Boyce/DiPrima/Meade 11th ed, Ch 6.1: Definition of Laplace Transform

- Many practical engineering problems involve mechanical or electrical systems acted upon by discontinuous or impulsive forcing terms.
- For such problems the methods described in Chapter 3 are difficult to apply.
- In this chapter we use the Laplace transform to convert a problem for an unknown function *f* into a simpler problem for *F*, solve for *F*, and then recover *f* from its transform *F*.
- Given a known function $K(s,t)$, an **integral transform** of a function *f* is a relation of the form

$$F(s) = \int_a^b K(s,t) f(t)\,dt, \quad -\infty \le a < b \le \infty$$

Improper Integrals

- The Laplace transform will involve an integral from zero to infinity. Such an integral is a type of improper integral.
- An improper integral over an unbounded interval is defined as the limit of an integral over a finite interval

$$\int_a^\infty f(t)dt = \lim_{A \to \infty} \int_a^A f(t)dt$$

  where A is a positive real number.
- If the integral from a to A exists for each $A > a$ and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise, the integral is said to **diverge** or fail to exist.

Example 1

- Consider the following improper integral.

$$\int_1^{\infty} \frac{dt}{t}$$

- We can evaluate this integral as follows:

$$\int_1^{\infty} \frac{dt}{t} = \lim_{A\to\infty} \int_1^{A} \frac{dt}{t} = \lim_{A\to\infty} \left(\ln A\right) \to \infty$$

- Therefore, the improper integral diverges.

Example 2

- Consider the following improper integral.

$$\int_0^\infty e^{ct}\,dt$$

- We can evaluate this integral as follows:

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\frac{1}{c}\left(e^{cA}-1\right)$$

- Note that if $c = 0$, then $e^{ct} = 1$. Thus the following two cases hold:

$$\int_0^\infty e^{ct}\,dt = -\frac{1}{c}, \text{ if } c<0; \text{ and}$$

$$\int_0^\infty e^{ct}\,dt \text{ diverges, if } c\geq 0.$$

Example 3

- Consider the following improper integral.

$$\int_1^{\infty} t^{-p}\ dt$$

- From Example 1, this integral diverges at $p = 1$
- We can evaluate this integral for $p \neq 1$ as follows:

$$\int_1^{\infty} t^{-p}\ dt = \lim_{A\to\infty}\int_1^{A} t^{-p}\ dt = \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right)$$

- The improper integral diverges at $p = 1$ and

$$\text{If } p > 1,\ \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right) = \frac{1}{p-1}$$

$$\text{If } p < 1,\ \lim_{A\to\infty}\frac{1}{1-p}\left(A^{1-p}-1\right) \to \infty$$

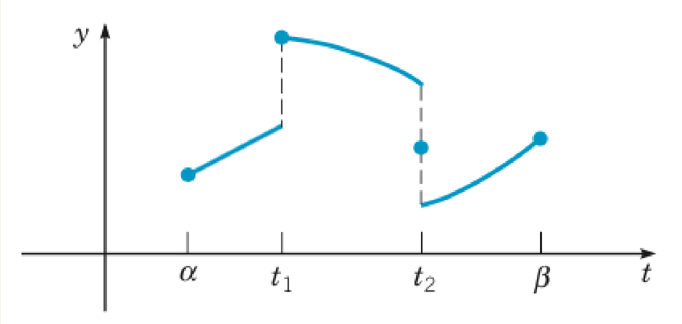
Piecewise Continuous Functions

- A function f is **piecewise continuous** on an interval $[a, b]$ if this interval can be partitioned by a finite number of points $a = t_0 < t_1 < \ldots < t_n = b$ such that

  (1) f is continuous on each (t_k, t_{k+1})

  (2) $\left| \lim_{t \to t_k^+} f(t) \right| < \infty, \quad k = 0, \ldots, n-1$

  (3) $\left| \lim_{t \to t_{k+1}^-} f(t) \right| < \infty, \quad k = 1, \ldots, n$



- In other words, f is piecewise continuous on $[a, b]$ if it is continuous there except for a finite number of jump discontinuities.

Theorem 6.1.1

- If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive M and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ also converges.
- On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ also diverges.

The Laplace Transform

- Let f be a function defined for $t > 0$, and satisfies certain conditions to be named later.
- The **Laplace Transform of *f*** is defined as an **integral transform**:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

- The **kernel** function is $K(s,t) = e^{-st}$.
- Since solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.
- Note that the Laplace Transform is defined by an improper integral, and thus must be checked for convergence.
- On the next few slides, we review examples of improper integrals and piecewise continuous functions.

Theorem 6.1.2

- Suppose that *f* is a function for which the following hold:
  (1) f is piecewise continuous on $[0, b]$ for all $b > 0$.
  (2) $| f(t) | \leq Ke^{at}$ when $t \geq M$, for constants a, K, M, with K, $M > 0$.
- Then the Laplace Transform of f exists for $s > a$.

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt \text{ finite}$$

- Note: A function *f* that satisfies the conditions specified above is said to to have **exponential order** as $t \to \infty$.

Example 4

- Let $f(t) = 1$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
L\{1\} &= \int_0^\infty e^{-st}dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}dt \\
&= -\lim_{b\to\infty} \left. \frac{e^{-st}}{s} \right|_0^b \\
&= \frac{1}{s}, \quad s > 0
\end{aligned}
$$

Example 5

- Let $f(t) = e^{at}$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-(s-a)t} dt$$

$$= -\lim_{b \to \infty} \left. \frac{e^{-(s-a)t}}{s-a} \right|_0^b$$

$$= \frac{1}{s-a}, \quad s > a$$

Example 6

- Consider the following piecewise-defined function *f*

$$f(t)=\begin{cases}1, & 0\le t\le 1\\ k, & t=1\\ 0 & t>1\end{cases}$$

where k is a constant. This represents a unit impulse.

- Noting that $f(t)$ is piecewise continuous, we can compute its Laplace transform

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^1 e^{-st}dt=\frac{1-e^{-s}}{s},\qquad s>0$$

- Observe that this result does not depend on k, the function value at the point of discontinuity.

Example 7

- Let $f(t) = \sin(at)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$$F(s) = L\{\sin(at)\} = \int_0^\infty e^{-st} \sin at dt = \lim_{b\to\infty} \int_0^b e^{-st} \sin at dt$$

$$= \lim_{b\to\infty} \left[-(e^{-st} \cos at)/a \Big|_0^b - \frac{s}{a} \int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{b\to\infty} \left[\int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a} \lim_{b\to\infty} \left[(e^{-st} \sin at)/a \Big|_0^b + \frac{s}{a} \int_0^b e^{-st} \sin at \right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2} F(s) \;\Rightarrow\; F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Linearity of the Laplace Transform

- Suppose f and g are functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively.
- Then, for s greater than the maximum of a_1 and a_2, the Laplace transform of $c_1 f(t) + c_2 g(t)$ exists. That is,

$$L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}\left[c_1 f(t) + c_2 g(t)\right]dt \quad \text{is finite}$$

with

$$\begin{aligned} L\{c_1 f(t) + c_2 g(t)\} &= c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt \\ &= c_1 L\{f(t)\} + c_2 L\{g(t)\} \end{aligned}$$

Example 8

- Let $f(t) = 5e^{-2t} - 3\sin(4t)$ for $t \geq 0$.
- Then by linearity of the Laplace transform, and using results of previous examples, the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
F(s) &= L\{f(t)\} \\
&= L\left\{5e^{-2t} - 3\sin(4t)\right\} \\
&= 5L\left\{e^{-2t}\right\} - 3L\left\{\sin(4t)\right\} \\
&= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0
\end{aligned}
$$

Boyce/DiPrima/Meade 11th ed, Ch 6.2: Solution of Initial Value Problems

- The Laplace transform is named for the French mathematician Laplace, who studied this transform in 1782.
- The techniques described in this chapter were developed primarily by Oliver Heaviside (1850 - 1925), an English electrical engineer.
- In this section we see how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients.
- The Laplace transform is useful in solving these differential equations because the transform of f' is related in a simple way to the transform of f, as stated in Theorem 6.2.1.

Theorem 6.2.1

- Suppose that f is a function for which the following hold:

  (1) f is continuous and f' is piecewise continuous on $[0, b]$ for all $b > 0$.

  (2) $|f(t)| \leq Ke^{at}$ when $t \geq M$, for constants a, K, M, with $K, M > 0$.

- Then the Laplace Transform of f' exists for $s > a$, with

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- **Proof** (outline): For f and f' continuous on $[0, b]$, we have

$$\begin{aligned}\lim_{b\to\infty}\int_0^b e^{-st} f'(t)dt &= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt \right] \\ &= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt \right]\end{aligned}$$

- Similarly for f' piecewise continuous on $[0, b]$, see text.

The Laplace Transform of f'

- Thus if f and f' satisfy the hypotheses of Theorem 6.2.1, then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

- Now suppose f' and f'' satisfy the conditions specified for f and f' of Theorem 6.2.1. We then obtain

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s\left[sL\{f(t)\} - f(0)\right] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

- Similarly, we can derive an expression for $L\{f^{(n)}\}$, provided f and its derivatives satisfy suitable conditions. This result is given in Corollary 6.2.2

Corollary 6.2.2

- Suppose that f is a function for which the following hold:

  (1) $f, f', f'', \ldots, f^{(n\text{-}1)}$ are continuous, and $f^{(n)}$ piecewise continuous, on $[0, b]$ for all $b > 0$.

  (2) $|f(t)| \leq Ke^{at}, |f'(t)| \leq Ke^{at}, \ldots, |f^{(n-1)}(t)| \leq Ke^{at}$ for $t \geq M$, for constants a, K, M, with $K, M > 0$.

  Then the Laplace Transform of $f^{(n)}$ exists for $s > a$, with

$$L\left\{f^{(n)}(t)\right\} = s^n L\left\{f(t)\right\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example 1: Chapter 3 Method (1 of 4)

- Consider the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = 1, \; y'(0) = 0$$

- Recall from Section 3.1:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 - r - 2 = 0 \;\Leftrightarrow\; (r-2)(r+1) = 0$$

- Thus r_1 = –2 and r_2 = –3, and general solution has the form

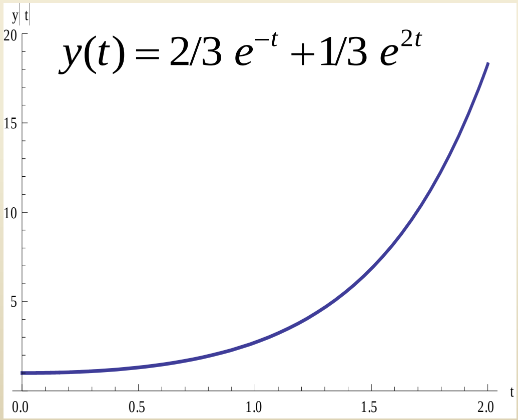
$$y(t) = c_1 e^{-t} + c_2 e^{2t}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + 2c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 2/3 \;,\; c_2 = 1/3$$

- Thus

$$y(t) = 2/3\; e^{-t} + 1/3\; e^{2t}$$



- We now solve this problem using Laplace Transforms.

$$y'' - y' - 2y = 0, \quad y(0) = 1, \; y'(0) = 0$$

Example 1: Laplace Transform Method (2 of 4)

- Assume that our IVP has a solution $f(t)$ and $f'(t)$ and $f''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y'' - y' - 2y\} = L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\} = 0$$

and hence

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] - \left[sL\{y\} - y(0)\right] - 2L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 - s - 2\right)Y(s) - (s-1)y(0) - y'(0) = 0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 - s - 2\right)Y(s) - (s-1) = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{s-1}{(s-2)(s+1)}$$

Example 1: Partial Fractions

- Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{s-1}{(s-2)(s+1)} = \frac{a}{(s-2)} + \frac{b}{(s+1)}$$

$$s-1 = a(s+1) + b(s-2)$$

$$s-1 = (a+b)s + (a-2b)$$

$$a+b=1,\ \ a-2b=-1$$

$$a = 1/3\ ,\ b = 2/3$$

- Thus

$$L\{y\} = Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)}$$

Example 1: Solution (4 of 4)

- Recall from Section 6.1:

$$L\left\{e^{at}\right\} = F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a$$

- Thus

$$Y(s) = \frac{1/3}{(s-2)} + \frac{2/3}{(s+1)} = 1/3\ L\{e^{2t}\} + 2/3\ L\{e^{-t}\}, \ \ s > 2$$

- Recalling $Y(s) = L\{y\}$, we have

$$L\{y\} = L\{2/3\ e^{-t} + 1/3\ e^{2t}\}$$

and hence

$$y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

General Laplace Transform Method

- Consider the constant coefficient equation

$$ay'' + by' + cy = f(t)$$

- Assume that the solution $y(t)$ satisfies the conditions of Corollary 6.2.2 for $n = 2$.
- We can take the transform of the above equation:

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = F(s)$$

where $F(s)$ is the transform of $f(t)$.

- Solving for $Y(s)$ gives:

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Algebraic Problem

- Thus the differential equation has been transformed into the the algebraic equation

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

for which we seek $y = f(t)$ such that $L\{f(t)\} = Y(s)$.

- Note that we do not need to solve the homogeneous and nonhomogeneous equations separately, nor do we have a separate step for using the initial conditions to determine the values of the coefficients in the general solution.

Characteristic Polynomial

- Using the Laplace transform, our initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \ y'(0) = y_0'$$

becomes

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

- The polynomial in the denominator is the characteristic polynomial associated with the differential equation.
- The partial fraction expansion of $Y(s)$ used to determine $f(t)$ requires us to find the roots of the characteristic equation.
- For higher order equations, this may be difficult, especially if the roots are irrational or complex.

Inverse Problem

- The main difficulty in using the Laplace transform method is determining the function $y = f(t)$ such that $L\{f(t)\} = Y(s)$.
- This is an inverse problem, in which we try to find $f(t)$ such that $f(t) = L^{-1}\{Y(s)\}$.
- There is a general formula for L^{-1}, but it requires knowledge of the theory of functions of a complex variable, and we do not consider it here.
- It can be shown that if f is continuous with $L\{f(t)\} = F(s)$, then f is the **unique** continuous function with $f(t) = L^{-1}\{F(s)\}$.
- Table 6.2.1 in the text lists many of the functions and their transforms that are encountered in this chapter.

Linearity of the Inverse Transform

- Frequently a Laplace transform $F(s)$ can be expressed as

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

- Let

$$f_1(t) = L^{-1}\{F_1(s)\}, \ldots, f_n(t) = L^{-1}\{F_n(s)\}$$

- Then the function

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t)$$

  has the Laplace transform $F(s)$, since L is linear.
- By the uniqueness result of the previous slide, no other continuous function f has the same transform $F(s)$.
- Thus L^{-1} is a linear operator with

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + \cdots + L^{-1}\{F_n(s)\}$$

Example 2: Nonhomogeneous Problem

- Consider the initial value problem

$$y'' + y = \sin 2t, \quad y(0) = 2, \; y'(0) = 1$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2 + 4)$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 + 1\right)Y(s) - sy(0) - y'(0) = 2/(s^2 + 4)$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 + 1\right)Y(s) - 2s - 1 = 2/(s^2 + 4)$$

- Thus

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}$$

Example 2: Solution (2 of 2)

- Using partial fractions,

$$Y(s)=\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4}$$

- Then

$$2s^3+s^2+8s+6=\left(As+B\right)\left(s^2+4\right)+\left(Cs+D\right)\left(s^2+1\right)$$
$$=(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)$$

- Solving, we obtain *A* = 2, *B* = 5/3, *C* = 0, and *D* = -2/3. Thus

$$Y(s)=\frac{2s}{s^2+1}+\frac{5/3}{s^2+1}-\frac{2/3}{s^2+4}$$

- Hence

$$y(t)=2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$$

Example 3: Solving a 4th Order IVP (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0, \qquad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$$

- Taking the Laplace transform of the differential equation, and assuming the conditions of Corollary 6.2.2 are met, we have

$$\left[s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)\right] + L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$ and substituting the initial values, we have

$$Y(s) = \frac{s^2}{(s^4 - 1)} = \frac{s^2}{(s^2 - 1)(s^2 + 1)}$$

- Using partial fractions

$$Y(s) = \frac{s^2}{(s^2 - 1)(s^2 + 1)} = \frac{as + b}{(s^2 - 1)} + \frac{cs + d}{(s^2 + 1)}$$

- Thus

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2$$

$$y^{(4)} - y = 0, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0$$

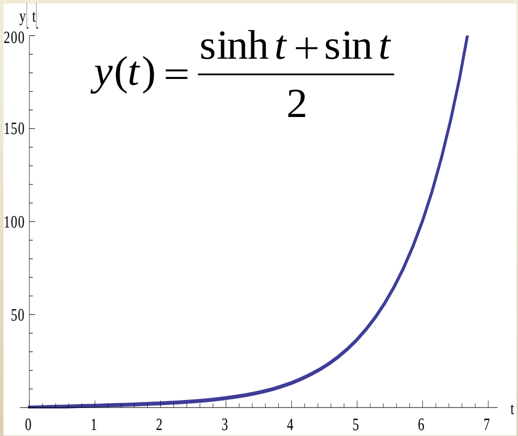
Example 3: Solving a 4th Order IVP (2 of 2)

- In the expression: $(as+b)(s^2+1)+(cs+d)(s^2-1)=s^2$
- Setting s = 1 and s = –1 enables us to solve for *a* and *b*:

  $$2(a+b)=1 \text{ and } 2(-a+b)=1 \Rightarrow a=0, b=1/2$$

- Setting s = 0, *b* – *d* = 0, so *d* = 1/2
- Equating the coefficients of s^3 in the first expression gives *a* + *c* = 0, so *c* = 0
- Thus $Y(s)=\dfrac{1/2}{(s^2-1)}+\dfrac{1/2}{(s^2+1)}$
- Using Table 6.2.1, the solution is

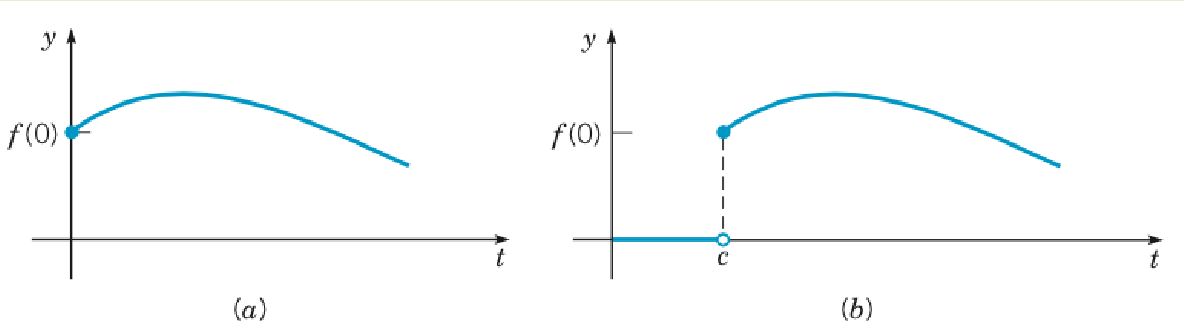
$$y(t)=\frac{\sinh t+\sin t}{2}$$



Boyce/DiPrima/Meade 11th ed, Ch 6.3: Step Functions

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade ©2017 by John Wiley & Sons, Inc.

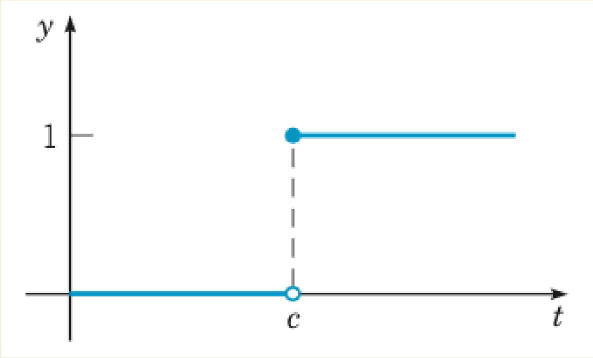
- Some of the most interesting elementary applications of the Laplace Transform method occur in the solution of linear equations with discontinuous or impulsive forcing functions.
- In this section, we will assume that all functions considered are piecewise continuous and of exponential order, so that their Laplace Transforms all exist, for *s* large enough.



Step Function definition

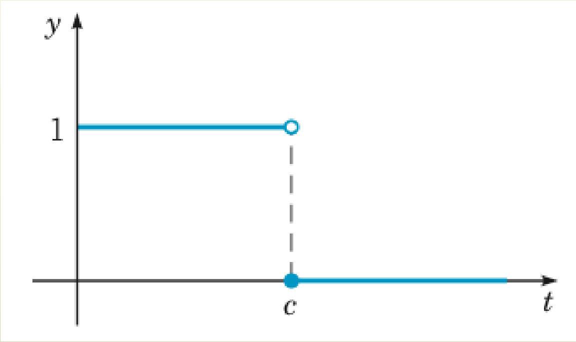
- Let $c > 0$. The **unit step function**, or Heaviside function, is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



- A negative step can be represented by

$$y(t) = 1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



Example 1

- Sketch the graph of $y = h(t)$, where

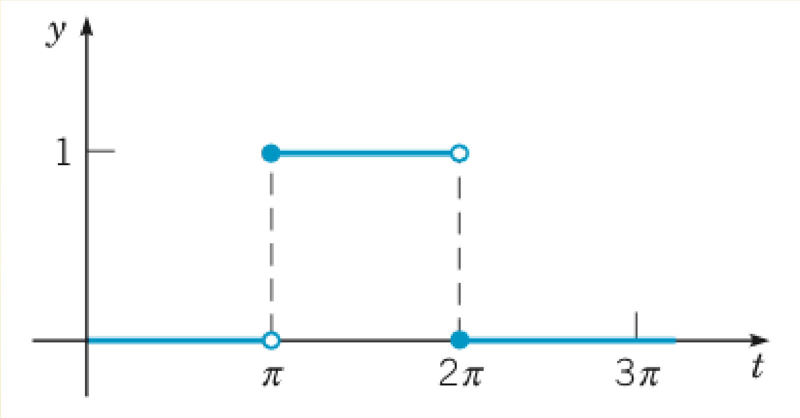
$$h(t) = u_\pi(t) - u_{2\pi}(t), \quad t \geq 0$$

- Solution: Recall that $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- Thus

$$h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0 & 2\pi \leq t < \infty \end{cases}$$



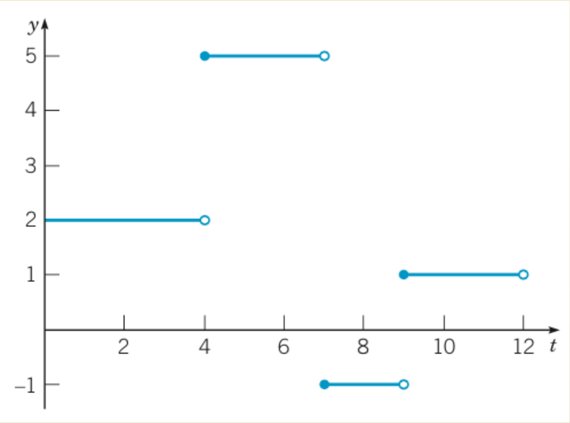
and hence the graph of $h(t)$ is a rectangular pulse.

Example 2

- For the function

$$h(t)=\begin{cases}2, & 0\le t<4\\ 5, & 4\le t<7\\ -1, & 7\le t<9\\ 1, & t\ge 9\end{cases}$$

whose graph is shown



- To write $h(t)$ in terms of $u_c(t)$, we will need $u_4(t)$, $u_7(t)$, and $u_9(t)$. We begin with the 2, then add 3 to get 5, then subtract 6 to get –1, and finally add 2 to get 1 – each quantity is multiplied by the appropriate $u_c(t)$

$$h(t)=2+3u_4(t)-6u_7(t)+2u_9(t),\quad t\ge 0$$

Laplace Transform of Step Function

- The Laplace Transform of $u_c(t)$ is

$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_c^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{e^{-bs}}{s} + \frac{e^{-cs}}{s} \right]$$

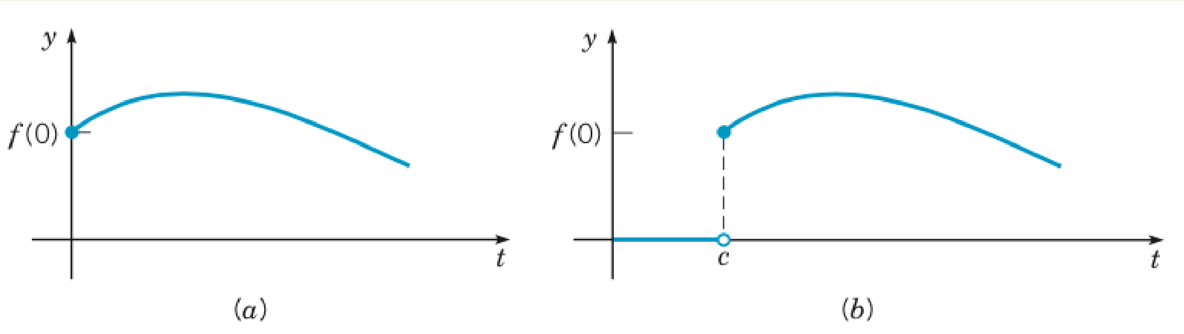
$$= \frac{e^{-cs}}{s}$$

Translated Functions

- Given a function $f(t)$ defined for $t \geq 0$, we will often want to consider the related function $g(t) = u_c(t) f(t - c)$:

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

- Thus g represents a translation of f a distance c in the positive t direction.
- In the figure below, the graph of f is given on the left, and the graph of g on the right.



Theorem 6.3.1

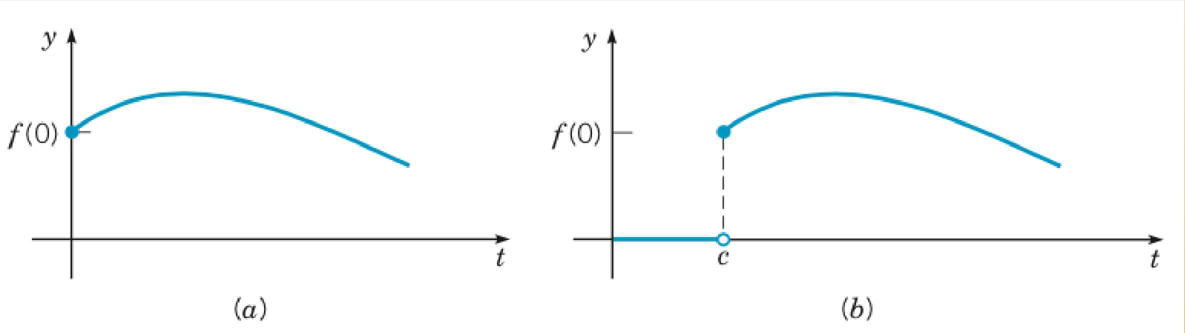
- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, then

$$L\left\{u_c(t)\,f(t-c)\right\} = e^{-cs}L\left\{f(t)\right\} = e^{-cs}F(s)$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$u_c(t)\,f(t-c) = L^{-1}\left\{e^{-cs}F(s)\right\}$$

- Thus the translation of $f(t)$ a distance c in the positive t direction corresponds to a multiplication of $F(s)$ by e^{-cs}.



Theorem 6.3.1: Proof Outline

- We need to show

$$L\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

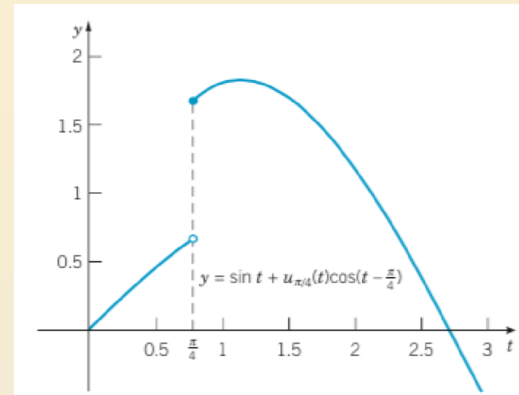
- Using the definition of the Laplace Transform, we have

$$\begin{aligned}
L\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&\overset{u=t-c}{=} \int_0^\infty e^{-s(u+c)}f(u)du \\
&= e^{-cs}\int_0^\infty e^{-su}f(u)du \\
&= e^{-cs}F(s)
\end{aligned}$$

Example 3

- Find $L\{f(t)\}$, where f is defined by

$$f(t)=\begin{cases}\sin t, & 0\le t<\frac{p}{4}\\ \sin t+\cos(t-\frac{p}{4}), & t\ge\frac{p}{4}\end{cases}$$



- Note that $f(t) = \sin(t) + u_{p/4}(t)\cos(t - p/4)$, and

$$\begin{aligned}L\{f(t)\} &= L\{\sin t\} + L\{u_{\pi/4}(t)\cos(t-\pi/4)\}\\ &= L\{\sin t\} + e^{-\pi s/4}L\{\cos t\}\\ &= \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1}\\ &= \frac{1+se^{-\pi s/4}}{s^2+1}\end{aligned}$$

Example 4

- Find $L^{-1}\{F(s)\}$, where

$$F(s) = \frac{1 - e^{-2s}}{s^2}$$

- Solution:

$$f(t) = L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t - u_2(t)\big(t - 2\big)$$

- The function may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$$

Theorem 6.3.2

- If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$L\left\{e^{ct} f(t)\right\} = F(s-c), \quad s > a+c$$

- Conversely, if $f(t) = L^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = L^{-1}\left\{F(s-c)\right\}$$

- Thus multiplication $f(t)$ by e^{ct} results in translating $F(s)$ a distance c in the positive t direction, and conversely.
- Proof Outline:

$$L\left\{e^{ct} f(t)\right\} = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c)$$

Example 5

- To find the inverse transform of

$$G(s) = \frac{1}{s^2 - 4s + 5}$$

- We first complete the square:

$$G(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{\left(s^2 - 4s + 4\right) + 1} = \frac{1}{(s-2)^2 + 1} = F(s-2)$$

- Since

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos t \text{ and } L^{-1}\{F(s-2)\} = e^{2t} f(t)$$

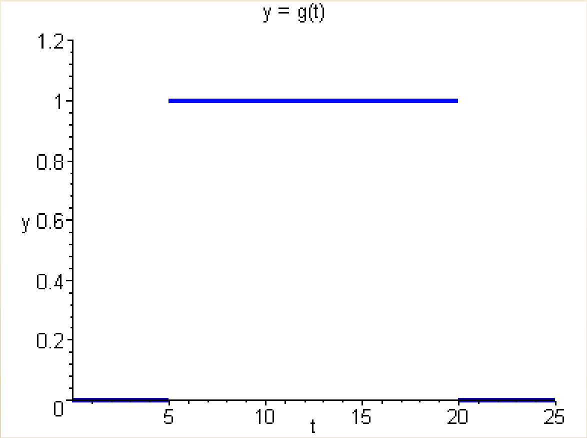
it follows that

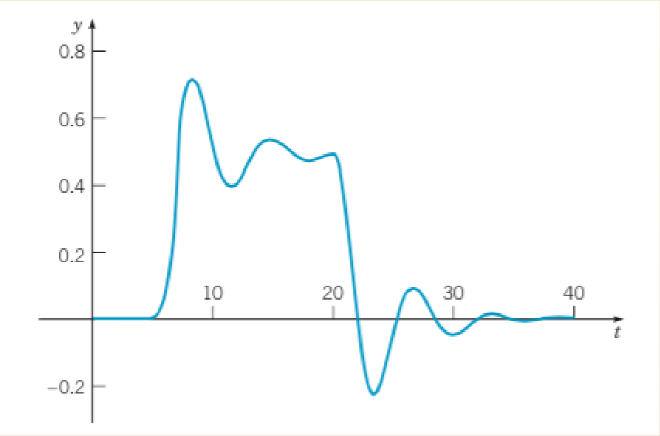
$$g(t) = L^{-1}\{G(s)\} = e^{2t} \cos t$$

Boyce/DiPrima/Meade 11th ed, Ch 6.4: Differential Equations with Discontinuous Forcing Functions

- In this section focus on examples of nonhomogeneous initial value problems in which the forcing function is discontinuous.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \; y'(0) = y_0'$$





Example 1: Initial Value Problem

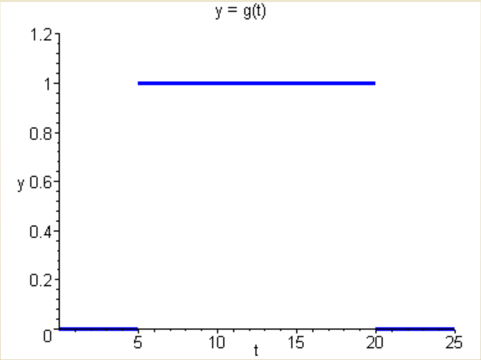
- Find the solution to the initial value problem

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

- Such an initial value problem might model the response of a damped oscillator subject to $g(t)$, or current in a circuit for a unit voltage pulse.



$$2y'' + y' + 2y = u_5(t) - u_{20}(t), \quad y(0) = 0, \; y'(0) = 0$$

Example 1: Laplace Transform (2 of 12)

- Assume the conditions of Corollary 6.2.2 are met. Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$

or

$$\left[2s^2 L\{y\} - 2sy(0) - 2y'(0)\right] + \left[sL\{y\} - y(0)\right] + 2L\{y\} = \frac{e^{-5s} - e^{-20s}}{s}$$

- Letting $Y(s) = L\{y\}$,

$$\left(2s^2 + s + 2\right)Y(s) - (2s+1)y(0) - 2y'(0) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)}$$

Example 1: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)} = \left(e^{-5s} - e^{-20s}\right)H(s)$$

where

$$H(s) = \frac{1}{s\left(2s^2 + s + 2\right)}$$

- If we let $h(t) = L^{-1}\{H(s)\}$, then

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

by Theorem 6.3.1.

Example 1: Partial Fractions

- Thus we examine $H(s)$, as follows.

$$H(s) = \frac{1}{s\left(2s^2 + s + 2\right)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2}$$

- This partial fraction expansion yields the equations

$$(2A + B)s^2 + (A + C)s + 2A = 1$$
$$\Rightarrow A = 1/2,\ B = -1,\ C = -1/2$$

- Thus

$$H(s) = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2}$$

Example 1: Completing the Square

- Completing the square,

$$
\begin{aligned}
H(s) &= \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1/16+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{(s+1/4)^2+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]
\end{aligned}
$$

Example 1: Solution (6 of 12)

- Thus

$$H(s)=\frac{1/2}{s}-\frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]$$

$$=\frac{1/2}{s}-\frac{1}{2}\left[\frac{(s+1/4)}{(s+1/4)^2+15/16}\right]-\frac{1}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2+15/16}\right]$$

and hence

$$h(t)=L^{-1}\{H(s)\}=\frac{1}{2}-\frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right)-\frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$\phi(t)=u_5(t)h(t-5)-u_{20}(t)h(t-20)$$

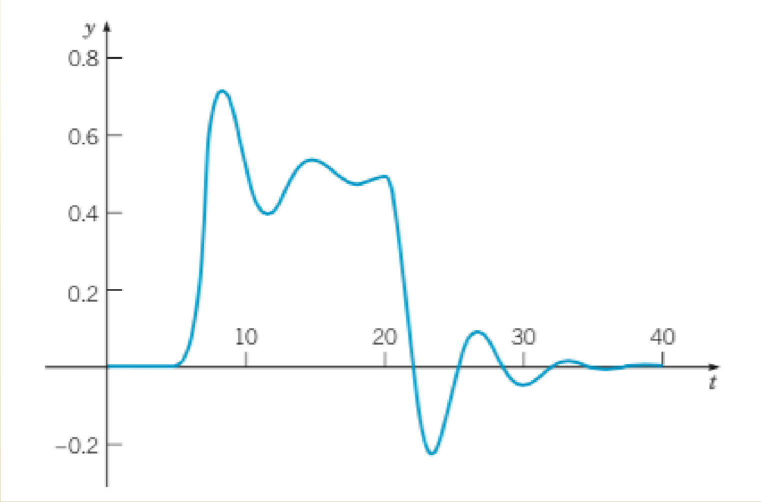
Example 1: Solution Graph (7 of 12)

- Thus the solution to the initial value problem is

$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20), \quad \text{where}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\sqrt{15}t/4\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\sqrt{15}t/4\right)$$

- The graph of this solution is given below.



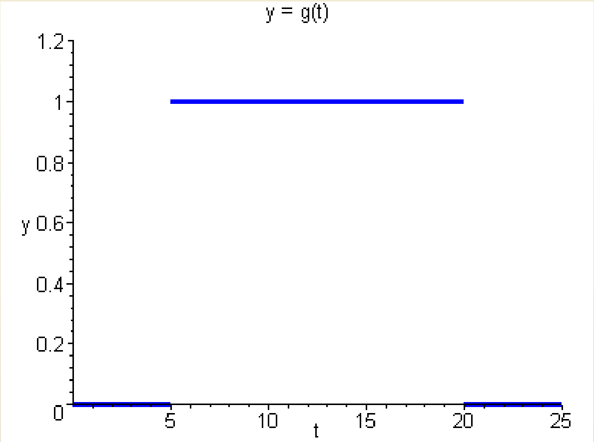
Example 1: Composite IVPs (8 of 12)

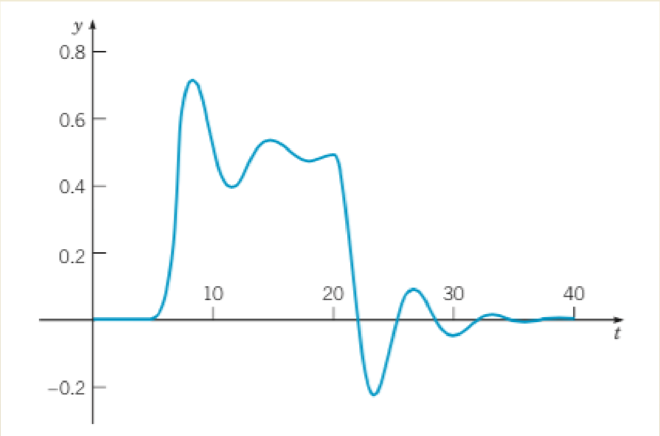
- The solution to original IVP can be viewed as a composite of three separate solutions to three separate IVPs:

$$0 \le t < 5: \quad 2y_1'' + y_1' + 2y_1 = 0, \quad y_1(0) = 0,\ y_1'(0) = 0$$

$$5 < t < 20: \quad 2y_2'' + y_2' + 2y_2 = 1, \quad y_2(5) = 0,\ y_2'(5) = 0$$

$$t > 20: \quad 2y_3'' + y_3' + 2y_3 = 0, \quad y_3(20) = y_2(20),\ y_3'(20) = y_2'(20)$$



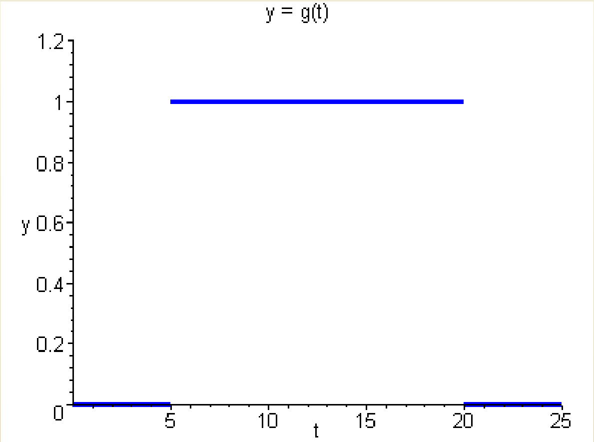


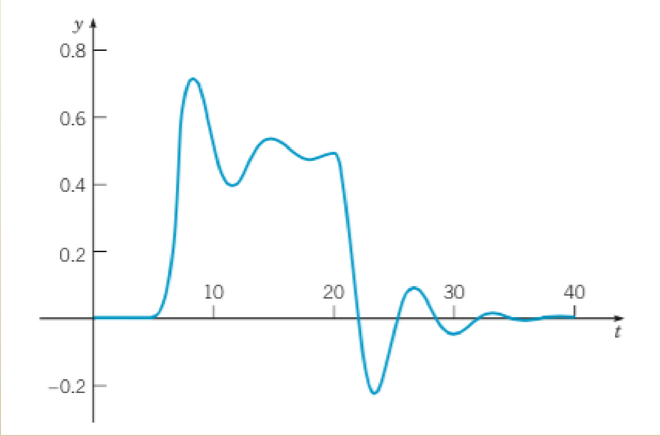
Example 1: First IVP (9 of 12)

- Consider the first initial value problem

$$2y_1'' + y_1' + 2y_1 = 0, \quad y_1(0) = 0, \; y_1'(0) = 0; \quad 0 \le t < 5$$

- From a physical point of view, the system is initially at rest, and since there is no external forcing, it remains at rest.
- Thus the solution over [0, 5) is $y_1 = 0$, and this can be verified analytically as well. See graphs below.





Example 1: Second IVP (10 of 12)

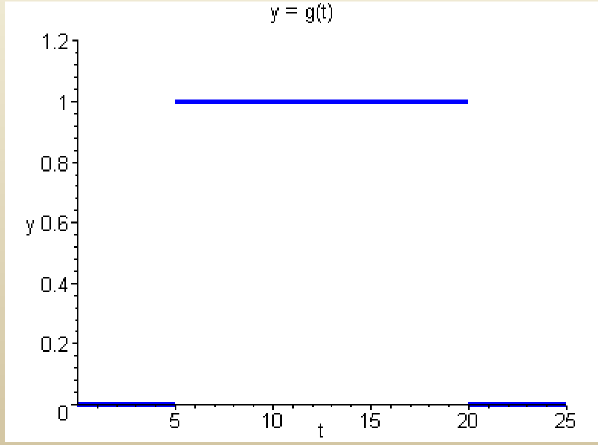
- Consider the second initial value problem

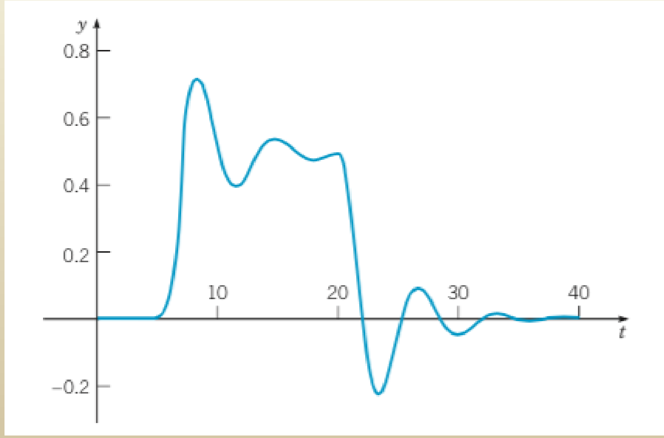
$$2y_2'' + y_2' + 2y_2 = 1, \quad y_2(5) = 0,\ y_2'(5) = 0; \quad 5 < t < 20$$

- Using methods of Chapter 3, the solution has the form

$$y_2 = c_1 e^{-t/4}\cos\left(\sqrt{15}t/4\right) + c_2 e^{-t/4}\sin\left(\sqrt{15}t/4\right) + 1/2$$

- Physically, the system responds with the sum of a constant (the response to the constant forcing function) and a damped oscillation, over the time interval (5, 20). See graphs below.





Example 1: Third IVP

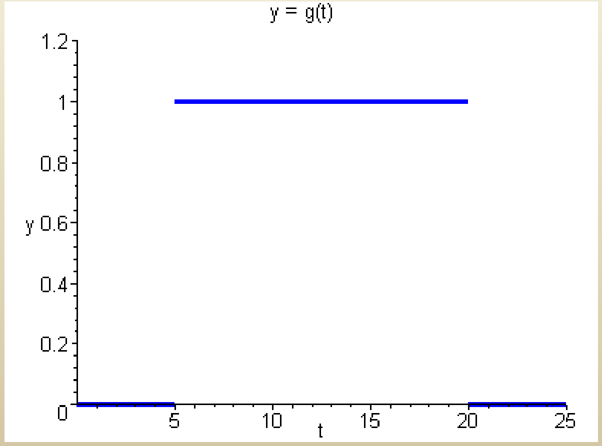
- Consider the third initial value problem

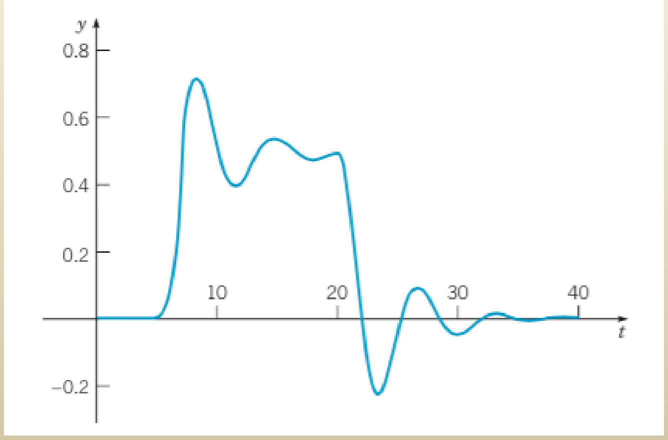
$$2y_3'' + y_3' + 2y_3 = 0, \quad y_3(20) = y_2(20),\ y_3'(20) = y_2'(20); \quad t > 20$$

- Using methods of Chapter 3, the solution has the form

$$y_3 = c_1 e^{-t/4} \cos\left(\sqrt{15}t/4\right) + c_2 e^{-t/4} \sin\left(\sqrt{15}t/4\right)$$

- Physically, since there is no external forcing, the response is a damped oscillation about $y = 0$, for $t > 20$. See graphs below.





Example 1: Solution Smoothness (12 of 12)

- Our solution is

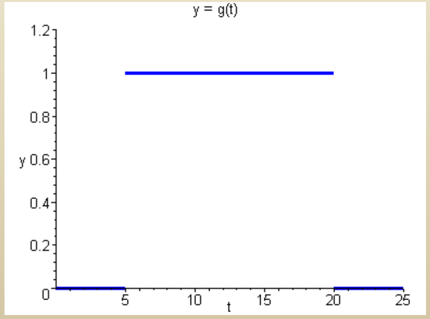
$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

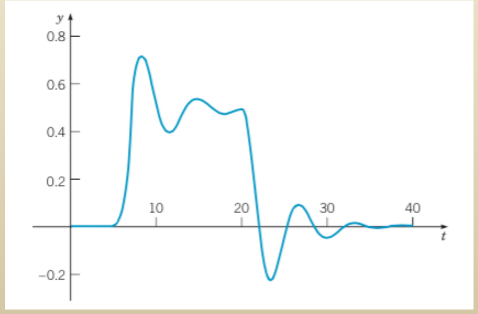
- It can be shown that ϕ and ϕ' are continuous at $t = 5$ and $t = 20$, and ϕ'' has a jump of 1/2 at $t = 5$ and a jump of –1/2 at $t = 20$:

$$\lim_{t\to 5^-} \phi''(t) = 0, \quad \lim_{t\to 5^+} \phi''(t) = 1/2$$

$$\lim_{t\to 20^-} \phi''(t) \cong -0.0072, \quad \lim_{t\to 20^+} \phi''(t) \cong -0.5072$$

- Thus jump in forcing term $g(t)$ at these points is balanced by a corresponding jump in highest order term $2y''$ in ODE.





Smoothness of Solution in General

- Consider a general second order linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p and q are continuous on some interval (a, b) but g is only piecewise continuous there.

- If $y = f(t)$ is a solution, then f and f' are continuous on (a, b) but f'' has jump discontinuities at the same points as g.
- Similarly for higher order equations, where the highest derivative of the solution has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous over (a, b).

Example 2: Initial Value Problem (1 of 12)

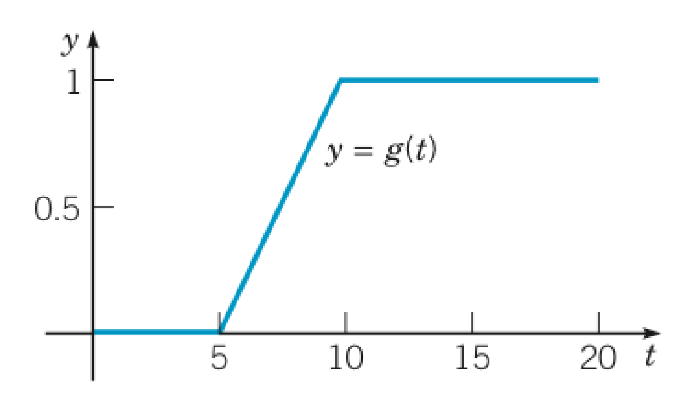
- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5 \\ \frac{1}{5}(t-5) & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

- The graph of forcing function $g(t)$ is given on right, and is known as ramp loading.



$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0) = 0, \ y'(0) = 0$$

Example 2: Laplace Transform

- Assume that this ODE has a solution $y = f(t)$ and that $f'(t)$ and $f''(t)$ satisfy the conditions of Corollary 6.2.2.

Then

$$L\{y''\} + 4L\{y\} = \left[L\{u_5(t)(t-5)\}\right]/5 - \left[L\{u_{10}(t)(t-10)\}\right]/5$$

or $$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2 + 4\right)Y(s) = \left(e^{-5s} - e^{-10s}\right)/5s^2$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2 + 4\right)}$$

Example 2: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2+4\right)} = \frac{e^{-5s} - e^{-10s}}{5} H(s)$$

where

$$H(s) = \frac{1}{s^2\left(s^2+4\right)}$$

- If we let $h(t) = L^{-1}\{H(s)\}$, then

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

by Theorem 6.3.1.

Example 2: Partial Fractions

- Thus we examine $H(s)$, as follows.

$$H(s)=\frac{1}{s^2\left(s^2+4\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}$$

- This partial fraction expansion yields the equations

$$(A+C)s^3+(B+D)s^2+4As+4B=1$$

$$\Rightarrow A=0,\ B=1/4,\ C=0,\ D=-1/4$$

- Thus

$$H(s)=\frac{1/4}{s^2}-\frac{1/4}{s^2+4}$$

Example 2: Solution (5 of 12)

- Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

$$= \frac{1}{4}\left[\frac{1}{s^2}\right] - \frac{1}{8}\left[\frac{2}{s^2+4}\right]$$

and hence

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

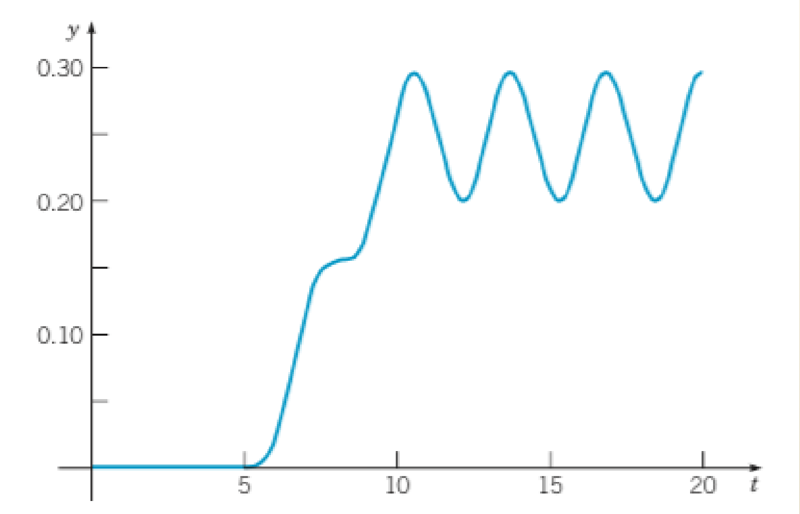
Example 2: Graph of Solution

- Thus the solution to the initial value problem is

$$\phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad \text{where}$$

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- The graph of this solution is given below.



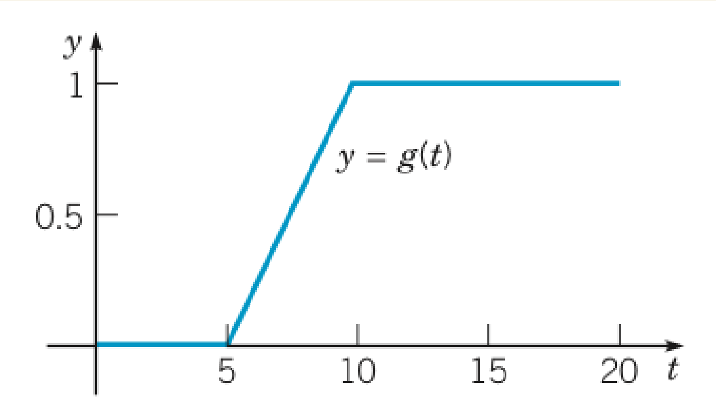
Example 2: Composite IVPs

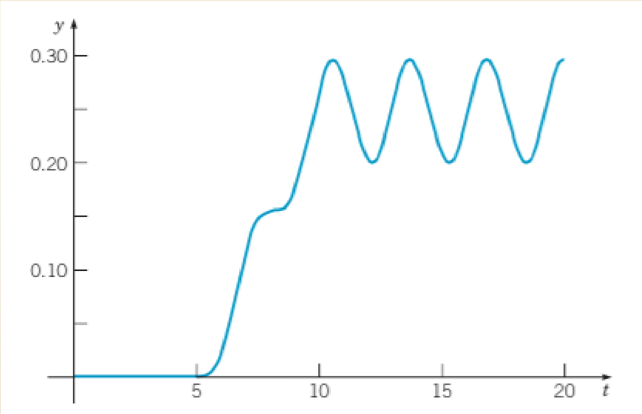
- The solution to original IVP can be viewed as a composite of three separate solutions to three separate IVPs (discuss):

$$0 \le t < 5: \quad y_1'' + 4y_1 = 0, \qquad y_1(0) = 0,\ y_1'(0) = 0$$

$$5 < t < 10: \quad y_2'' + 4y_2 = (t-5)/5, \qquad y_2(5) = 0,\ y_2'(5) = 0$$

$$t > 10: \quad y_3'' + 4y_3 = 1, \qquad y_3(10) = y_2(10),\ y_3'(10) = y_2'(10)$$



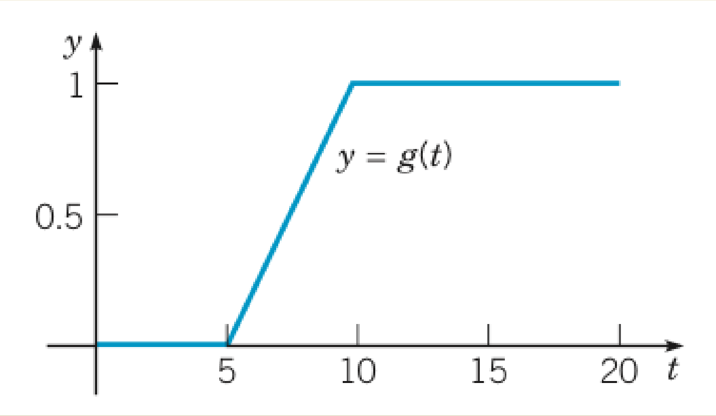


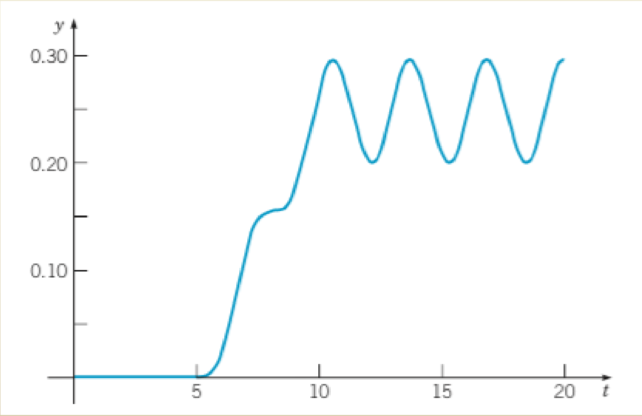
Example 2: First IVP (8 of 12)

- Consider the first initial value problem

$$y_1'' + 4y_1 = 0, \qquad y_1(0) = 0,\ y_1'(0) = 0; \quad 0 \le t < 5$$

- From a physical point of view, the system is initially at rest, and since there is no external forcing, it remains at rest.
- Thus the solution over [0, 5) is $y_1 = 0$, and this can be verified analytically as well. See graphs below.





Example 2: Second IVP (9 of 12)

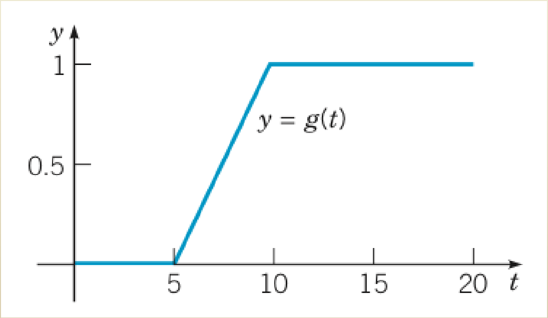
- Consider the second initial value problem

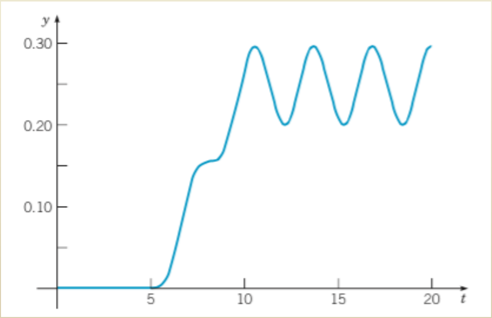
$$y_2'' + 4y_2 = (t-5)/5, \qquad y_2(5) = 0,\ y_2'(5) = 0; \quad 5 < t < 10$$

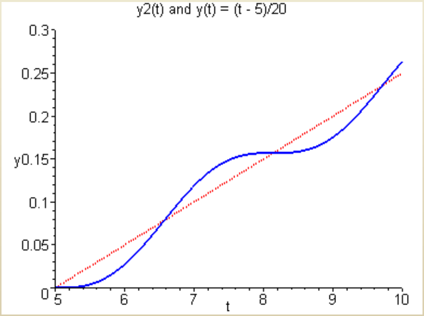
- Using methods of Chapter 3, the solution has the form

$$y_2 = c_1 \cos(2t) + c_2 \sin(2t) + t/20 - 1/4$$

- Thus the solution is an oscillation about the line (t – 5)/20, over the time interval (5, 10). See graphs below.







Example 2: Third IVP

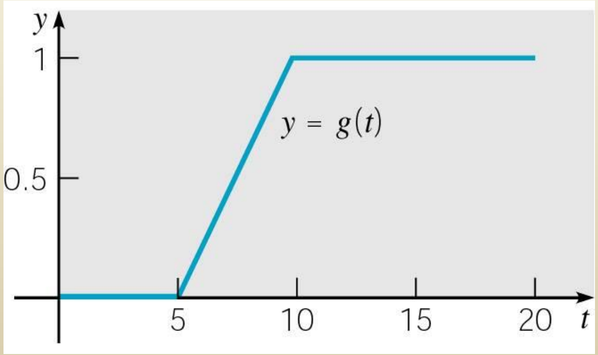
- Consider the third initial value problem

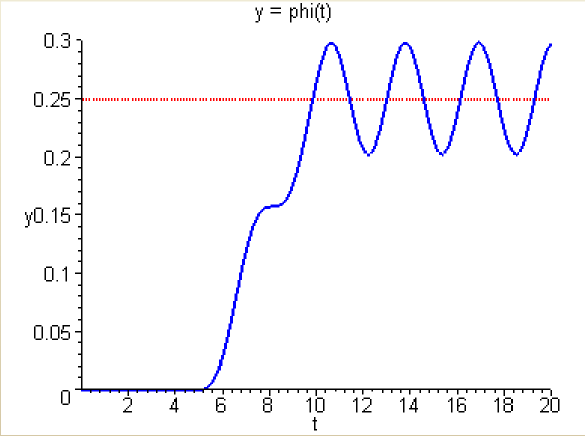
$$y_3'' + 4y_3 = 1, \quad y_3(10) = y_2(10),\ y_3'(10) = y_2'(10); \quad t > 10$$

- Using methods of Chapter 3, the solution has the form

$$y_3 = c_1 \cos(2t) + c_2 \sin(2t) + 1/4$$

- Thus the solution is an oscillation about $y = 1/4$, for $t > 10$. See graphs below.



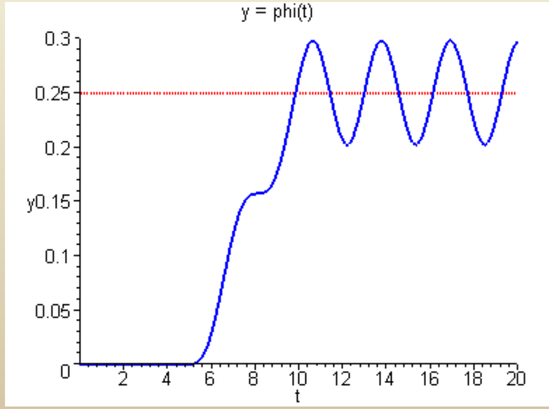


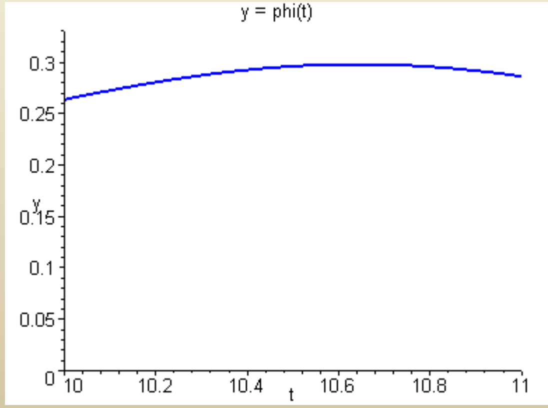
Example 2: Amplitude

- Recall that the solution to the initial value problem is

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- To find the amplitude of the eventual steady oscillation, we locate one of the maximum or minimum points for $t > 10$.
- Solving $y' = 0$, the first maximum is (10.642, 0.2979).
- Thus the amplitude of the oscillation is about 0.0479.



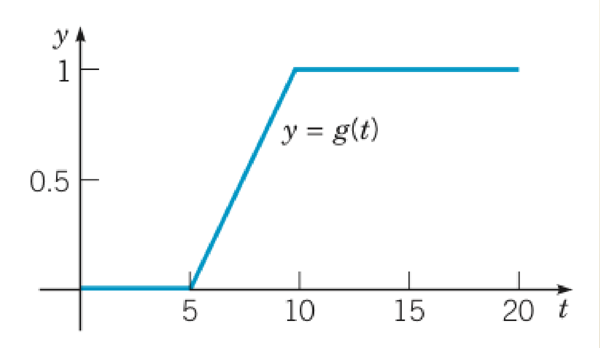


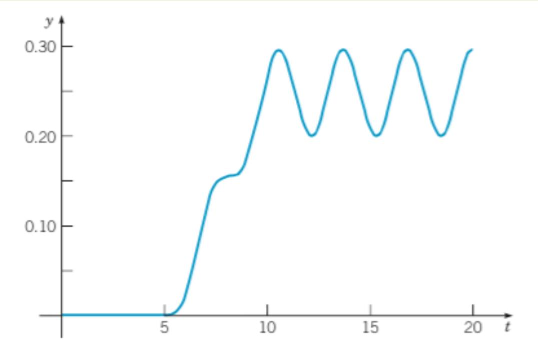
Example 2: Solution Smoothness (12 of 12)

- Our solution is

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- In this example, the forcing function g is continuous but g' is discontinuous at $t = 5$ and $t = 10$.
- It follows that f and its first two derivatives are continuous everywhere, but f''' has discontinuities at $t = 5$ and $t = 10$ that match the discontinuities of g' at $t = 5$ and $t = 10$.





Boyce/DiPrima/Meade 11th ed, Ch 6.5: Impulse Functions

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade ©2017 by John Wiley & Sons, Inc.

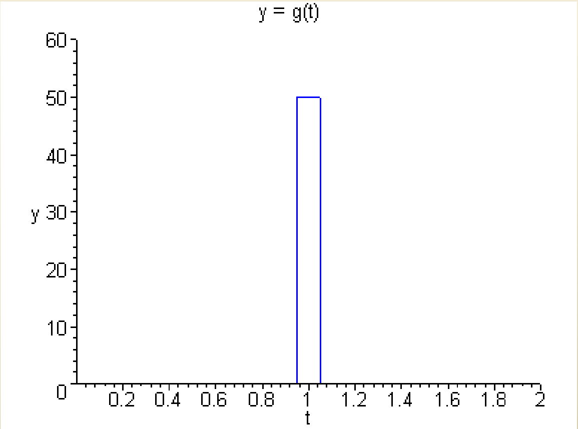
- In some applications, it is necessary to deal with phenomena of an impulsive nature.
- For example, an electrical circuit or mechanical system subject to a sudden voltage or force $g(t)$ of large magnitude that acts over a short time interval about t_0. The differential equation will then have the form

$$ay'' + by' + cy = g(t),$$

where

$$g(t) = \begin{cases} \text{big}, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

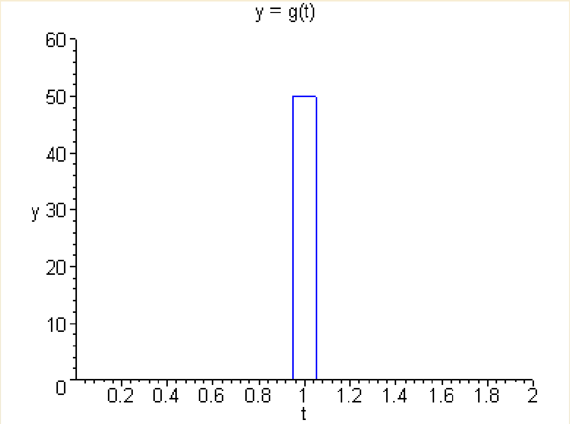
and $\tau > 0$ is small.



Measuring Impulse

- In a mechanical system, where $g(t)$ is a force, the total **impulse** of this force is measured by the integral

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$



- Note that if $g(t)$ has the form

$$g(t) = \begin{cases} c, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

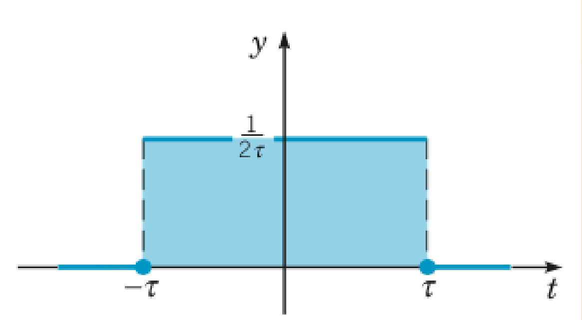
then
$$I(\tau) = \int_{-\infty}^{\infty} g(t)\,dt = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt = 2\tau c, \quad \tau > 0$$

- In particular, if $c = 1/(2\tau)$, then $I(\tau) = 1$ (independent of τ).

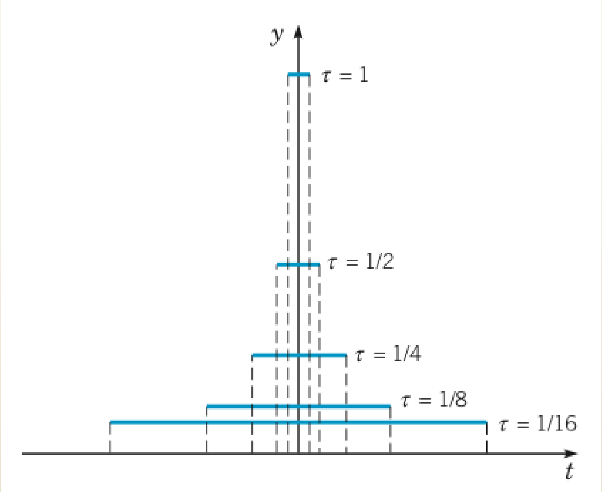
Unit Impulse Function

- Suppose the forcing function $d_t(t)$ has the form

$$d_t(t) = \begin{cases} \dfrac{1}{2t}, & -t < t < t \\ 0, & \text{otherwise} \end{cases}$$



- Then as we have seen, $I(t) = 1$.
- We are interested $d_t(t)$ acting over shorter and shorter time intervals (i.e., $t \to 0$). See graph on right.



- Note that $d_t(t)$ gets taller and narrower as $t \to 0$. Thus for $t \neq 0$, we have

$$\lim_{\tau \to 0} d_\tau(t) = 0, \text{ and } \lim_{\tau \to 0} I(\tau) = 1$$

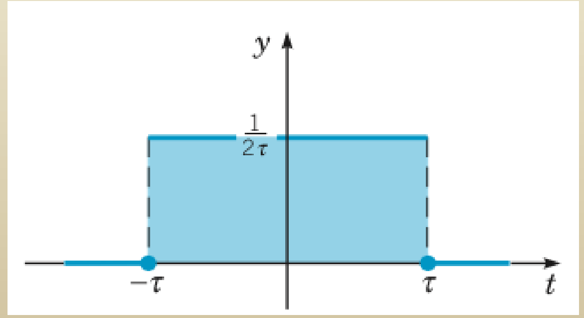
Dirac Delta Function

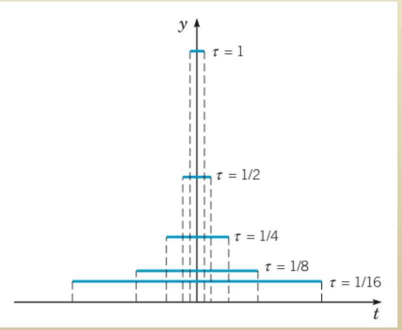
- Thus for $t \neq 0$, we have $\lim_{\tau \to 0} d_\tau(t) = 0$, and $\lim_{\tau \to 0} I(\tau) = 1$
- The **unit impulse function** δ is defined to have the properties

$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- The unit impulse function is an example of a generalized function and is usually called the **Dirac delta function**.
- In general, for a unit impulse at an arbitrary point t_0,

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \text{ and } \int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$





Laplace Transform of δ

- The Laplace Transform of δ is defined by

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} L\{d_\tau(t-t_0)\},\ \ t_0 > 0$$

and thus

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0}\int_0^\infty e^{-st} d_\tau(t-t_0)dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}dt$$

$$= \lim_{\tau\to 0}\left.\frac{-e^{-st}}{2s\tau}\right|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau\to 0}\frac{1}{2s\tau}\left[-e^{-s(t_0+\tau)} + e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0}\frac{e^{-st_0}}{s\tau}\left[\frac{e^{s\tau}-e^{-s\tau}}{2}\right] = e^{-st_0}\left[\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}\right]$$

$$= e^{-st_0}\left[\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s}\right] = e^{-st_0}$$

Laplace Transform of δ (2 of 2)

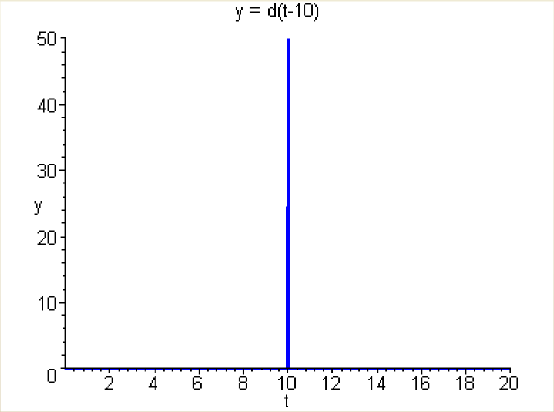
- Thus the Laplace Transform of δ is

$$L\{\delta(t-t_0)\} = e^{-st_0}, \ t_0 > 0$$

- For Laplace Transform of δ at $t_0 = 0$, take limit as follows:

$$L\{\delta(t)\} = \lim_{t_0 \to 0} L\{d_\tau(t-t_0)\} = \lim_{\tau_0 \to 0} e^{-st_0} = 1$$

- For example, when $t_0 = 10$, we have $L\{\delta(t - 10)\} = e^{-10s}$.



Product of Continuous Functions and d

- The product of the delta function and a continuous function f can be integrated, using the mean value theorem for integrals:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt &= \lim_{\tau\to 0} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)dt \\
&= \lim_{\tau\to 0} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)dt \\
&= \lim_{\tau\to 0} \frac{1}{2\tau} \left[2\tau\, f(t^*)\right] \quad (\text{where } t_0-\tau < t^* < t_0+\tau) \\
&= \lim_{\tau\to 0} f(t^*) \\
&= f(t_0)
\end{aligned}
$$

- Thus

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt = f(t_0)$$

Example 1: Initial Value Problem (1 of 3)

- Consider the solution to the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \quad y(0) = 0, \; y'(0) = 0$$

- Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-5)\}$$

- Letting $Y(s) = L\{y\}$,

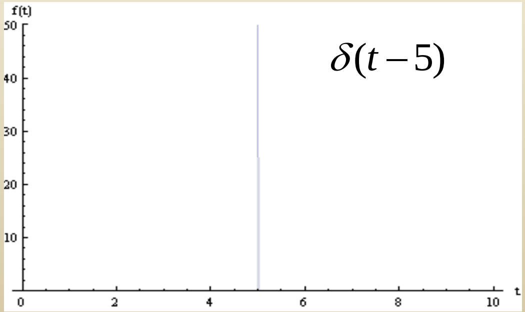
$$\left[2s^2Y(s) - 2sy(0) - 2y'(0)\right] + \left[sY(s) - y(0)\right] + 2Y(s) = e^{-5s}$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = e^{-5s}$$

or

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$



Example 1: Solution (2 of 3)

- We have

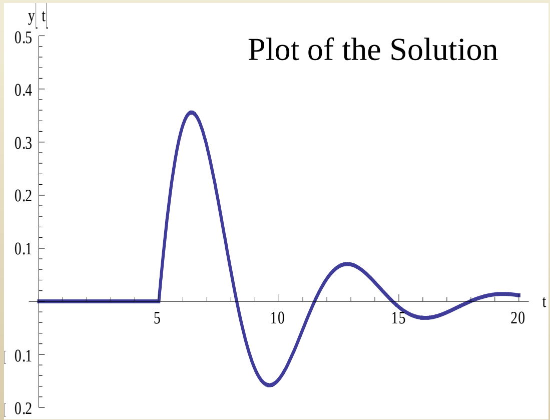
$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

- The partial fraction expansion of $Y(s)$ yields

$$Y(s) = \frac{e^{-5s}}{2\sqrt{15}} \left[\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16} \right]$$

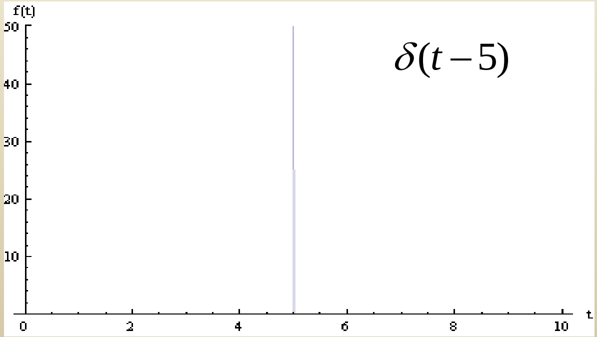
and hence

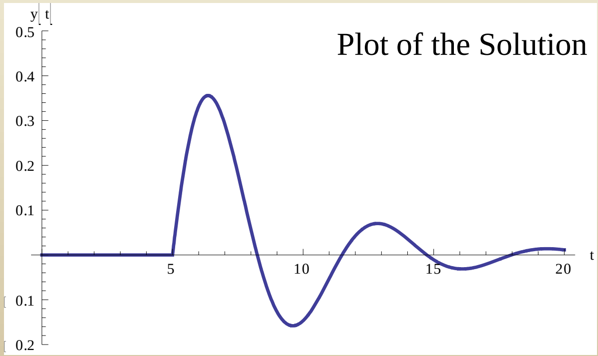
$$y(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin\left(\frac{\sqrt{15}}{4}(t-5) \right)$$



Example 1: Solution Behavior (3 of 3)

- With homogeneous initial conditions at $t = 0$ and no external excitation until $t = 5$, there is no response on (0, 5).
- The impulse at $t = 5$ produces a decaying oscillation that persists indefinitely.
- Response is continuous at $t = 5$ despite singularity in forcing function. Since y' has a jump discontinuity at $t = 5$, y'' has an infinite discontinuity there. Thus a singularity in the forcing function is balanced by a corresponding singularity in y''.





Boyce/DiPrima/Meade 11th ed, Ch 6.6: The Convolution Integral

- Sometimes it is possible to write a Laplace transform $H(s)$ as $H(s) = F(s)G(s)$, where $F(s)$ and $G(s)$ are the transforms of known functions f and g, respectively.
- In this case we might expect $H(s)$ to be the transform of the product of f and g. That is, does

$$H(s) = F(s)G(s) = L\{f\}L\{g\} = L\{f\,g\}?$$

- On the next slide we give an example that shows that this equality does not hold, and hence the Laplace transform cannot in general be commuted with ordinary multiplication.
- In this section we examine the **convolution** of f and g, which can be viewed as a generalized product, and one for which the Laplace transform does commute.

Observation

- Let $f(t) = 1$ and $g(t) = \sin(t)$. Recall that the Laplace Transforms of f and g are

$$L\{f(t)\} = L\{1\} = \frac{1}{s}, \quad L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

- Thus

$$L\{f(t)g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

and

$$L\{f(t)\}L\{g(t)\} = \frac{1}{s(s^2+1)}$$

- Therefore for these functions it follows that

$$L\{f(t)g(t)\} \neq L\{f(t)\}L\{g(t)\}$$

Theorem 6.6.1

- Suppose $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$. Then $H(s) = F(s)G(s) = L\{h(t)\}$ for $s > a$, where

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(t)g(t-\tau)d\tau$$

- The function $h(t)$ is known as the **convolution** of f and g and the integrals above are known as **convolution integrals**.

- Note that the equality of the two convolution integrals can be seen by making the substitution $u = t - \chi$.
- The convolution integral defines a "generalized product" and can be written as $h(t) = (f * g)(t)$. See text for more details.

Theorem 6.6.1 Proof Outline

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)du \int_0^\infty e^{-s\tau} g(\tau)d\tau$$

$$= \int_0^\infty g(\tau)d\tau \int_0^\infty e^{-s(\tau+u)} f(u)du$$

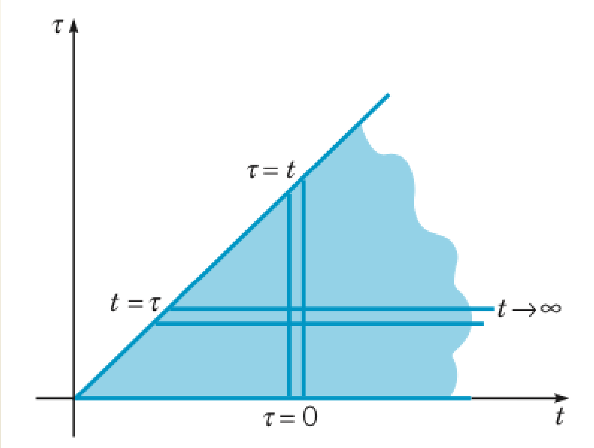
$$= \int_0^\infty g(\tau)d\tau \int_\tau^\infty e^{-st} f(t-\tau)dt \qquad (t = \tau + u)$$

$$= \int_0^\infty \int_\tau^\infty e^{-st} g(\tau) f(t-\tau)dtd\tau$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)d\tau\, dt$$

$$= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt$$

$$= L\{h(t)\}$$



Example 1: Find Inverse Transform (1 of 2)

- Find the inverse Laplace Transform of $H(s)$, given below.

$$H(s) = \frac{a}{s^2(s^2 + a^2)}$$

- Solution: Let $F(s) = 1/s^2$ and $G(s) = a/(s^2 + a^2)$, with

$$f(t) = L^{-1}\{F(s)\} = t$$
$$g(t) = L^{-1}\{G(s)\} = \sin(at)$$

- Thus by Theorem 6.6.1,

$$L^{-1}\{H(s)\} = h(t) = \int_0^t (t-\tau)\sin(a\tau)\,d\tau$$

$$L^{-1}\left\{H(s)\right\} = h(t) = \int_0^t (t-\tau)\sin(a\tau)d\tau$$

Example 1: Solution *h*(*t*) (2 of 2)

- We can integrate to simplify *h*(*t*), as follows.

$$h(t) = \int_0^t (t-\tau)\sin(a\tau)d\tau \;= t\int_0^t \sin(a\tau)d\tau - \int_0^t \tau\sin(a\tau)d\tau$$

$$= -\frac{1}{a}t\cos(a\tau)\Big|_0^t - \left[-\frac{1}{a}\tau\cos(a\tau)\Big|_0^t + \frac{1}{a}\int_0^t \cos(a\tau)d\tau\right]$$

$$= -\frac{1}{a}t\left[\cos(at)-1\right] - \left[-\frac{1}{a}t\left[\cos(at)\right] + \frac{1}{a^2}\left[\sin(at)\right]\right]$$

$$= \frac{1}{a}t - \frac{1}{a^2}\sin(at)$$

$$= \frac{at - \sin(at)}{a^2}$$

Example 2: Initial Value Problem (1 of 4)

- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = -1$$

- Solution:

$$L\{y''\} + 4L\{y\} = L\{g(t)\}$$

- or

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = G(s)$$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2 + 4\right)Y(s) = 3s - 1 + G(s)$$

- Thus

$$Y(s) = \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4}$$

Example 2: Solution (2 of 4)

- We have

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$

$$= 3\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\left[\frac{2}{s^2+4}\right]G(s)$$

- Thus

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)d\tau$$

- Note that if $g(t)$ is given, then the convolution integral can be evaluated.

$$y'' + 4y = g(t), \quad y(0) = 3,\ y'(0) = -1$$

Example 2: Laplace Transform of Solution (3 of 4)

- Recall that the Laplace Transform of the solution *y* is

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} = \Phi(s) + \Psi(s)$$

- Note $F(s)$ depends only on system coefficients and initial conditions, while $Y(s)$ depends only on system coefficients and forcing function $g(t)$.
- Further, $f(t) = L^{-1}[F(s)]$ solves the homogeneous IVP

$$y'' + 4y = 0, \quad y(0) = 3,\ y'(0) = -1$$

while $y(t) = L^{-1}\{Y(s)\}$ solves the nonhomogeneous IVP

$$y'' + 4y = g(t), \quad y(0) = 0,\ y'(0) = 0$$

Example 2: Transfer Function (4 of 4)

- Examining $Y(s)$ more closely,

$$\Psi(s) = \frac{G(s)}{s^2+4} = H(s)G(s), \text{ where } H(s) = \frac{1}{s^2+4}$$

- The function $H(s)$ is known as the **transfer function**, and depends only on system coefficients.
- The function $G(s)$ depends only on external excitation $g(t)$ applied to system.
- If $G(s) = 1$, then $g(t) = \delta(t)$ and hence $h(t) = L^{-1}\{H(s)\}$ solves the nonhomogeneous initial value problem

$$y'' + 4y = \delta(t), \quad y(0) = 0, \ y'(0) = 0$$

- Thus $h(t)$ is response of system to unit impulse applied at $t = 0$, and hence $h(t)$ is called the **impulse response** of system.