solution implicitly; that is, for each value of t we must solve equation (26) to find the corresponding value of y. If equation (26) is simple enough, it may be possible to solve it for y by analytical means and thereby obtain an explicit formula for the solution. However, more frequently this will not be possible, and you will have to resort to a numerical calculation to determine (approximately) the value of y for a given value of t. Once several pairs of values of t and y have been calculated, it is often helpful to plot them and then to sketch the integral curve that passes through them. You should take advantage of the wide range of computational and graphical utilities available to carry out these calculations and to create the graph of one or more integral curves.

Examples 2, 3, and 4 involve nonlinear problems in which it is easy to solve for an explicit formula for the solution $y = \phi(t)$. On the other hand, Examples 1 and 3 in Section 2.2 are cases in which it is better to leave the solution in implicit form and to use numerical means to evaluate it for particular values of the independent variable. The latter situation is more typical; unless the implicit relation is quadratic in y or has some other particularly simple form, it is unlikely that it can be solved exactly by analytical methods. Indeed, more often than not, it is impossible even to find an implicit expression for the solution of a first-order nonlinear equation.

Graphical or Numerical Construction of Integral Curves. Because of the difficulty in obtaining exact analytical solutions of nonlinear differential equations, methods that yield approximate solutions or other qualitative information about solutions are of correspondingly greater importance. We have already described, in Section 1.1, how the direction field of a differential equation can be constructed. The direction field can often show the qualitative form of solutions and can also be helpful in identifying regions of the ty-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigation. Graphical methods for first-order differential equations are discussed further in Section 2.5. An introduction to numerical methods for first-order equations is given in Section 2.7, and a systematic discussion of numerical methods appears in Chapter 8. However, it is not necessary to study the numerical algorithms themselves in order to use effectively one of the many software packages that generate and plot numerical approximations to solutions of initial value problems.

Summary. The linear equation $y' + p(t)y = g(t)$ has several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. There is an expression for the solution, namely, equation (7) or equation (8). Moreover, although it involves two integrations, the expression is an explicit one for the solution $y = \phi(t)$ rather than an equation that defines ϕ implicitly.
3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t, then the solution also exists and is differentiable for all t.

None of these statements are true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as on the differential equation.

Problems

In each of Problems 1 through 4, determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, \quad y(1) = 2$
2. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$
3. $(4-t^2)y' + 2ty = 3t^2, \quad y(-3) = 1$
4. $(\ln t)y' + y = \cot t, \quad y(2) = 3$

In each of Problems 5 through 8, state where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied.

5. $y' = (1 - t^2 - y^2)^{1/2}$
6. $y' = \dfrac{\ln|ty|}{1 - t^2 + y^2}$
7. $y' = (t^2 + y^2)^{3/2}$
8. $y' = \dfrac{1+t^2}{3y - y^2}$

In each of Problems 9 through 12, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

9. $y' = -4t/y, \quad y(0) = y_0$
10. $y' = 2ty^2, \quad y(0) = y_0$
11. $y' + y^3 = 0, \quad y(0) = y_0$
12. $y' = \dfrac{t^2}{y(1+t^3)}, \quad y(0) = y_0$

In each of Problems 13 through 16, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases and how their behavior depends on the initial value y_0 when $t = 0$.

G 13. $y' = ty(3 - y)$

G 14. $y' = y(3 - ty)$

G 15. $y' = -y(3 - ty)$

G 16. $y' = t - 1 - y^2$

17. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.
    a. Is there a solution that passes through the point $(1, 1)$? If so, find it.
    b. Is there a solution that passes through the point $(2, 1)$? If so, find it.
    c. Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at $t = 2$.
18. a. Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

Where are these solutions valid?

b. Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
c. Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part a for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

19. **a.** Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c.
b. Show that $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$ for $t > 0$, but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part b is nonlinear, while that of part a is linear.

20. Show that if $y = \phi(t)$ is a solution of $y' + p(t)y = 0$, then $y = c\phi(t)$ is also a solution for any value of the constant c.

21. Let $y = y_1(t)$ be a solution of

$$y' + p(t)y = 0, \tag{27}$$

and let $y = y_2(t)$ be a solution of

$$y' + p(t)y = g(t). \tag{28}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of equation (28).

22. **a.** Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \tag{29}$$

where c is an arbitrary constant.
b. Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \tag{30}$$

corresponding to $g(t) = 0$.
c. Show that y_2 is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher-order linear equations have a pattern similar to equation (29).

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n,$$

and is called a Bernoulli equation after Jakob Bernoulli. Problems 23 and 25 deal with equations of this type.

23. **a.** Solve Bernoulli's equation when $n = 0$; when $n = 1$.
b. Show that if $n \neq 0, 1$, then the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was formulated by Leibniz in 1696.

In each of Problems 24 through 25, the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 23b.

24. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.

25. $y' = \epsilon y - \sigma y^3$, $\epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make y' continuous at t_0.

26. Solve the initial value problem

$$y' + 2y = g(t), \quad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases}$$

27. Solve the initial value problem

$$y' + p(t)y = 0, \quad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases}$$

2.5 Autonomous Differential Equations and Population Dynamics

An important class of first-order equations consists of those in which the independent variable does not appear explicitly. Such equations are called **autonomous** and have the form

$$dy/dt = f(y). \tag{1}$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Sections 1.1 and 1.2 we considered the special case of equation (1) in which $f(y) = ay + b$.

Equation (1) is separable, so the discussion in Section 2.2 is applicable to it, but the main purpose of this section is to show how geometric methods can be used to obtain important qualitative information directly from the differential equation without solving the equation. Of

fundamental importance in this effort are the concepts of stability and instability of solutions of differential equations. These ideas were introduced informally in Chapter 1, but without using this terminology. They are discussed further here and will be examined in greater depth and in a more general setting in Chapter 9.

Exponential Growth. Let $y = \phi(t)$ be the population of the given species at time t. The simplest hypothesis concerning the variation of population is that the rate of change of y is proportional[10] to the current value of y; that is,

$$\frac{dy}{dt} = ry, \tag{2}$$

where the constant of proportionality r is called the **rate of growth** or **decline**, depending on whether r is positive or negative. Here, we assume that the population is growing, so $r > 0$.

Solving equation (2) subject to the initial condition[11]

$$y(0) = y_0, \tag{3}$$

we obtain

$$y = y_0 e^{rt}. \tag{4}$$

Thus the mathematical model consisting of the initial value problem (1), (2) with $r > 0$ predicts that the population will grow exponentially for all time, as shown in Figure 2.5.1 for several values of y_0. Under ideal conditions, equation (4) has been observed to be reasonably accurate for many populations, at least for limited periods of time. However, it is clear that such ideal conditions cannot continue indefinitely; eventually, limitations on space, food supply, or other resources will reduce the growth rate and bring an end to uninhibited exponential growth.

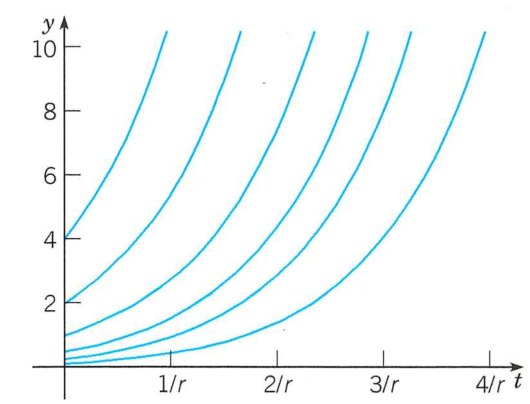

FIGURE 2.5.1 Exponential growth: y versus t for $dy/dt = ry$ $(r > 0)$.

Logistic Growth. To take account of the fact that the growth rate actually depends on the population, we replace the constant r in equation (2) by a function $h(y)$ and thereby obtain the modified equation

$$\frac{dy}{dt} = h(y)y. \tag{5}$$

We now want to choose $h(y)$ so that $h(y) \cong r > 0$ when y is small, $h(y)$ decreases as y grows larger, and $h(y) < 0$ when y is sufficiently large. The simplest function that has these properties is $h(y) = r - ay$, where a is also a positive constant. Using this function in equation (5), we obtain

$$\frac{dy}{dt} = (r - ay)y. \tag{6}$$

[10]It was apparently the British economist Thomas Malthus (1766–1834) who first observed that many biological populations increase at a rate proportional to the population. His first paper on populations appeared in 1798.

[11]In this section, because the unknown function is a population, we assume $y_0 > 0$.

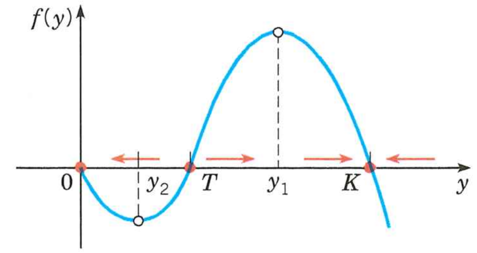

FIGURE 2.5.7 $f(y)$ versus y for $dy/dt = -r(1-y/T)(1-y/K)y$.

The phase line for equation (17) is shown in Figure 2.5.8*a*, and the graphs of some solutions are sketched in Figure 2.5.8*b*. You should make sure that you understand the relation between these two figures, as well as the relation between Figures 2.5.7 and 2.5.8*a*. From Figure 2.5.8*b* we see that if y starts below the threshold T, then y declines to ultimate extinction. On the other hand, if y starts above T, then y eventually approaches the carrying capacity K. The inflection points on the graphs of y versus t in Figure 2.5.8*b* correspond to the maximum and minimum points, y_1 and y_2, respectively, on the graph of $f(y)$ versus y in Figure 2.5.7. These values can be obtained by differentiating the right-hand side of equation (17) with respect to y, setting the result equal to zero, and solving for y. We obtain

$$y_{1,2} = (K + T \pm \sqrt{K^2 - KT + T^2})/3, \tag{18}$$

where the plus sign yields y_1 and the minus sign y_2.

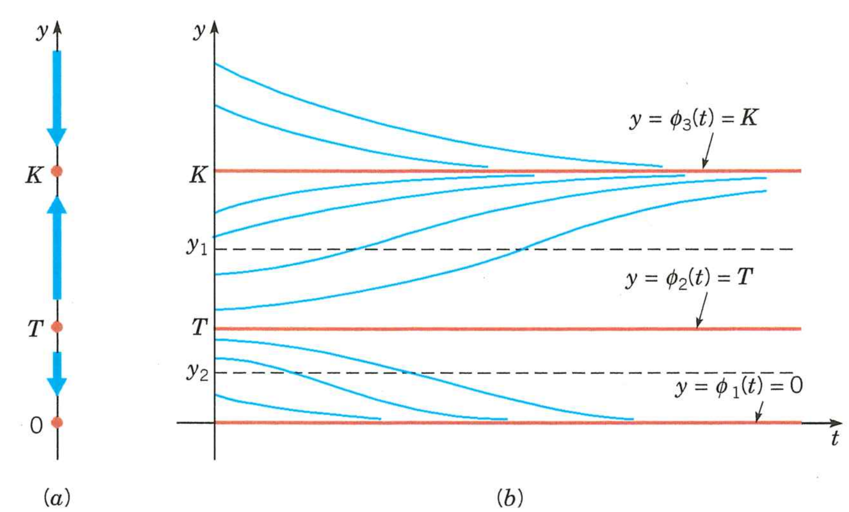

FIGURE 2.5.8 Logistic growth with a threshold: $dy/dt = -r(1-y/T)(1-y/K)y$; $y = \phi_1(t) = 0$ and $y = \phi_3(t) = K$ are asymptotically stable equilibria and $y = \phi_2(t) = T$ is an asymptotically unstable equilibrium. (*a*) The phase line. (*b*) Plots of y versus t.

A model of this general sort apparently describes the population of the passenger pigeon,[14] which was present in the United States in vast numbers until the late nineteenth century. It was heavily hunted for food and for sport, and consequently its numbers were drastically reduced by the 1880s. Unfortunately, the passenger pigeon could apparently breed successfully only when present in a large concentration, corresponding to a relatively high threshold T. Although a reasonably large number of individual birds remained alive in the late 1880s, there were not enough in any one place to permit successful breeding, and the population rapidly declined to extinction. The last passenger pigeon died in 1914. The precipitous decline in the passenger pigeon population from huge numbers to extinction in a few decades was one of the early factors contributing to a concern for conservation in this country.

[14] See, for example, Oliver L. Austin, Jr., *Birds of the World* (New York: Golden Press, 1983), pp. 143–145.

Problems

Problems 1 through 4 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.

G **1.** $dy/dt = ay + by^2, \quad a > 0, \; b > 0, \; -\infty < y_0 < \infty$

G **2.** $dy/dt = y(y-1)(y-2), \quad y_0 \ge 0$

G **3.** $dy/dt = e^y - 1, \quad -\infty < y_0 < \infty$

G **4.** $dy/dt = e^{-y} - 1, \quad -\infty < y_0 < \infty$

5. Semistable Equilibrium Solutions. Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side depart from it (see Figure 2.5.9). In this case the equilibrium solution is said to be **semistable**.

a. Consider the equation

$$dy/dt = k(1-y)^2, \tag{19}$$

where k is a positive constant. Show that $y = 1$ is the only critical point, with the corresponding equilibrium solution $\phi(t) = 1$.

G **b.** Sketch $f(y)$ versus y. Show that y is increasing as a function of t for $y < 1$ and also for $y > 1$. The phase line has upward-pointing arrows both below and above $y = 1$. Thus solutions below the equilibrium solution approach it, and those above it grow farther away. Therefore, $\phi(t) = 1$ is semistable.

c. Solve equation (19) subject to the initial condition $y(0) = y_0$ and confirm the conclusions reached in part b.

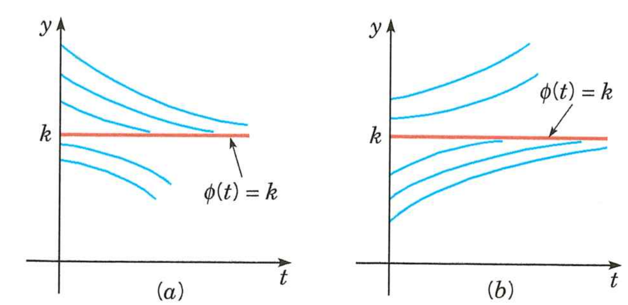

FIGURE 2.5.9 In both cases the equilibrium solution $\phi(t) = k$ is semistable. (*a*) $dy/dt \le 0$; (*b*) $dy/dt \ge 0$.

Problems 6 through 9 involve equations of the form $dy/dt = f(y)$. In each problem sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable (see Problem 5). Draw the phase line, and sketch several graphs of solutions in the ty-plane.

G **6.** $dy/dt = y^2(y^2-1), \quad -\infty < y_0 < \infty$

G **7.** $dy/dt = y(1-y^2), \quad -\infty < y_0 < \infty$

G **8.** $dy/dt = y^2(4-y^2), \quad -\infty < y_0 < \infty$

G **9.** $dy/dt = y^2(1-y)^2, \quad -\infty < y_0 < \infty$

10. Complete the derivation of the explicit formula for the solution (11) of the logistic model by solving equation (10) for y.

11. In Example 1, complete the manipulations needed to arrive at equation (13). That is, solve the solution (11) for t.

12. Complete the derivation of the location of the vertical asymptote in the solution (15) when $y_0 > T$. That is, derive formula (16) by finding the value of t when the denominator of the right-hand side of equation (15) is zero.

13. Complete the derivation of formula (18) for the locations of the inflection points of the solution of the logistic growth model with a threshold (17). *Hint:* Follow the steps outlined on p. 66.

14. Consider the equation $dy/dt = f(y)$ and suppose that y_1 is a critical point—that is, $f(y_1) = 0$. Show that the constant equilibrium solution $\phi(t) = y_1$ is asymptotically stable if $f'(y_1) < 0$ and unstable if $f'(y_1) > 0$.

15. Suppose that a certain population obeys the logistic equation $dy/dt = ry(1-(y/K))$.

a. If $y_0 = K/3$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

b. If $y_0/K = \alpha$, find the time T at which $y(T)/K = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.

G **16.** Another equation that has been used to model population growth is the Gompertz[15] equation

$$\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right),$$

where r and K are positive constants.

a. Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.

b. For $0 \le y \le K$, determine where the graph of y versus t is concave up and where it is concave down.

c. For each y in $0 < y \le K$, show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.

17. a. Solve the Gompertz equation

$$\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right),$$

subject to the initial condition $y(0) = y_0$.

Hint: You may wish to let $u = \ln(y/K)$.

b. For the data given in Example 1 in the text ($r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $y_0/K = 0.25$), use the Gompertz model to find the predicted value of $y(2)$.

c. For the same data as in part b, use the Gompertz model to find the time τ at which $y(\tau) = 0.75K$.

[15] Benjamin Gompertz (1779–1865) was an English actuary. He developed his model for population growth, published in 1825, in the course of constructing mortality tables for his insurance company.

18. A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

a. Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3},$$

where α is the coefficient of evaporation.

b. Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

c. Find a condition that must be satisfied if the pond is not to overflow.

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y.$$

Although it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level and possibly even driven to extinction. Problems 19 and 20 explore some of the questions involved in formulating a rational strategy for managing the fishery.[16]

19. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: the more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - Ey. \tag{20}$$

This equation is known as the **Schaefer model** after the biologist M. B. Schaefer, who applied it to fish populations.

a. Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.

b. Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

c. A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E; the graph of this function is known as the yield–effort curve.

d. Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

20. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y - h. \tag{21}$$

The assumption of a constant catch rate h may be reasonable when y is large but becomes less so when y is small.

a. If $h < rK/4$, show that equation (21) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.

b. Show that y_1 is unstable and y_2 is asymptotically stable.

c. From a plot of $f(y)$ versus y, show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.

d. If $h > rK/4$, show that y decreases to zero as t increases, regardless of the value of y_0.

e. If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 5). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 19d. The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

Epidemics. The use of mathematical methods to study the spread of contagious diseases goes back at least to some work by Daniel Bernoulli in 1760 on smallpox. In more recent years many mathematical models have been proposed and studied for many different diseases.[17] Problems 21 through 23 deal with a few of the simpler models and the conclusions that can be drawn from them. Similar models have also been used to describe the spread of rumors and of consumer products.

21. Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x + y = 1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1 - y$, we obtain the initial value problem

$$\frac{dy}{dt} = \alpha y(1 - y), \quad y(0) = y_0, \tag{22}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

a. Find the equilibrium points for the differential equation (22) and determine whether each is asymptotically stable, semistable, or unstable.

b. Solve the initial value problem 22 and verify that the conclusions you reached in part a are correct. Show that $y(t) \to 1$ as $t \to \infty$, which means that ultimately the disease spreads through the entire population.

22. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease but who exhibit no overt symptoms. Let x and y denote the proportions of susceptibles and carriers, respectively, in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$\frac{dy}{dt} = -\beta y. \tag{23}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$\frac{dx}{dt} = -\alpha xy. \tag{24}$$

a. Determine y at any time t by solving equation (23) subject to the initial condition $y(0) = y_0$.

b. Use the result of part a to find x at any time t by solving equation (24) subject to the initial condition $x(0) = x_0$.

c. Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

[16] An excellent treatment of this kind of problem, which goes far beyond what is outlined here, may be found in the book by Clark mentioned previously, especially in the first two chapters. Numerous additional references are given there.

[17] A standard source is the book by Bailey listed in the references. The models in Problems 21, 22, and 23 are discussed by Bailey in Chapters 5, 10, and 20, respectively.

23. Daniel Bernoulli's work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at that time was a major threat to public health. His model applies equally well to any other disease that, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year ($t = 0$), and let $n(t)$ be the number of these individuals surviving t years later. Let $x(t)$ be the number of members of this cohort who have not had smallpox by year t and who are therefore still susceptible. Let β be the rate at which susceptibles contract smallpox, and let ν be the rate at which people who contract smallpox die from the disease. Finally, let $\mu(t)$ be the death rate from all causes other than smallpox. Then dx/dt, the rate at which the number of susceptibles declines, is given by

$$\frac{dx}{dt} = -(\beta + \mu(t))x. \tag{25}$$

The first term on the right-hand side of equation (25) is the rate at which susceptibles contract smallpox, and the second term is the rate at which they die from all other causes. Also

$$\frac{dn}{dt} = -\nu\beta x - \mu(t)n, \tag{26}$$

where dn/dt is the death rate of the entire cohort, and the two terms on the right-hand side are the death rates due to smallpox and to all other causes, respectively.

a. Let $z = x/n$, and show that z satisfies the initial value problem

$$\frac{dz}{dt} = -\beta z(1 - \nu z), \quad z(0) = 1. \tag{27}$$

Observe that the initial value problem (27) does not depend on $\mu(t)$.

b. Find $z(t)$ by solving equation (27).

c. Bernoulli estimated that $\nu = \beta = 1/8$. Using these values, determine the proportion of 20-year-olds who have not had smallpox.

Note: On the basis of the model just described and the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated ($\nu = 0$), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years, 7 months. He therefore supported the inoculation program.

Bifurcation Points. For an equation of the form

$$\frac{dy}{dt} = f(a, y), \tag{28}$$

where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a **bifurcation point**, critical points come together, or separate, and equilibrium solutions may be either lost or gained. Bifurcation points are of great interest in many applications, because near them the nature of the solution of the underlying differential equation is undergoing an abrupt change. For example, in fluid mechanics a smooth (laminar) flow may break up and become turbulent. Or an axially loaded column may suddenly buckle and exhibit a large lateral displacement. Or, as the amount of one of the chemicals in a certain mixture is increased, spiral wave patterns of varying color may suddenly emerge in an originally quiescent fluid. Problems 24 through 26 describe three types of bifurcations that can occur in simple equations of the form (28).

24. Consider the equation

$$\frac{dy}{dt} = a - y^2. \tag{29}$$

a. Find all of the critical points for equation (29). Observe that there are no critical points if $a < 0$, one critical point if $a = 0$, and two critical points if $a > 0$.

G b. Draw the phase line in each case and determine whether each critical point is asymptotically stable, semistable, or unstable.

G c. In each case sketch several solutions of equation (29) in the ty-plane.

Note: If we plot the location of the critical points as a function of a in the ay-plane, we obtain Figure 2.5.10. This is called the **bifurcation diagram** for equation (29). The bifurcation at $a = 0$ is called a **saddle–node** bifurcation. This name is more natural in the context of second-order systems, which are discussed in Chapter 9.

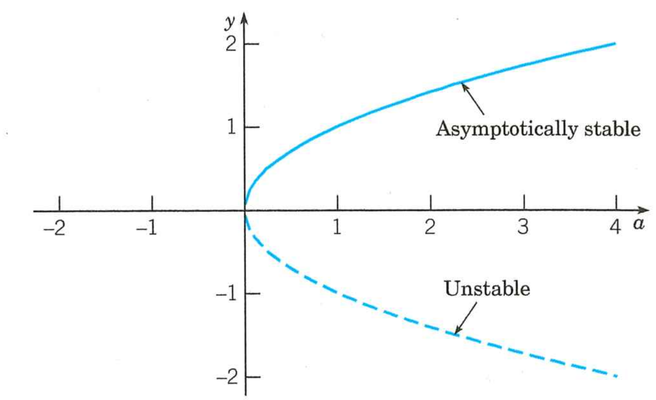

FIGURE 2.5.10 Bifurcation diagram for $y' = a - y^2$.

25. Consider the equation

$$\frac{dy}{dt} = ay - y^3 = y(a - y^2). \tag{30}$$

G a. Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

G b. In each case sketch several solutions of equation (30) in the ty-plane.

G c. Draw the bifurcation diagram for equation (30)—that is, plot the location of the critical points versus a.

Note: For equation (30) the bifurcation point at $a = 0$ is called a **pitchfork bifurcation**. Your diagram may suggest why this name is appropriate.

26. Consider the equation

$$\frac{dy}{dt} = ay - y^2 = y(a - y). \tag{31}$$

a. Again consider the cases $a < 0$, $a = 0$, and $a > 0$. In each case find the critical points, draw the phase line, and determine whether each critical point is asymptotically stable, semistable, or unstable.

b. In each case sketch several solutions of equation (31) in the ty-plane.

c. Draw the bifurcation diagram for equation (31).

Note: Observe that for equation (31) there are the same number of critical points for $a < 0$ and $a > 0$ but that their stability has changed. For $a < 0$ the equilibrium solution $y = 0$ is asymptotically stable and $y = a$ is unstable, while for $a > 0$ the situation is reversed. Thus there has been an **exchange of stability** as a passes through the bifurcation point $a = 0$. This type of bifurcation is called a **transcritical bifurcation**.

27. Chemical Reactions. A second-order chemical reaction involves the interaction (collision) of one molecule of a substance P with one molecule of a substance Q to produce one molecule of a new substance X; this is denoted by $P + Q \to X$. Suppose that p and q, where $p \neq q$, are the initial concentrations of P and Q, respectively, and let $x(t)$ be the concentration of X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of P and Q at time t, and the rate at which the reaction occurs is given by the equation

$$\frac{dx}{dt} = \alpha(p - x)(q - x), \tag{32}$$

where α is a positive constant.

a. If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and find $x(t)$ for any t.

b. If the substances P and Q are the same, then $p = q$ and equation (32) is replaced by

$$\frac{dx}{dt} = \alpha(p - x)^2. \tag{33}$$

If $x(0) = 0$, determine the limiting value of $x(t)$ as $t \to \infty$ without solving the differential equation. Then solve the initial value problem and determine $x(t)$ for any t.

2.6 Exact Differential Equations and Integrating Factors

For first-order differential equations there are a number of integration methods that are applicable to various classes of problems. The most important of these are linear equations and separable equations, which we have discussed previously. Here, we consider a class of equations known as exact differential equations for which there is also a well-defined method of solution. Keep in mind, however, that the first-order differential equations that can be solved by elementary integration methods are rather special; most first-order equations cannot be solved in this way.

EXAMPLE 1

Solve the differential equation

$$2x + y^2 + 2xyy' = 0. \tag{1}$$

Solution:

The equation is neither linear nor separable, so the methods suitable for those types of equations are not applicable here. However, observe that the function $\psi(x, y) = x^2 + xy^2$ has the property that

$$2x + y^2 = \frac{\partial \psi}{\partial x}, \quad 2xy = \frac{\partial \psi}{\partial y}. \tag{2}$$

Therefore, the differential equation can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Assuming that y is a function of x, we can use the chain rule to write the left-hand side of equation (3) as $d\psi(x, y)/dx$. Then equation (3) has the form

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4}$$

Integrating equation (4) we obtain

$$\psi(x, y) = x^2 + xy^2 = c, \tag{5}$$

where c is an arbitrary constant. The level curves of $\psi(x, y)$ are the integral curves of equation (1). Solutions of equation (1) are defined implicitly by equation (5).

In solving equation (1) the key step was the recognition that there is a function ψ that satisfies equations (2). More generally, let the differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{6}$$

be given. Suppose that we can identify a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y), \tag{7}$$

and such that $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x.[18]

When there is a function $\psi(x, y)$ such that $\psi_x = M$ and $\psi_y = N$, we can write

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x, \phi(x))$$

and the differential equation (6) becomes

$$\frac{d}{dx}\psi(x, \phi(x)) = 0. \tag{8}$$

In this case equation (6) is said to be an **exact differential equation** because it can be expressed exactly as the derivative of a specific function. Solutions of equation (6), or the equivalent equation (8), are given implicitly by

$$\psi(x, y) = c, \tag{9}$$

where c is an arbitrary constant.

In Example 1 it was relatively easy to see that the differential equation was exact and, in fact, easy to find its solution, at least implicitly, by recognizing the required function ψ. For more complicated equations it may not be possible to do this so easily. How can we tell whether a given equation is exact, and if it is, how can we find the function $\psi(x, y)$? The following theorem answers the first question, and its proof provides a way of answering the second.

Theorem 2.6.1

Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular[19] region $R: \alpha < x < \beta, \gamma < y < \delta$. Then equation (6)

$$M(x, y) + N(x, y)y' = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \tag{10}$$

at each point of R. That is, there exists a function ψ satisfying equations (7),

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy equation (10).

The proof of this theorem has two parts. First, we show that if there is a function ψ such that equations (7) are true, then it follows that equation (10) is satisfied. Computing M_y and N_x from equations (7), we obtain

$$M_y(x, y) = \psi_{xy}(x, y), \quad N_x(x, y) = \psi_{yx}(x, y). \tag{11}$$

Since M_y and N_x are continuous, it follows that ψ_{xy} and ψ_{yx} are also continuous. This guarantees their equality, and equation (10) is valid.

We now show that if M and N satisfy equation (10), then equation (6) is exact. The proof involves the construction of a function ψ satisfying equations (7)

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y).$$

[18] While a complete discussion of when $\psi(x, y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of x is beyond the scope and focus of this course, in general terms this condition is satisfied, locally, at points (x, y), where $\partial\psi/\partial y(x, y) \neq 0$. More details can be found in most books on advanced calculus.

[19] It is not essential that the region be rectangular, only that it be simply connected. In two dimensions this means that the region has no holes in its interior. Thus, for example, rectangular or circular regions are simply connected, but an annular region is not. More details can be found in most books on advanced calculus.

Since M and N are given functions, equation (25) states that the integrating factor μ must satisfy the first-order partial differential equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{26}$$

If a function μ satisfying equation (26) can be found, then equation (24) will be exact. The solution of equation (24) can then be obtained by the method described in the first part of this section. The solution found in this way also satisfies equation (23), since the integrating factor μ can be canceled out of equation (24).

A partial differential equation of the form (26) may have more than one solution; if this is the case, any such solution may be used as an integrating factor of equation (23). This possible nonuniqueness of the integrating factor is illustrated in Example 4.

Unfortunately, equation (26), which determines the integrating factor μ, is ordinarily at least as hard to solve as the original equation (23). Therefore, although in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

Let us determine conditions on M and N so that equation (23) has an integrating factor μ that depends on x only. If we assume that μ is a function of x only, then the partial derivative μ_x reduces to the ordinary derivative $d\mu/dx$ and $\mu_y = 0$. Making these substitutions in equation (26), we find that

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{27}$$

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends only on x; further, $\mu(x)$ can be found by solving differential equation (27), which is both linear and separable.

A similar procedure can be used to determine a condition under which equation (23) has an integrating factor depending only on y; see Problem 17.

EXAMPLE 4

Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0 \tag{19}$$

and then solve the equation.

Solution:

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on x only. On computing the quantity $(M_y - N_x)/N$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{28}$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{29}$$

Hence (see Problem 7 in Section 2.2)

$$\mu(x) = x. \tag{30}$$

Multiplying equation (19) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{31}$$

Equation (31) is exact, since

$$\frac{\partial}{\partial y}(3x^2y + xy^2) = 3x^2 + 2xy = \frac{\partial}{\partial x}(x^3 + x^2y).$$

Thus there is a function ψ such that

$$\psi_x(x,y) = 3x^2y + xy^2, \quad \psi_y(x,y) = x^3 + x^2y. \tag{32}$$

Integrating the first of equations (32) with respect to x, we obtain

$$\psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Substituting this expression for $\psi(x,y)$ in the second of equations (32), we find that

$$x^3 + x^2y + h'(y) = x^3 + x^2y,$$

so $h'(y) = 0$ and $h(y)$ is a constant. Thus the solutions of equation (31), and hence of equation (19), are given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c. \tag{33}$$

Solutions may also be found in explicit form since equation (33) is quadratic in y.

You may also verify that a second integrating factor for equation (19) is

$$\mu(x,y) = \frac{1}{xy(2x+y)}$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 22).

Problems

Determine whether each of the equations in Problems 1 through 8 is exact. If it is exact, find the solution.

1. $(2x + 3) + (2y - 2)y' = 0$
2. $(2x + 4y) + (2x - 2y)y' = 0$
3. $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
4. $\dfrac{dy}{dx} = -\dfrac{ax + by}{bx + cy}$
5. $\dfrac{dy}{dx} = -\dfrac{ax - by}{bx - cy}$
6. $(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x) + (xe^{xy}\cos(2x) - 3)y' = 0$
7. $(y/x + 6x) + (\ln x - 2)y' = 0, \quad x > 0$
8. $\dfrac{x}{(x^2+y^2)^{3/2}} + \dfrac{y}{(x^2+y^2)^{3/2}}\dfrac{dy}{dx} = 0$

In each of Problems 9 and 10, solve the given initial value problem and determine at least approximately where the solution is valid.

9. $(2x - y) + (2y - x)y' = 0, \quad y(1) = 3$
10. $(9x^2 + y - 1) - (4y - x)y' = 0, \quad y(1) = 0$

In each of Problems 11 and 12, find the value of b for which the given equation is exact, and then solve it using that value of b.

11. $(xy^2 + bx^2y) + (x + y)x^2y' = 0$
12. $(ye^{2xy} + x) + bxe^{2xy}y' = 0$
13. Assume that equation (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a possible function $\psi(x,y)$ is

$$\psi(x,y) = \int_{x_0}^{x} M(s, y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt,$$

where (x_0, y_0) is a point in R.

14. Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

In each of Problems 15 and 16, show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

15. $x^2y^3 + x(1 + y^2)y' = 0, \quad \mu(x,y) = 1/(xy^3)$
16. $(x + 2)\sin y + (x\cos y)y' = 0, \quad \mu(x,y) = xe^x$
17. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp\int Q(y)\,dy.$$

In each of Problems 18 through 21, find an integrating factor and solve the given equation.

18. $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$
19. $y' = e^{2x} + y - 1$
20. $1 + (x/y - \sin y)y' = 0$
21. $y + (2xy - e^{-2y})y' = 0$
22. Solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

using the integrating factor $\mu(x,y) = (xy(2x + y))^{-1}$. Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

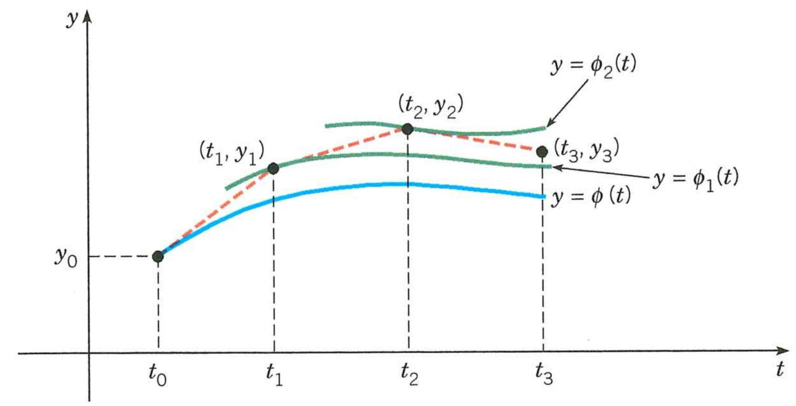

FIGURE 2.7.4 The Euler method.

In Example 2 the general solution of the differential equation is

$$y = 14 - 4t + ce^{-t/2} \tag{17}$$

and the solution of the initial value problem (9) corresponds to $c = -13$. The family of solutions (17) is a converging family since the term involving the arbitrary constant c approaches zero as $t \to \infty$. It does not matter very much which solutions we are approximating by tangent lines in the implementation of Euler's method, since all the solutions are getting closer and closer to each other as t increases.

On the other hand, in Example 3 the general solution of the differential equation is

$$y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}, \tag{18}$$

and, because the term involving the arbitrary constant c grows without bound as $t \to \infty$, this is a diverging family. Note that solutions corresponding to two nearby values of c become arbitrarily far apart as t increases. In Example 3 we are trying approximate the solution for $c = 11/4$, but in the use of Euler's method we are actually at each step following another solution that separates from the desired one faster and faster as t increases. This explains why the errors in Example 3 are so much larger than those in Example 2.

In using a numerical procedure such as the Euler method, you must always keep in mind the question of whether the results are accurate enough to be useful. In the preceding examples, the accuracy of the numerical results could be determined directly by a comparison with the solution obtained analytically. Of course, usually the analytical solution is not available if a numerical procedure is to be employed, so what we usually need are bounds for, or at least estimates of, the error that do not require a knowledge of the exact solution. You should also keep in mind that the best that we can expect, or hope for, from a numerical approximation is that it reflects the behavior of the actual solution. Thus a member of a diverging family of solutions will always be harder to approximate than a member of a converging family.

If you wish to read more about numerical approximations to solutions of initial value problems, you may go directly to Chapter 8 at this point. There, we present some information on the analysis of errors and also discuss several algorithms that are computationally much more efficient than the Euler method.

Problems

Note about Variations of Computed Results. Most of the problems in this section call for fairly extensive numerical computations. To handle these problems you need suitable computing hardware and software. Keep in mind that numerical results may vary somewhat, depending on how your program is constructed and on how your computer executes arithmetic steps, rounds off, and so forth. Minor variations in the last decimal place may be due to such causes and do not necessarily indicate that something is amiss. Answers in the back of the book are recorded to six digits in most cases, although more digits were retained in the intermediate calculations.

In each of Problems 1 through 4:

N **a.** Find approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3,$ and 0.4 using the Euler method with $h = 0.1$.

N **b.** Repeat part (a) with $h = 0.05$. Compare the results with those found in a.

N **c.** Repeat part a with $h = 0.025$. Compare the results with those found in a and b.

N **d.** Find the solution $y = \phi(t)$ of the given problem and evaluate $\phi(t)$ at $t = 0.1, 0.2, 0.3,$ and 0.4. Compare these values with the results of a, b, and c.

1. $y' = 3 + t - y, \quad y(0) = 1$

2. $y' = 2y - 1, \quad y(0) = 1$

3. $y' = 0.5 - t + 2y, \quad y(0) = 1$

4. $y' = 3\cos t - 2y, \quad y(0) = 0$

In each of Problems 5 through 8, draw a direction field for the given differential equation and state whether you think that the solutions are converging or diverging.

G **5.** $y' = 5 - 3\sqrt{y}$

G **6.** $y' = y(3 - ty)$

G **7.** $y' = -ty + 0.1y^3$

G **8.** $y' = t^2 + y^2$

In each of Problems 9 and 10, use Euler's method to find approximate values of the solution of the given initial value problem at $t = 0.5$, 1, 1.5, 2, 2.5, and 3: (a) With $h = 0.1$, (b) With $h = 0.05$, (c) With $h = 0.025$, (d) With $h = 0.01$.

N **9.** $y' = 5 - 3\sqrt{y}, \quad y(0) = 2$

N **10.** $y' = y(3 - ty), \quad y(0) = 0.5$

11. Consider the initial value problem

$$y' = \frac{3t^2}{3y^2 - 4}, \quad y(1) = 0.$$

N **a.** Use Euler's method with $h = 0.1$ to obtain approximate values of the solution at $t = 1.2, 1.4, 1.6,$ and 1.8.

N **b.** Repeat part a with $h = 0.05$.

c. Compare the results of parts a and b. Note that they are reasonably close for $t = 1.2, 1.4,$ and 1.6 but are quite different for $t = 1.8$. Also note (from the differential equation) that the line tangent to the solution is parallel to the y-axis when $y = \pm 2/\sqrt{3} \cong \pm 1.155$. Explain how this might cause such a difference in the calculated values.

N **12.** Consider the initial value problem

$$y' = t^2 + y^2, \quad y(0) = 1.$$

Use Euler's method with $h = 0.1, 0.05, 0.025,$ and 0.01 to explore the solution of this problem for $0 \le t \le 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field in Problem 8?

13. Consider the initial value problem

$$y' = -ty + 0.1y^3, \quad y(0) = \alpha,$$

where α is a given number.

G **a.** Draw a direction field for the differential equation (or reexamine the one from Problem 7). Observe that there is a critical value of α in the interval $2 \le \alpha \le 3$ that separates converging solutions from diverging ones. Call this critical value α_0.

N **b.** Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

14. Consider the initial value problem

$$y' = y^2 - t^2, \quad y(0) = \alpha,$$

where α is a given number.

G **a.** Draw a direction field for the differential equation. Note that there is a critical value of α in the interval $0 \le \alpha \le 1$ that separates converging solutions from diverging ones. Call this critical value α_0.

N **b.** Use Euler's method with $h = 0.01$ to estimate α_0. Do this by restricting α_0 to an interval $[a, b]$, where $b - a = 0.01$.

15. Convergence of Euler's Method. It can be shown that under suitable conditions on f, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ converges to the exact solution as the step size h decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \quad y(t_0) = y_0.$$

a. Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

N **b.** Using the Euler formula, show that

$$y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \ldots.$$

c. Noting that $y_1 = (1 + h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1 + h)^n(y_0 - t_0) + t_n \tag{19}$$

for each positive integer n.

d. Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in equation (19) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.

Hint: $\lim_{n\to\infty} (1 + a/n)^n = e^a$.

In each of Problems 16 and 17, use the technique discussed in Problem 15 to show that the approximation obtained by the Euler method converges to the exact solution at any fixed point as $h \to 0$.

16. $y' = y, \quad y(0) = 1$

17. $y' = 2y - 1, \quad y(0) = 1 \quad$ *Hint:* $y_1 = (1 + 2h)/2 + 1/2$

2.8 The Existence and Uniqueness Theorem

In this section we discuss the proof of Theorem 2.4.2, the fundamental existence and uniqueness theorem for first-order initial value problems. Recall that this theorem states that under certain conditions on $f(t, y)$, the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{1}$$

has a unique solution in some interval containing the point t_0.

4. *Are there other solutions of the integral equation (3) besides $y = \phi(t)$?*
To show the uniqueness of the solution $y = \phi(t)$, we can proceed much as in the example. First, assume the existence of another solution $y = \psi(t)$. It is then possible to show (see Problem 18) that the difference $\phi(t) - \psi(t)$ satisfies the inequality

$$|\phi(t) - \psi(t)| \le A \int_0^t |\phi(s) - \psi(s)|\, ds \tag{30}$$

for $0 \le t \le h$ and a suitable positive number A. From this point the argument is identical to that given in the example, and we conclude that there is no solution of the initial value problem (2) other than the one generated by the method of successive approximations.

Problems

In each of Problems 1 and 2, transform the given initial value problem into an equivalent problem with the initial point at the origin.

1. $dy/dt = t^2 + y^2, \quad y(1) = 2$
2. $dy/dt = 1 - y^3, \quad y(-1) = 3$

In each of Problems 3 through 4, let $\phi_0(t) = 0$ and define $\{\phi_n(t)\}$ by the method of successive approximations.

a. Determine $\phi_n(t)$ for an arbitrary value of n.
G b. Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.
c. Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.
G d. Plot $|\phi(t) - \phi_n(t)|$ for $n = 1, \ldots, 4$. For each of $\phi_1(t), \ldots, \phi_4(t)$, estimate the interval in which it is a reasonably good approximation to the actual solution.

N 3. $y' = 2(y + 1), \quad y(0) = 0$
N 4. $y' = -y/2 + t, \quad y(0) = 0$

In each of Problems 5 and 6, let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem.

a. Determine $\phi_n(t)$ for an arbitrary value of n.
G b. Plot $\phi_n(t)$ for $n = 1, \ldots, 4$. Observe whether the iterates appear to be converging.
c. Show that the sequence $\{\phi_n(t)\}$ converges.

5. $y' = ty + 1, \quad y(0) = 0$
6. $y' = t^2 y - t, \quad y(0) = 0$

In each of Problems 7 and 8, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

a. Calculate $\phi_1(t), \ldots, \phi_3(t)$.
G b. Plot $\phi_1(t), \ldots, \phi_3(t)$. Observe whether the iterates appear to be converging.

7. $y' = t^2 + y^2, \quad y(0) = 0$
8. $y' = 1 - y^3, \quad y(0) = 0$

In each of Problems 9 and 10, let $\phi_0(t) = 0$ and use the method of successive approximations to approximate the solution of the given initial value problem.

a. Calculate $\phi_1(t), \ldots, \phi_4(t)$, or (if necessary) Taylor approximations to these iterates. Keep terms up to order six.
G b. Plot the functions you found in part a and observe whether they appear to be converging.

9. $y' = -\sin y + 1, \quad y(0) = 0$
10. $y' = \dfrac{3t^2 + 4t + 2}{2(y - 1)}, \quad y(0) = 0$

11. Let $\phi_n(x) = x^n$ for $0 \le x \le 1$ and show that

$$\lim_{n\to\infty} \phi_n(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

This example shows that a sequence of continuous functions may converge to a limit function that is discontinuous.

12. Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}, \quad 0 \le x \le 1$.
a. Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \le x \le 1$; hence

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\,dx = 0.$$

b. Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\,dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\,dx \ne \int_a^b \lim_{n\to\infty} \phi_n(x)\,dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.

13. a. Verify that $\phi(t) = \sum_{k=1}^{\infty} \dfrac{t^{2k}}{k!}$ is a solution of the integral equation (9).
b. Verify that $\phi(t)$ is also a solution of the initial value problem (6).
c. Use the fact that $\sum_{k=0}^{\infty} \dfrac{t^k}{k!} = e^t$ to evaluate $\phi(t)$ in terms of elementary functions.
d. Solve initial value problem (6) as a separable equation.
e. Solve initial value problem (6) as a first order linear equation.

In Problems 14 through 17, we indicate how to prove that the sequence $\{\phi_n(t)\}$, defined by equations (4) through (7), converges.

a. Verify that $\phi(t) = \sum_{k=1}^{\infty} \dfrac{t^{2k}}{k!}$ is a solution of the integral equation (9).
b. Verify that $\phi(t)$ is also a solution of the initial value problem (6).
c. Use the fact that $\sum_{k=0}^{\infty} \dfrac{t^k}{k!} = e^t$ to evaluate $\phi(t)$ in terms of elementary functions.
d. Solve initial value problem (6) as a separable equation.
e. Solve initial value problem (6) as a first order linear equation.

14. If $\partial f/\partial y$ is continuous in the rectangle D, show that there is a positive constant K such that

$$|f(t, y_1) - f(t, y_2)| \le K|y_1 - y_2|, \tag{31}$$

where (t, y_1) and (t, y_2) are any two points in D having the same t coordinate. This inequality is known as a Lipschitz[22] condition.
Hint: Hold t fixed and use the mean value theorem on f as a function of y only. Choose K to be the maximum value of $|\partial f/\partial y|$ in D.

15. If $\phi_{n-1}(t)$ and $\phi_n(t)$ are members of the sequence $\{\phi_n(t)\}$, use the result of Problem 14 to show that

$$\left|f(t, \phi_n(t)) - f\big(t, \phi_{n-1}(t)\big)\right| \le K\left|\phi_n(t) - \phi_{n-1}(t)\right|.$$

16. a. Show that if $|t| \le h$, then

$$|\phi_1(t)| \le M|t|,$$

where M is chosen so that $|f(t, y)| \le M$ for (t, y) in D.
b. Use the results of Problem 15 and part a of Problem 16 to show that

$$|\phi_2(t) - \phi_1(t)| \le \frac{MK|t|^2}{2}.$$

c. Show, by mathematical induction, that

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{MK^{n-1}|t|^n}{n!} \le \frac{MK^{n-1}h^n}{n!}.$$

17. Note that
$\phi_n(t) = \phi_1(t) + \big(\phi_2(t) - \phi_1(t)\big) + \cdots + \big(\phi_n(t) - \phi_{n-1}(t)\big).$

a. Show that

$$|\phi_n(t)| \le |\phi_1(t)| + |\phi_2(t) - \phi_1(t)| + \cdots + |\phi_n(t) - \phi_{n-1}(t)|.$$

b. Use the results of Problem 16 to show that

$$|\phi_n(t)| \le \frac{M}{K}\left(Kh + \frac{(Kh)^2}{2!} + \cdots + \frac{(Kh)^n}{n!}\right).$$

c. Show that the sum in part b converges as $n \to \infty$ and, hence, the sum in part a also converges as $n \to \infty$. Conclude therefore that the sequence $\{\phi_n(t)\}$ converges since it is the sequence of partial sums of a convergent infinite series.

18. In this problem we deal with the question of uniqueness of the solution of the integral equation (3)

$$\phi(t) = \int_0^t f(s, \phi(s))\,ds.$$

a. Suppose that ϕ and ψ are two solutions of equation (3). Show that, for $t \ge 0$,

$$\phi(t) - \psi(t) = \int_0^t (f(s, \phi(s)) - f(s, \psi(s)))\,ds.$$

b. Show that

$$|\phi(t) - \psi(t)| \le \int_0^t (f(s, \phi(s)) - f(s, \psi(s)))\,ds.$$

c. Use the result of Problem 14 to show that

$$|\phi(t) - \psi(t)| \le K \int_0^t |\phi(s) - \psi(s)|\,ds,$$

where K is an upper bound for $|\partial f/\partial y|$ in D. This is the same as equation (30), and the rest of the proof may be constructed as indicated in the text.

[22]The German mathematician Rudolf Lipschitz (1832–1903), professor at the University of Bonn for many years, worked in several areas of mathematics. The inequality (i) can replace the hypothesis that $\partial f/\partial y$ is continuous in Theorem 2.8.1; this results in a slightly stronger theorem.

2.9 First-Order Difference Equations

Although a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more natural. For instance, the continuous model of compound interest used in Section 2.3 is only an approximation to the actual discrete process. Similarly, sometimes population growth may be described more accurately by a discrete model than by a continuous model. This is true, for example, of species whose generations do not overlap and that propagate at regular intervals, such as at particular times of the calendar year. Then the population y_{n+1} of the species in the year $n + 1$ is some function of n and the population y_n in the preceding year; that is,

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots. \tag{1}$$

Equation (1) is called a **first-order difference equation.** It is first-order because the value of y_{n+1} depends on the value of y_n but not on earlier values y_{n-1}, y_{n-2}, and so forth. As for differential equations, the difference equation (1) is **linear** if f is a linear function of y_n; otherwise, it is **nonlinear**. A **solution** of the difference equation (1) is a sequence of numbers $y_0, y_1, y_2, \ldots$ that satisfy the equation for each n. In addition to the difference equation itself, there may also be an **initial condition**

$$y_0 = \alpha \tag{2}$$

that prescribes the value of the first term of the solution sequence.

We now assume temporarily that the function f in equation (1) depends only on y_n, but not on n. In this case

$$y_{n+1} = f(y_n), \quad n = 0, 1, 2, \ldots. \tag{3}$$

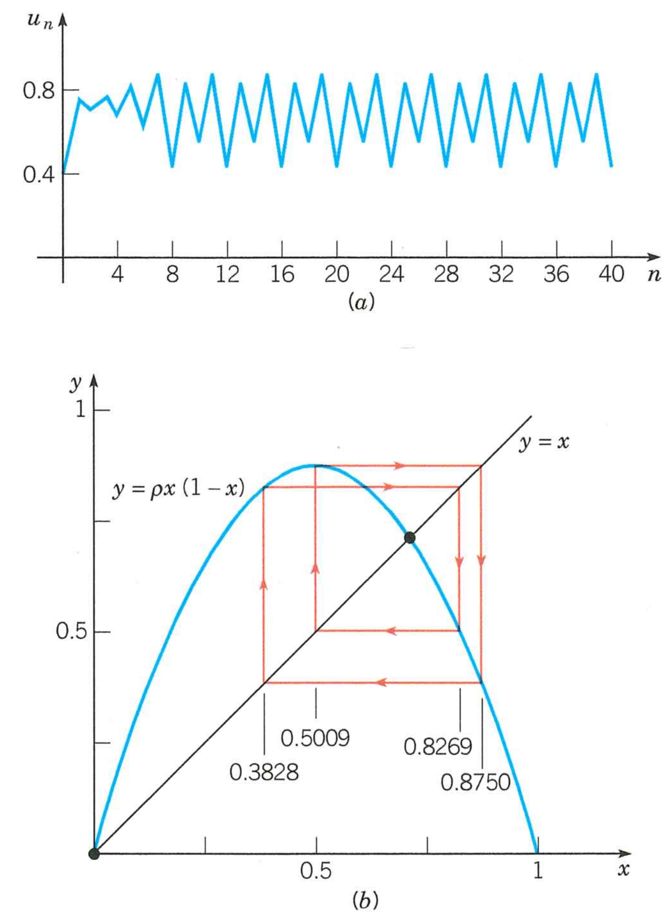

FIGURE 2.9.5 A solution of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.5$; period 4. (a) u_n versus n; (b) the cobweb diagram shows the iterates are in a four-cycle.

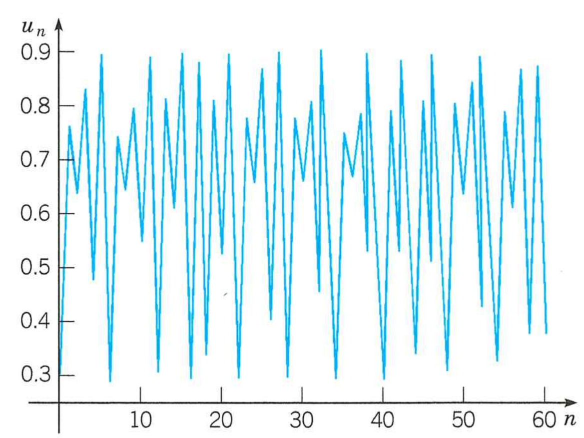

FIGURE 2.9.6 A solution of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.65$; a chaotic solution.

It is only comparatively recently that chaotic solutions of difference and differential equations have become widely known. Equation (20) was one of the first instances of mathematical chaos to be found and studied in detail, by Robert May[23] in 1974. On the basis

[23]Robert M. May (1936–) was born in Sydney, Australia, and received his education at the University of Sydney with a doctorate in theoretical physics in 1959. His interests soon turned to population dynamics and theoretical ecology; the work cited in the text is described in two papers listed in the References at the end of this chapter. He has held professorships at Sydney, at Princeton, at Imperial College (London), and (since 1988) at Oxford.

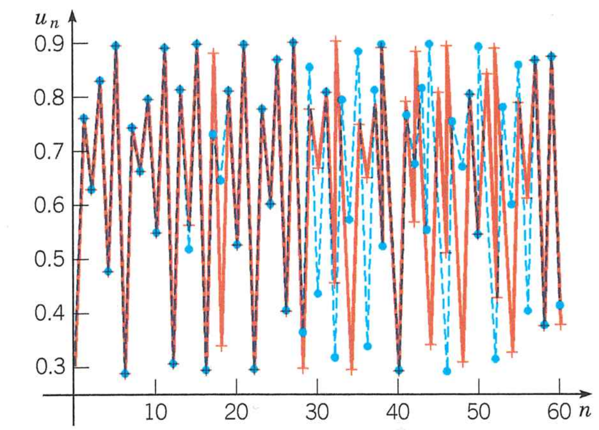

FIGURE 2.9.7 Two solutions of $u_{n+1} = \rho u_n(1 - u_n)$ for $\rho = 3.65$; $u_0 = 0.3$ and $u_0 = 0.305$.

of his analysis of this equation as a model of the population of certain insect species, May suggested that if the growth rate ρ is too large, then it will be impossible to make effective long-range predictions about these insect populations. The occurrence of chaotic solutions in seemingly simple problems has stimulated an enormous amount of research, but many questions remain unanswered. It is increasingly clear, however, that chaotic solutions are much more common than was suspected at first and that they may be a part of the investigation of a wide range of phenomena.

Problems

In each of Problems 1 through 4, solve the given difference equation in terms of the initial value y_0. Describe the behavior of the solution as $n \to \infty$.

1. $y_{n+1} = -0.9y_n$

2. $y_{n+1} = \sqrt{\dfrac{n+3}{n+1}}\, y_n$

3. $y_{n+1} = (-1)^{n+1} y_n$

4. $y_{n+1} = 0.5y_n + 6$

5. An investor deposits \$1000 in an account paying interest at a rate of 8%, compounded monthly, and also makes additional deposits of \$25 per month. Find the balance in the account after 3 years.

6. A certain college graduate borrows \$8000 to buy a car. The lender charges interest at an annual rate of 10%. What monthly payment rate is required to pay off the loan in 3 years? Compare your result with that of Problem 7 in Section 2.3.

7. A homebuyer takes out a mortgage of \$100,000 with an interest rate of 9%. What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?

8. If the interest rate on a 20-year mortgage is fixed at 10% and if a monthly payment of \$1000 is the maximum that the buyer can afford, what is the maximum mortgage loan that can be made under these conditions?

9. A homebuyer wishes to finance the purchase with a \$95,000 mortgage with a 20-year term. What is the maximum interest rate the buyer can afford if the monthly payment is not to exceed \$900?

The Logistic Difference Equation. Problems 10 through 15 deal with the difference equation (21), $u_{n+1} = \rho u_n(1 - u_n)$.

10. Carry out the details in the linear stability analysis of the equilibrium solution $u_n = (\rho - 1)/\rho$. That is, derive the difference equation (26) in the text for the perturbation v_n.

11. N **a.** For $\rho = 3.2$, plot or calculate the solution of the logistic equation (21) for several initial conditions, say, $u_0 = 0.2$, 0.4, 0.6, and 0.8. Observe that in each case the solution approaches a steady oscillation between the same two values. This illustrates that the long-term behavior of the solution is independent of the initial conditions.

N **b.** Make similar calculations and verify that the nature of the solution for large n is independent of the initial condition for other values of ρ, such as 2.6, 2.8, and 3.4.

12. Assume that $\rho > 1$ in equation (21).

G **a.** Draw a qualitatively correct stairstep diagram and thereby show that if $u_0 < 0$, then $u_n \to -\infty$ as $n \to \infty$.

G **b.** In a similar way, determine what happens as $n \to \infty$ if $u_0 > 1$.

13. The solutions of equation (21) change from convergent sequences to periodic oscillations of period 2 as the parameter ρ passes through the value 3. To see more clearly how this happens, carry out the following calculations.

N **a.** Plot or calculate the solution for $\rho = 2.9$, 2.95, and 2.99, respectively, using an initial value u_0 of your choice in the interval $(0, 1)$. In each case estimate how many iterations are required for the solution to get "very close" to the limiting value. Use any convenient interpretation of what "very close" means in the preceding sentence.

N **b.** Plot or calculate the solution for $\rho = 3.01$, 3.05, and 3.1, respectively, using the same initial condition as in part a. In each case estimate how many iterations are needed to reach a steady-state oscillation. Also find or estimate the two values in the steady-state oscillation.

N **14.** By calculating or plotting the solution of equation (21) for different values of ρ, estimate the value of ρ at which the solution changes from an oscillation of period 2 to one of period 4. In the same way, estimate the value of ρ at which the solution changes from period 4 to period 8.

N **15.** Let ρ_k be the value of ρ at which the solution of equation (21) changes from period 2^{k-1} to period 2^k. Thus, as noted in the text, $\rho_1 = 3$, $\rho_2 \cong 3.449$, and $\rho_3 \cong 3.544$.

a. Using these values of ρ_1, ρ_2, and ρ_3, or those you found in Problem 14, calculate $(\rho_2 - \rho_1)/(\rho_3 - \rho_2)$.

b. Let $\delta_n = (\rho_n - \rho_{n-1})/(\rho_{n+1} - \rho_n)$. It can be shown that δ_n approaches a limit δ as $n \to \infty$, where $\delta \cong 4.6692$ is known as the Feigenbaum[24] number. Determine the percentage difference between the limiting value δ and δ_2, as calculated in part a.

c. Assume that $\delta_3 = \delta$ and use this relation to estimate ρ_4, the value of ρ at which solutions of period 16 appear.

G **d.** By plotting or calculating solutions near the value of ρ_4 found in part c, try to detect the appearance of a period 16 solution.

e. Observe that

$$\rho_n = \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + \cdots + (\rho_n - \rho_{n-1}).$$

Assuming that

$$\rho_4 - \rho_3 = (\rho_3 - \rho_2)\delta^{-1}, \quad \rho_5 - \rho_4 = (\rho_3 - \rho_2)\delta^{-2},$$

and so forth, express ρ_n as a geometric sum. Then find the limit ρ_n as $n \to \infty$. This is an estimate of the value of ρ at which the onset of chaos occurs in the solution of the logistic equation (21).

[24] This result for the logistic difference equation was discovered in August 1975 by Mitchell Feigenbaum (1944–), while he was working at the Los Alamos National Laboratory. Within a few weeks he had established that the same limiting value also appears in a large class of period-doubling difference equations. Feigenbaum, who has a doctorate in physics from M.I.T., is now at Rockefeller University.

Chapter Review Problems

Miscellaneous Problems. One of the difficulties in solving first-order differential equations is that there are several methods of solution, each of which can be used on a certain type of equation. It may take some time to become proficient in matching solution methods with equations. The first 24 of the following problems are presented to give you some practice in identifying the method or methods applicable to a given equation. The remaining problems involve certain types of equations that can be solved by specialized methods.

In each of Problems 1 through 24, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

1. $\dfrac{dy}{dx} = \dfrac{x^3 - 2y}{x}$

2. $\dfrac{dy}{dx} = \dfrac{1 + \cos x}{2 - \sin y}$

3. $\dfrac{dy}{dx} = \dfrac{2x + y}{3 + 3y^2 - x}, \quad y(0) = 0$

4. $\dfrac{dy}{dx} = 3 - 6x + y - 2xy$

5. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$

6. $x\dfrac{dy}{dx} + xy = 1 - y, \quad y(1) = 0$

7. $x\dfrac{dy}{dx} + 2y = \dfrac{\sin x}{x}, \quad y(2) = 1$

8. $\dfrac{dy}{dx} = -\dfrac{2xy + 1}{x^2 + 2y}$

9. $(x^2y + xy - y) + (x^2y - 2x^2)\dfrac{dy}{dx} = 0$

10. $(x^2 + y) + (x + e^y)\dfrac{dy}{dx} = 0$

11. $(x + y) + (x + 2y)\dfrac{dy}{dx} = 0, \quad y(2) = 3$

12. $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$

13. $\dfrac{dy}{dx} = \dfrac{e^{-x}\cos y - e^{2y}\cos x}{-e^{-x}\sin y + 2e^{2y}\sin x}$

14. $\dfrac{dy}{dx} = e^{2x} + 3y$

15. $\dfrac{dy}{dx} + 2y = e^{-x^2-2x}, \quad y(0) = 3$

16. $\dfrac{dy}{dx} = \dfrac{3x^2 - 2y - y^3}{2x + 3xy^2}$

17. $y' = e^{x+y}$

18. $\dfrac{dy}{dx} + \dfrac{2y^2 + 6xy - 4}{3x^2 + 4xy + 3y^2} = 0$

19. $t\dfrac{dy}{dt} + (t + 1)y = e^{2t}$

20. $xy' = y + xe^{y/x}$

21. $\dfrac{dy}{dx} = \dfrac{x}{x^2y + y^3}$ *Hint:* Let $u = x^2$.

22. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

23. $(3y^2 + 2xy) - (2xy + x^2)\dfrac{dy}{dx} = 0$

24. $xy' + y - y^2e^{2x} = 0, \quad y(1) = 2$

25. Riccati Equations. The equation

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati[25] equation. Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution

$$y = y_1(t) + \frac{1}{v(t)}.$$

Show that $v(t)$ satisfies the first-order *linear* equation

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.$$

Note that $v(t)$ will contain a single arbitrary constant.

26. Verify that the given function is a particular solution of the given Riccati equation. Then use the method of Problem 25 to solve the following Riccati equations:

a. $y' = 1 + t^2 - 2ty + y^2; \quad y_1(t) = t$

b. $y' = -\dfrac{1}{t^2} - \dfrac{y}{t} + y^2; \quad y_1(t) = \dfrac{1}{t}$

c. $\dfrac{dy}{dt} = \dfrac{2\cos^2 t - \sin^2 t + y^2}{2\cos t}; \quad y_1(t) = \sin t$

27. The propagation of a single action in a large population (for example, drivers turning on headlights at sunset) often depends partly on external circumstances (gathering darkness) and partly on a tendency to imitate others who have already performed the action in question. In this case the proportion $y(t)$ of people who have performed the action can be described[26] by the equation

$$dy/dt = (1 - y)(x(t) + by), \tag{28}$$

where $x(t)$ measures the external stimulus and b is the imitation coefficient.

a. Observe that equation (28) is a Riccati equation and that $y_1(t) = 1$ is one solution. Use the transformation suggested in Problem 25, and find the linear equation satisfied by $v(t)$.

b. Find $v(t)$ in the case that $x(t) = at$, where a is a constant. Leave your answer in the form of an integral.

[25] Riccati equations are named for Jacopo Francesco Riccati (1676–1754), a Venetian nobleman, who declined university appointments in Italy, Austria, and Russia to pursue his mathematical studies privately at home. Riccati studied these equations extensively; however, it was Euler (in 1760) who discovered the result stated in this problem.

[26] See Anatol Rapoport, "Contribution to the Mathematical Theory of Mass Behavior: I. The Propagation of Single Acts," *Bulletin of Mathematical Biophysics 14* (1952), pp. 159–169, and John Z. Hearon, "Note on the Theory of Mass Behavior," *Bulletin of Mathematical Biophysics 17* (1955), pp. 7–13.

Some Special Second-Order Differential Equations. Second-order differential equations involve the second derivative of the unknown function and have the general form $y'' = f(t, y, y')$. Usually, such equations cannot be solved by methods designed for first-order equations. However, there are two types of second-order equations that can be transformed into first-order equations by a suitable change of variable. The resulting equation can sometimes be solved by the methods presented in this chapter. Problems 28 through 37 deal with these types of equations.

Equations with the Dependent Variable Missing. For a second-order differential equation of the form $y'' = f(t, y')$, the substitution $v = y'$, $v' = y''$ leads to a first-order differential equation of the form $v' = f(t, v)$. If this equation can be solved for v, then y can be obtained by integrating $dy/dt = v$. Note that one arbitrary constant is obtained in solving the first-order equation for v, and a second is introduced in the integration for y. In each of Problems 28 through 31, use this substitution to solve the given equation.

28. $t^2y'' + 2ty' - 1 = 0, \quad t > 0$

29. $ty'' + y' = 1, \quad t > 0$

30. $y'' + t(y')^2 = 0$

31. $2t^2y'' + (y')^3 = 2ty', \quad t > 0$

Equations with the Independent Variable Missing. Consider second-order differential equations of the form $y'' = f(y, y')$, in which the independent variable t does not appear explicitly. If we let $v = y'$, then we obtain $dv/dt = f(y, v)$. Since the right-hand side of this equation depends on y and v, rather than on t and v, this equation contains too many variables. However, if we think of y as the independent variable, then by the chain rule, $dv/dt = (dv/dy)(dy/dt) = v(dv/dy)$. Hence the original differential equation can be written as $v(dv/dy) = f(y, v)$. Provided that this first-order equation can be solved, we obtain v as a function of y. A relation between y and t results from solving $dy/dt = v(y)$, which is a separable equation. Again, there are two arbitrary constants in the final result. In each of Problems 32 through 35, use this method to solve the given differential equation.

32. $yy'' + (y')^2 = 0$

33. $y'' + y = 0$

34. $yy'' - (y')^3 = 0$

35. $y'' + (y')^2 = 2e^{-y}$

Hint: In Problem 35 the transformed equation is a Bernoulli equation. See Problem 23 in Section 2.4.

In each of Problems 36 through 37, solve the given initial value problem using the methods of Problems 28 through 35.

36. $y'y'' = 2, \quad y(0) = 1, \quad y'(0) = 2$

37. $(1 + t^2)y'' + 2ty' + 3t^{-2} = 0, \quad y(1) = 2, \quad y'(1) = -1$

References

The two books mentioned in Section 2.5 are

Bailey, N. T. J., *The Mathematical Theory of Infectious Diseases and Its Applications* (2nd ed.) (New York: Hafner Press, 1975).

Clark, Colin W., *Mathematical Bioeconomics* (2nd ed.) (New York: Wiley-Interscience, 1990).

A good introduction to population dynamics, in general, is

Frauenthal, J. C., *Introduction to Population Modeling* (Boston: Birkhauser, 1980).

A fuller discussion of the proof of the fundamental existence and uniqueness theorem can be found in many more advanced books on differential equations. Two that are reasonably accessible to elementary readers are