Boyce/DiPrima/Meade 11th ed, Ch 5.1: Review of Power Series

- Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation.
- So far, we have a systematic procedure for constructing fundamental solutions if equation has constant coefficients.
- For a larger class of equations with variable coefficients, we must search for solutions beyond the familiar elementary functions of calculus.
- The principal tool we need is the representation of a given function by a power series.
- Then, similar to the undetermined coefficients method, we assume the solutions have power series representations, and then determine the coefficients so as to satisfy the equation.

Convergent Power Series

- A **power series** about the point x_0 has the form

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$$

and is said to **converge** at a point x if

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n \left(x - x_0\right)^n$$

exists for that x.

- Note that the series converges for $x = x_0$. It may converge for all x, or it may converge for some values of x and not others.

Absolute Convergence

- A power series about the point x_0

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$$

is said to **converge absolutely** at a point x if the series

$$\sum_{n=0}^{\infty} \left|a_n \left(x - x_0\right)^n\right| = \sum_{n=0}^{\infty} \left|a_n\right| \left|x - x_0\right|^n$$

converges.

- If a series converges absolutely, then the series also converges. The converse, however, is not necessarily true.

Ratio Test

- One of the most useful tests for the absolute convergence of a power series

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$$

is the ratio test. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$

then the power series converges absolutely at that value of x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. The test is inconclusive if $|x - x_0|L = 1$.

Example 1

- Find which values of x does power series below converge.

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-2)^n$$

- Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{n+1}{n} = |x-2| < 1, \text{ for } 1 < x < 3$$

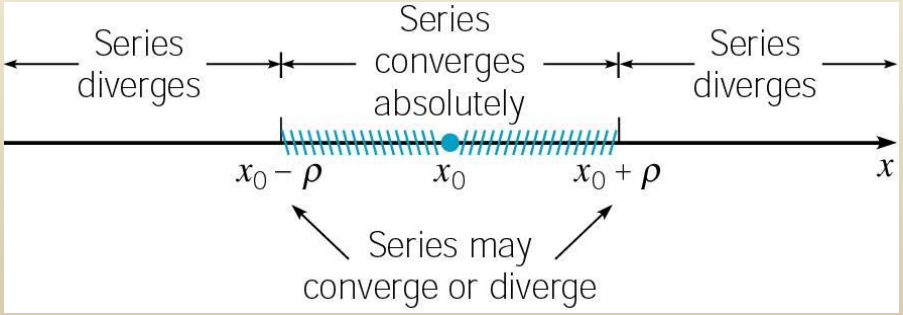
- At $x = 1$ and $x = 3$, the corresponding series are, respectively,

$$\sum_{n=1}^{\infty}(1-2)^n = \sum_{n=1}^{\infty}(-1)^n, \qquad \sum_{n=1}^{\infty}(3-2)^n = \sum_{n=1}^{\infty}(1)^n$$

- Both series diverge, since the nth terms do not approach zero.
- Therefore the interval of convergence is (1, 3).

Radius of Convergence

- There is a nonnegative number r, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely for all x satisfying $|x - x_0| < r$ and diverges for $|x - x_0| > r$.
- For a series that converges only at x_0, we define r to be zero.
- For a series that converges for all x, we say that r is infinite.
- If $r > 0$, then $|x - x_0| < r$ is called the **interval of convergence**.
- The series may either converge or diverge when $|x - x_0| = r$.



Example 2

- Find the radius of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$$

- Using the ratio test, we obtain

$$\lim_{n\to\infty}\left|\frac{(n)\,2^n(x+1)^{n+1}}{(n+1)2^{n+1}(x+1)^n}\right| = \frac{|x+1|}{2}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x+1|}{2} < 1, \text{ for } -3 < x < 1$$

- At $x = -3$ and $x = 1$, the corresponding series are, respectively,

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty}\frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

- The alternating series on the left is convergent but not absolutely convergent. The series on the right, called the harmonic series is divergent. Therefore the interval of convergence is [–3, 1), and hence the radius of convergence is $r = 2$.

Taylor Series

- Suppose that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges to $f(x)$ for $|x - x_0| < r$.
- Then the value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and the series is called the **Taylor series** for f about $x = x_0$.

- Also, if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then f is continuous and has derivatives of all orders on the interval of convergence. Further, the derivatives of f can be computed by differentiating the relevant series term by term.

Analytic Functions

- A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

  with a radius of convergence $r > 0$, is said to be **analytic** at x_0.
- All of the familiar functions of calculus are analytic.
- For example, $\sin x$ and e^x are analytic everywhere, while $1/x$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\pi/2$.
- If f and g are analytic at x_0, then so are $f \pm g$, fg, and f/g; see text for details on these arithmetic combinations of series.

Series Equality

- If two power series are equal, that is,

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n = \sum_{n=1}^{\infty} b_n (x - x_0)^n$$

for each x in some open interval with center x_0, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$

- In particular, if

$$\sum_{n=1}^{\infty} a_n (x - x_0)^n = 0$$

then $a_n = 0$ for $n = 0, 1, 2, 3, \ldots$

Shifting Index of Summation

- The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable.
- Thus it is immaterial which letter is used for the index of summation:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

- Just as we make changes in the variable of integration in a definite integral, we find it convenient to make changes of summation in calculating series solutions of differential equations.

Example 3: Shifting Index of Summation

- We are asked to rewrite the series below as one starting with the index $n = 0$.

$$\sum_{n=2}^{\infty} a_n (x)^n$$

By letting $m = n - 2$ in this series. $n = 2$ corresponds to $m = 0$, and hence

$$\sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2}$$

- Replacing the dummy index m with n, we obtain

$$\sum_{n=2}^{\infty} a_n (x)^n = \sum_{n=0}^{\infty} a_{n+2} (x)^{n+2}$$

as desired.

Example 4: Rewriting Generic Term

- We can write the following series

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2}$$

as a sum whose generic term involves $(x-x_0)^n$ by letting $m = n - 2$. Then $n = 2$ corresponds to $m = 0$.

- It follows that

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_n(x-x_0)^{n-2} = \sum_{m=0}^{\infty}(m+4)(m+3)a_{m+2}(x-x_0)^m$$

- Replacing the dummy index m with n, we obtain

$$\sum_{n=0}^{\infty}(n+4)(n+3)a_{n+2}(x-x_0)^n$$

as desired.

Example 5: Rewriting Generic Term

- We can write the following series

$$x^2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

as a series whose generic term involves x^{r+n}

- Begin by taking x^2 inside the summation and letting $m = n+1$

$$x^2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n+1} = \sum_{m=1}^{\infty} (r+m-1) a_{m-1} x^{r+m}$$

- Replacing the dummy index m with n, we obtain the desired result:

$$\sum_{n=1}^{\infty} (r+n-1) a_{n-1} x^{r+n}$$

Example 6: Determining Coefficients (1 of 2)

- Assume that $$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$
- Determine what this implies about the coefficients.
- Begin by writing both series with the same powers of x. As before, for the series on the left, let $m = n - 1$, then replace m by as we have been doing. The above equality becomes:

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n \Rightarrow (n+1) a_{n+1} = a_n \Rightarrow a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, 3, \ldots$

Example 6: Determining Coefficients (2 of 2)

- Using the recurrence relationship just derived:

$$a_{n+1} = \frac{a_n}{n+1}$$

- we can solve for the coefficients successively by letting $n = 0, 1, 2, \ldots$

$$a_1 = a_0, a_2 = \frac{a_1}{2} = \frac{a_0}{2}, a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots, a_n = \frac{a_0}{n!}$$

- Using these coefficients in the original series, we get a recognizable Taylor series:

$$a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

Boyce/DiPrima/Meade 11th ed, Ch 5.2: Series Solutions Near an Ordinary Point, Part I

- In Chapter 3, we examined methods of solving second order linear differential equations with constant coefficients.
- We now consider the case where the coefficients are functions of the independent variable, which we will denote by x.
- It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

  since the method for the nonhomogeneous case is similar.
- We primarily consider the case when P, Q, R are polynomials, and hence also continuous.
- However, as we will see, the method of solution is also applicable when P, Q and R are general analytic functions.

Ordinary Points

- Assume P, Q, R are polynomials with no common factors, and that we want to solve the equation below in a neighborhood of a point of interest x_0:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

- The point x_0 is called an **ordinary point** if $P(x_0) \neq 0$. Since P is continuous, $P(x) \neq 0$ for all x in some interval about x_0. For x in this interval, divide the differential equation by P to get

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- Since p and q are continuous, Theorem 3.2.1 says there is a unique solution, given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$

Singular Points

- Suppose we want to solve the equation below in some neighborhood of a point of interest x_0:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \text{ where } p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

- The point x_0 is called an **singular point** if $P(x_0) = 0$.
- Since P, Q, R are polynomials with no common factors, it follows that $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, or both.
- Then at least one of p or q becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this situation.
- Sections 5.4 through 5.8 deal with finding solutions in the neighborhood of a singular point.

Series Solutions Near Ordinary Points

- In order to solve our equation near an ordinary point x_0,

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

we will assume a series representation of the unknown solution function y:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- As long as we are within the interval of convergence, this representation of y is continuous and has derivatives of all orders.

Example 1: Series Solution (1 of 8)

- Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = 1$. Thus every point x is an ordinary point. We will take $x_0 = 0$.
- Assume a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

- Differentiate term by term to obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Example 1: Combining Series (2 of 8)

- Our equation is

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

- Shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

Example 1: Recurrence Relation (3 of 8)

- Our equation is

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n=0$$

- For this equation to be valid for all x, the coefficient of each power of x must be zero, and hence

$$(n+2)(n+1)a_{n+2}+a_n=0,\quad n=0,1,2,\ldots$$

or

$$a_{n+2}=\frac{-a_n}{(n+2)(n+1)},\quad n=0,1,2,\ldots$$

- This type of equation is called a **recurrence relation**.
- Next, we find the individual coefficients a_0, a_1, a_2, …

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

Example 1: Even Coefficients (4 of 8)

- To find a_2, a_4, a_6, …., we proceed as follows:

$$a_2 = -\frac{a_0}{2 \cdot 1},$$

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1},$$

$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},$$

$$\vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \quad k = 1, 2, 3, \ldots$$

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

Example: Odd Coefficients (5 of 8)

- To find a_3, a_5, a_7, …., we proceed as follows:

$$a_3 = -\frac{a_1}{3 \cdot 2},$$

$$a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},$$

$$a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},$$

$$\vdots$$

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}, \quad k = 1, 2, 3, \ldots$$

Example 1: Solution (6 of 8)

- We now have the following information:

$$y(x)=\sum_{n=0}^{\infty}a_n x^n,\ \text{where } a_{2k}=\frac{(-1)^k}{(2k)!}a_0,\quad a_{2k+1}=\frac{(-1)^k}{(2k+1)!}a_1$$

- Thus

$$y(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

- Note: a_0 and a_1 are determined by the initial conditions. (Expand series a few terms to see this.)
- Also, by the ratio test it can be shown that these two series converge absolutely on (-¥,¥), and hence the manipulations we performed on the series at each step are valid.

Example 1: Functions Defined by IVP (7 of 8)

- Our solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- From Calculus, we know this solution is equivalent to

$$y(x) = a_0 \cos x + a_1 \sin x$$

- In hindsight, we see that cos *x* and sin *x* are indeed fundamental solutions to our original differential equation

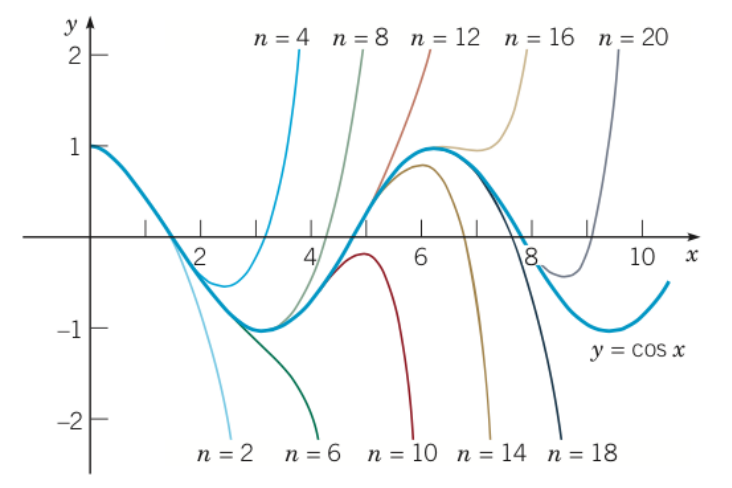
$$y'' + y = 0, \quad -\infty < x < \infty$$

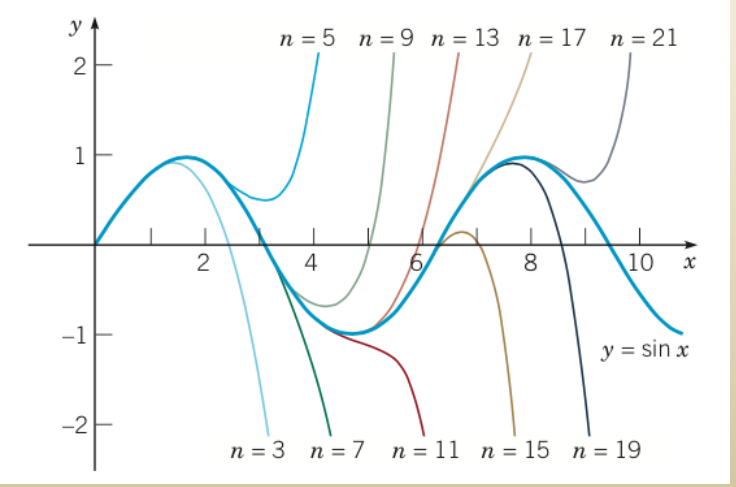
- While we are familiar with the properties of cos *x* and sin *x*, many important functions are defined by the initial value problem that they solve.

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Example 1: Graphs (8 of 8)

- The graphs below show the partial sum approximations of $\cos x$ and $\sin x$.
- As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each x in this interval the accuracy improves.
- However, the truncated power series provides only a local approximation in the neighborhood of $x = 0$.





Example 2: Airy's Equation (1 of 10)

- Find a series solution of Airy's equation about $x_0 = 0$:

$$y'' - xy = 0, \; -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$. Thus every point x is an ordinary point. We will take $x_0 = 0$.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \;\; y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \;\; y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substituting these expressions into the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Example 2: Combine Series (2 of 10)

- Our equation is

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

- Shifting the indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

or

$$2 \cdot 1 \cdot a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1} \right] x^n = 0$$

Example 2: Recurrence Relation (3 of 10)

- Our equation is

$$2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n = 0$$

- For this equation to be valid for all x, the coefficient of each power of x must be zero; hence $a_2 = 0$ and

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

or

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots$$

Example 2: Coefficients (4 of 10)

- We have $a_2 = 0$ and

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \quad n = 0, 1, 2, \ldots$$

- For this recurrence relation, note that $a_2 = a_5 = a_8 = \ldots = 0$.
- Next, we find the coefficients a_0, a_3, a_6, ….
- We do this by finding a formula a_{3n}, $n = 1, 2, 3, \ldots$
- After that, we find a_1, a_4, a_7, …, by finding a formula for a_{3n+1}, $n = 1, 2, 3, \ldots$

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}$$

Example 2: Find a_{3n} (5 of 10)

- Find a_3, a_6, a_9, ….

$$a_3 = \frac{a_0}{2 \cdot 3}, \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \quad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \cdots$$

- The general formula for this sequence is

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)(3n-1)(3n)}, \quad n \geq 4$$

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}$$

Example 2: Find a_{3n+1} (6 of 10)

- Find $a_4, a_7, a_{10}, \ldots$

$$a_4 = \frac{a_1}{3 \cdot 4}, \quad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \quad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, \ldots$$

- The general formula for this sequence is

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3)(3n-2)(3n)(3n+1)}, \quad n \geq 4$$

Example 2: Series and Coefficients (7 of 10)

- We now have the following information:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=3}^{\infty} a_n x^n$$

where a_0, a_1 are arbitrary, and

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4)(3n-3)(3n-1)(3n)}, \quad n \geq 4$$

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3)(3n-2)(3n)(3n+1)}, \quad n \geq 4$$

Example 2: Solution (8 of 10)

- Thus our solution is

$$y(x) = a_0\left[1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right]+a_1\left[x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

  where a_0, a_1 are arbitrary (determined by initial conditions).

- Consider the two cases

  (1) $a_0 = 1$, $a_1 = 0$ and $y(0) = 1, y'(0) = 0$

  (2) $a_0 = 0$, $a_1 = 1$ and $y(0) = 0, y'(0) = 1$

- The corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, since $W(y_1, y_2)(0) = 1 \neq 0$, where

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = y_1(0)y_2'(0) - y_1'(0)y_2(0)$$

Example 2: Fundamental Solutions (9 of 10)

- Our solution:

$$y(x) = a_0\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)}\right] + a_1\left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)}\right]$$

- For the cases

(1) $a_0 = 1$, $a_1 = 0$ and $y(0) = 1, y'(0) = 0$

(2) $a_0 = 0$, $a_1 = 1$ and $y(0) = 0, y'(0) = 1$,

the corresponding solutions $y_1(x)$, $y_2(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = c_1 y_1(x) + c_1 y_2(x)$$

Example 2: Graphs (10 of 10)

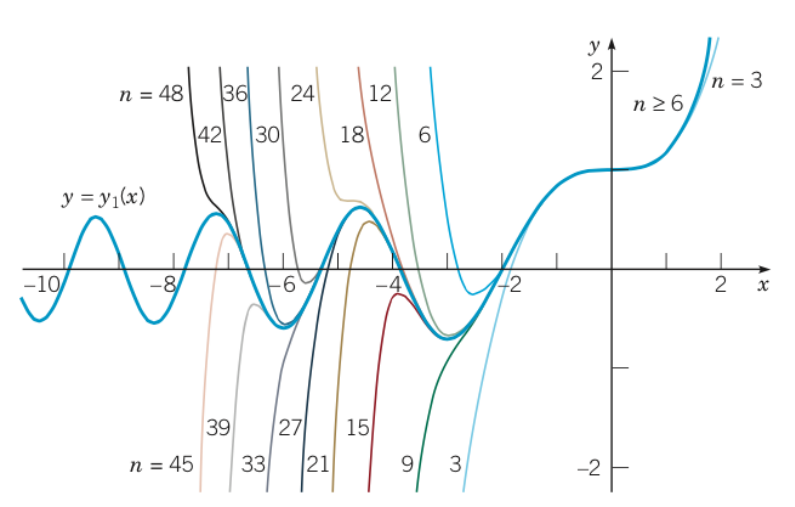
- Thus given the initial conditions

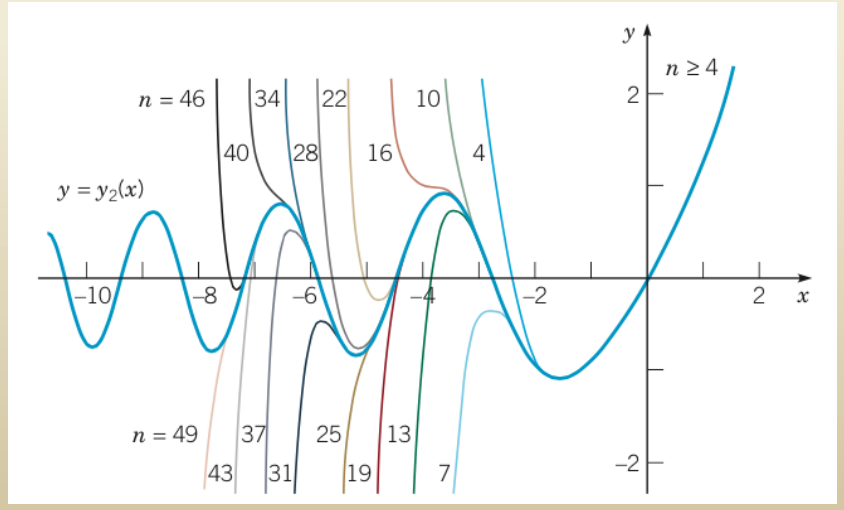
$$y(0) = 1,\ y'(0) = 0 \quad \text{and} \quad y(0) = 0,\ y'(0) = 1$$

the solutions are, respectively,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)},\ y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}$$

- The graphs of y_1 and y_2 are given below. Note the approximate intervals of accuracy for each partial sum





Example 3: Airy's Equation (1 of 7)

- Find a series solution of Airy's equation in powers of $x-1$ (i.e. about $x_0 = 1$):

$$y'' - xy = 0, \; -\infty < x < \infty$$

- Here, $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$. Thus every point x is an ordinary point. We will take $x_0 = 1$.
- Assuming a series solution and differentiating, we obtain

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n, \;\; y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \;\; y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

- Substituting these into ODE & shifting indices, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n$$

Example 3: Rewriting Series Equation (2 of 7)

- Our equation is

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = x\sum_{n=0}^{\infty}a_n(x-1)^n$$

- The x on right side can be written as $1 + (x - 1)$; and thus

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = [1+(x-1)]\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1}$$

$$= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n$$

Example 3: Recurrence Relation (3 of 7)

- Thus our equation becomes

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = a_0 + \sum_{n=1}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n$$

- Thus the recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \geq 1)$$

- Equating like powers of $x - 1$, we obtain

$$2a_2 = a_0 \qquad \Rightarrow a_2 = \frac{a_0}{2},$$

$$(3\cdot 2)a_3 = a_1 + a_0 \quad \Rightarrow a_3 = \frac{a_0}{6} + \frac{a_1}{6},$$

$$(4\cdot 3)a_4 = a_2 + a_1 \quad \Rightarrow a_4 = \frac{a_0}{24} + \frac{a_1}{12},$$

$$\vdots$$

Example 3: Solution (4 of 7)

- We now have the following information:

$$y(x)=\sum_{n=0}^{\infty}a_n(x-1)^n$$

and

$$y(x)=a_0\left[1+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}+\cdots\right]$$
$$+a_1\left[(x-1)+\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}+\cdots\right]$$

$$a_0=\text{arbitrary}$$
$$a_1=\text{arbitrary}$$
$$a_2=\frac{a_0}{2},$$
$$a_3=\frac{a_0}{6}+\frac{a_1}{6},$$
$$a_4=\frac{a_0}{24}+\frac{a_1}{12},$$
$$\vdots$$

Example 3: Solution and Recursion (5 of 7)

- Our solution:

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right] + a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

$$a_0 = \text{arbitrary}$$
$$a_1 = \text{arbitrary}$$
$$a_2 = \frac{a_0}{2},$$
$$a_3 = \frac{a_0}{6} + \frac{a_1}{6},$$
$$a_4 = \frac{a_0}{24} + \frac{a_1}{12},$$
$$\vdots$$

- The recursion has three terms,

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1},\ (n \geq 1)$$

and determining a general formula for the coefficients a_n can be difficult or impossible.

- However, we can generate as many coefficients as we like, preferably with the help of a computer algebra system.

Example 3: Solution and Convergence (6 of 7)

- Our solution:

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right]$$
$$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

- Since we don't have a general formula for the a_n, we cannot use a convergence test (i.e., ratio test) on our power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

- This means our manipulations of the power series to arrive at our solution are suspect. However, the results of Section 5.3 will confirm the convergence of our solution.

Example 3: Fundamental Solutions (7 of 7)

- Our solution:

$$y(x) = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \cdots\right]$$
$$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \cdots\right]$$

or

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

- It can be shown that the solutions $y_3(x)$, $y_4(x)$ are linearly independent, and thus are fundamental solutions for Airy's equation, with general solution

$$y(x) = a_0 y_3(x) + a_1 y_4(x)$$

Boyce/DiPrima/Meade 11th ed, Ch 5.3: Series Solutions Near an Ordinary Point, Part II

- A function p is **analytic** at x_0 if it has a Taylor series expansion that converges to p in some interval about x_0

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$

- The point x_0 is an **ordinary point** of the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

  if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0. Otherwise x_0 is a **singular point**.

- If x_0 is an ordinary point, then p and q are analytic and have derivatives of all orders at x_0, and this enables us to solve for a_n in the solution expansion $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. See text.

Theorem 5.3.1

- If x_0 is an ordinary point of the differential equation

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=0$$

then the general solution for this equation is

$$y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0y_1(x)+a_1y_2(x)$$

where a_0 and a_1 are arbitrary, and y_1, y_2 are linearly independent series solutions that are analytic at x_0.

- Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Radius of Convergence

- Thus if x_0 is an ordinary point of the differential equation, then there exists a series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.
- Further, the radius of convergence of the series solution is at least as large as the minimum of the radii of convergence of the series for p and q.
- These radii of convergence can be found in two ways:

1. Find the series for p and q, and then determine their radii of convergence using a convergence test.
2. If P, Q and R are polynomials with no common factors, then it can be shown that Q/P and R/P are analytic at x_0 if $P(x_0) \neq 0$, and the radius of convergence of the power series for Q/P and R/P about x_0 is the distance to the nearest zero of P (including complex zeros).

Example 1 (1 of 2)

- Let $y = f(x)$ be a solution of the initial value problem:

$$(1+x^2)y''+2xy'+4x^2y=0,\ y'(0)=1$$

- Determine $f''(0),\ f'''(0),$ and $f^{(4)}(0)$

- To find $f''(0)$, evaluate the equation when $x = 0$:

$$(1+0^2)y''+2(0)y'+4(0)^2y=0$$

so $f''(0)=0$

Example 1 (2 of 2)

- To find $f'''(0)$, differentiate the equation with respect to x:

$$(1+x^2)f'''(x)+2xf''(x)+2xf''(x)+2f'(x)+4x^2f'(x)+8xf(x)=0$$

- Then evaluate at $x = 0$:

$$f'''(0)+2f'(0)=0$$

- Thus $f'''(0)=-2f'(0)=0$
- Differentiating the equation above with respect to x:

$$(1+x^2)f^{(4)}(x)+2xf'''(x)+4xf'''(x)+4f''(x)+(4x^2+2)f''(x)$$
$$+8xf'(x)+8xf'(x)+8f(x)=0$$

- And evaluating using $f(0)=1,\ f'(0)=f''(0)=f'''(0)=0$ gives us $f^{(4)}(0)=-8$.

Example 2

- Let $f(x) = (1 + x^2)^{-1}$. Find the radius of convergence of the Taylor series of f about $x_0 = 0$.
- The Taylor series of f about $x_0 = 0$ is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

- Using the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} x^2 < 1, \text{ for } |x| < 1$$

- Thus the radius of convergence is $r = 1$.
- Alternatively, note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to i or $-i$ is 1, we see again that $r = 1$.

Example 3

- Find the radius of convergence of the Taylor series for $(x^2 - 2x + 1)^{-1}$ about $x_0 = 0$ and about $x_0 = 1$. First observe:

$$(x^2 - 2x + 1) = 0 \Rightarrow x = 1 \pm i$$

- Since the denominator cannot be zero, this establishes the bounds over which the function can be defined.
- In the complex plane, the distance from $x_0 = 0$ to $1 \pm i$ is $\sqrt{2}$, so the radius of convergence for the Taylor series expansion about $x_0 = 0$ is $r = \sqrt{2}$.
- In the complex plane, the distance from $x_0 = 1$ to $1 \pm i$ is 1, so the radius of convergence for the Taylor series expansion about $x_0 = 0$ is $r = 1$.

Example 4: Legendre Equation (1 of 2)

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the Legendre equation

  $$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad \alpha \text{ a constant.}$$

- Here, $P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$.
- Thus $x_0 = 0$ is an ordinary point, since $p(x) = -2x/(1 - x^2)$ and $q(x) = \alpha(\alpha+1)/(1 - x^2)$ are analytic at $x_0 = 0$.
- Also, p and q have singular points at $x = \pm 1$.
- Thus the radius of convergence for the Taylor series expansions of p and q about $x_0 = 0$ is $r = 1$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is at least $r = 1$.

Example 4: Legendre Equation (2 of 2)

- Thus, for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

the radius of convergence for the series solution about $x_0 = 0$ is at least $r = 1$.

- It can be shown that if r is a positive integer, then one of the series solutions terminates after a finite number of terms, and hence converges for all x, not just for $|x| < 1$.

Example 5: Radius of Convergence (1 of 2)

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the equation

$$(1+x^2)y'' + 2xy' + 4x^2 y = 0$$

- Here, $P(x) = 1 + x^2$, $Q(x) = 2x$, $R(x) = 4x^2$.
- Thus $x_0 = 0$ is an ordinary point, since $p(x) = 2x/(1 + x^2)$ and $q(x) = 4x^2/(1 + x^2)$ are analytic at $x_0 = 0$.
- Also, p and q have singular points at $x = \pm i$.
- Thus the radius of convergence for the Taylor series expansions of p and q about $x_0 = 0$ is $r = 1$.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is at least $r = 1$.

Example 5: Solution Theory (2 of 2)

- Thus for the equation

$$(1+x^2)y'' + 2xy' + 4x^2y = 0,$$

the radius of convergence for the series solution about $x_0 = 0$ is at least $r = 1$, by Theorem 5.3.1.

- Suppose that initial conditions $y(0) = y_0$ and $y(0) = y_0'$ are given. Since $1 + x^2 \neq 0$ for all x, there exists a unique solution of the initial value problem on $(-\infty, \infty)$, by Theorem 3.2.1.
- On the other hand, Theorem 5.3.1 only guarantees a solution of the form $\sum_{n=0}^{\infty} a_n x^n$ for $-1 < x < 1$, where $a_0 = y_0$ and $a_1 = y_0'$.
- Thus the unique solution on $(-\infty, \infty)$ may not have a power series about $x_0 = 0$ that converges for all x.

Example 6

- Determine a lower bound for the radius of convergence of the series solution about $x_0 = 0$ for the equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0$$

- Here, $P(x) = 1$, $Q(x) = \sin x$, $R(x) = 1 + x^2$.
- Note that $p(x) = \sin x$ is not a polynomial, but recall that it does have a Taylor series about $x_0 = 0$ that converges for all x.
- Similarly, $q(x) = 1 + x^2$ has a Taylor series about $x_0 = 0$, namely$1 + x^2$, which converges for all x.
- Therefore, by Theorem 5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is infinite.

Boyce/DiPrima/Meade 11th ed, Ch 5.4: Euler Equations; Regular Singular Points

- Recall that the point x_0 is an **ordinary point** of the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

  if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at at x_0. Otherwise x_0 is a **singular point**.

- Thus, if P, Q and R are polynomials having no common factors, then the singular points of the differential equation are the points for which $P(x) = 0$.

Euler Equations

- A relatively simple differential equation that has a regular singular point is the **Euler equation**,

$$L[y] = x^2 y'' + \alpha xy' + \beta y = 0$$

  where a and b are constants.
- Note that $x_0 = 0$ is a regular singular point.
- The solution of the Euler equation is typical of the solutions of all differential equations with regular singular points, and hence we examine Euler equations before discussing the more general problem.

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

Solutions of the Form $y = x^r$

- In any interval not containing the origin, the general solution of the Euler equation has the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

- Suppose $x > 0$, and assume a solution of the form $y = x^r$. Then

$$y = x^r,\ y' = r\,x^{r-1},\ y'' = r(r-1)x^{r-2}$$

- Substituting these into the differential equation, we obtain

$$L[x^r] = r(r-1)x^r + \alpha\, r\, x^r + \beta\, x^r = 0$$

or $$L[x^r] = x^r\left[r(r-1) + \alpha r + \beta\right] = 0$$

or $$L[x^r] = x^r\left[r^2 + (\alpha - 1)r + \beta\right] = 0$$

Quadratic Equation

- Thus, after substituting $y = x^r$ into our differential equation, we arrive at

$$x^r[r(r-1)+ar+b=0, \quad x>0$$

- and hence

$$r_1, r_2 = \frac{-(a-1)\pm\sqrt{(a-1)^2-4b}}{2}$$

- Let $F(r)$ be defined by

$$F(r)=r^2+(\alpha-1)r+\beta=(r-r_1)(r-r_2)$$

- We now examine the different cases for the roots r_1, r_2.

Real, Distinct Roots

- If $F(r)$ has real roots $r_1 \neq r_2$, then

$$y_1(x) = x^{r_1},\ y_2(x) = x^{r_2}$$

are solutions to the Euler equation. Note that

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix}$$

$$= r_2 x^{r_1+r_2-1} - r_1 x^{r_1+r_2-1}$$

$$= (r_2 - r_1) x^{r_1+r_2-1} \neq 0 \text{ for all } x > 0.$$

- Thus y_1 and y_2 are linearly independent, and the general solution to our differential equation is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0$$

Example 1

- Consider the equation

$$2x^2y'' + 3xy' - y = 0, \quad x > 0$$

- Substituting $y = x^r$ into this equation, we obtain

$$y = x^r,\ y' = r\,x^{r-1}, y'' = r(r-1)x^{r-2}$$

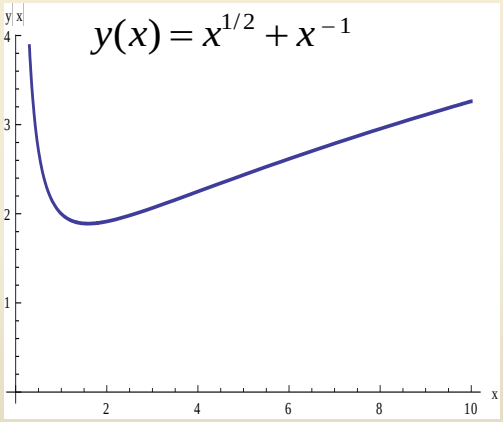
and

$$2r(r-1)x^r + 3rx^r - x^r = 0$$

$$x^r\left[2r(r-1) + 3r - 1\right] = 0$$

$$x^r\left[2r^2 + r - 1\right] = 0$$

$$x^r(2r-1)(r+1) = 0$$



- Thus $r_1 = 1/2$, $r_2 = -1$, and our general solution is

$$y(x) = c_1x^{1/2} + c_2x^{-1}, \quad x > 0$$

Equal Roots

- If $F(r)$ has equal roots $r_1 = r_2$, then we have one solution

$$y_1(x) = x^{r_1}$$

- We could use reduction of order to get a second solution; instead, we will consider an alternative method.
- Since $F(r)$ has a double root r_1, $F(r) = (r - r_1)^2$, and $F'(r_1) = 0$.
- This suggests differentiating $L[x^r]$ with respect to r and then setting r equal to r_1, as follows:

$$L[x^r] = x^r\left[r^2 + (\alpha - 1)r + \beta\right] = x^r\left(r - r_1\right)^2$$

$$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r} x^r\left(r - r_1\right)^2$$

$$L[x^r \ln x] = x^r \ln x\left(r - r_1\right)^2 + 2\left(r - r_1\right)x^r$$

$$\Rightarrow y_2(x) = x^{r_1} \ln x, \quad x > 0$$

Equal Roots

- Thus in the case of equal roots $r_1 = r_2$, we have two solutions

$$y_1(x) = x^{r_1},\ y_2(x) = x^{r_1} \ln x$$

- Now

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{r_1} & x^{r_1} \ln x \\ r_1 x^{r_1-1} & x^{r_1-1}(r_1 \ln x + 1) \end{vmatrix}$$

$$= x^{2r_1-1}(r_1 \ln x + 1) - r_1 x^{2r_1-1} \ln x$$

$$= x^{2r_1-1} \neq 0 \ \text{ for all } x > 0.$$

- Thus y_1 and y_2 are linearly independent, and the general solution to our differential equation is

$$y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln x = (c_1 + c_2 \ln x) x^{r_1}, \quad x > 0$$

Example 2

- Consider the equation

$$x^2 y'' + 5xy' + 4y = 0, \quad x > 0$$

- Then

$$y = x^r, \; y' = r\,x^{r-1}, y'' = r(r-1)x^{r-2}$$

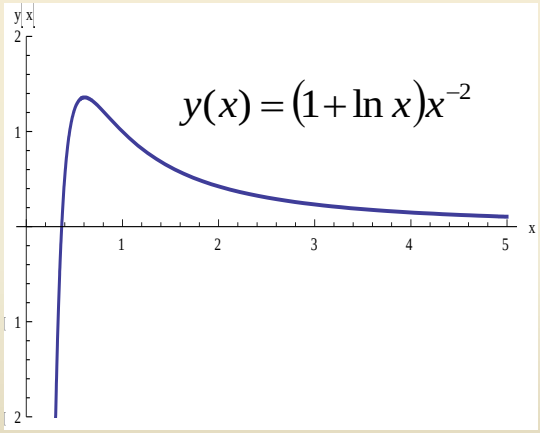
and

$$r(r-1)x^r + 5rx^r + 4x^r = 0$$

$$x^r(r(r-1) + 5r + 4) = 0$$

$$x^r(r^2 + 4r + 4) = 0$$

$$x^r(r+2)^2 = 0$$



- Thus $r_1 = r_2 = -2$, our general solution is

$$y(x) = (c_1 + c_2 \ln x)x^{-2}, \quad x > 0$$

Complex Roots

- Suppose $F(r)$ has complex roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, $\mu \neq 0$. Then

$$x^r = e^{\ln x^r} = e^{r \ln x} = e^{(\lambda + i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x}$$

$$= e^{\ln x^\lambda} e^{i\mu \ln x} = x^\lambda \left[\cos(\mu \ln x) + i \sin(\mu \ln x)\right], \quad x > 0$$

- Thus x^r is defined for complex r, and it can be shown that the general solution to the differential equation has the form

$$y(x) = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu}, \quad x > 0$$

- However, these solutions are complex-valued. It can be shown that the following functions are solutions as well:

$$y_1(x) = x^\lambda \cos(\mu \ln x), \; y_2(x) = x^\lambda \sin(\mu \ln x)$$

Complex Roots

- The following functions are solutions to our equation:

$$y_1(x) = x^{\lambda}\cos(\mu \ln x),\ y_2(x) = x^{\lambda}\sin(\mu \ln x)$$

- Using the Wronskian, it can be shown that y_1 and y_2 are linearly independent,

$$W\left[x^{\lambda}\cos(\mu \ln x), x^{\lambda}\sin(\mu \ln x)\right] = \mu x^{2\lambda-1} \neq 0 \text{ for } x > 0$$

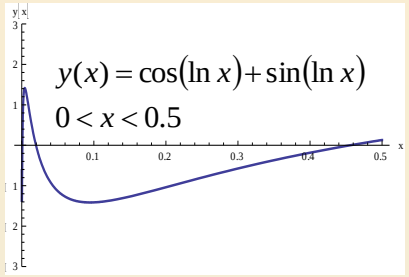
- and thus the general solution to our differential equation can be written as

$$y(x) = c_1 x^{\lambda}\cos(\mu \ln x) + c_2 x^{\lambda}\sin(\mu \ln x),\ x > 0$$

Example 3

- Consider the equation

$$x^2 y'' + xy' + y = 0, \quad x > 0$$



- Then

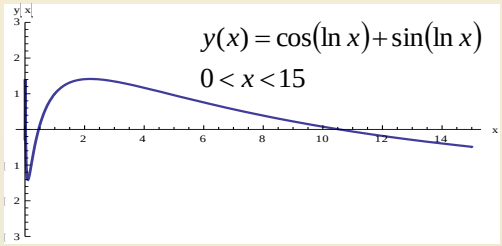
$$y = x^r, \; y' = r\,x^{r-1}, y'' = r(r-1)x^{r-2}$$

and

$$r(r-1)x^r + rx^r + x^r = 0$$

$$x^r\left[r(r-1) + r + \right] = 0$$

$$x^r\left[r^2 + 1\right] = 0$$



- Thus $r_1 = -i$, $r_2 = i$, and our general solution is

$$y(x) = c_1 x^0 \cos(\ln x) + c_2 x^0 \sin(\ln x)$$

$$= c_1 \cos(\ln x) + c_2 \sin(\ln x), \quad x > 0$$

Solution Behavior

- Recall that the solution to the Euler equation

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

depends on the roots:

$r_1 \neq r_2$: $\quad y(x) = c_1 x^{r_1} + c_2 x^{r_2}$

$r_1 = r_2$: $\quad y(x) = (c_1 + c_2 \ln x) x^{r_1}$

r_1, r_2 complex : $y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x)$,

where $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

- The qualitative behavior of these solutions near the singular point $x = 0$ depends on the nature of r_1 and r_2.
- Also, we obtain similar forms of solution when $t < 0$. Overall results are summarized on the next slide.

General Solution of the Euler Equation

- The general solution to the Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r^2 + (\alpha - 1)r + \beta = (r - r_1)(r - r_2)$$

according to the following cases:

$r_1 \neq r_2$: $\quad y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}$

$r_1 = r_2$: $\quad y(x) = \left(c_1 + c_2 \ln|x|\right)|x|^{r_1}$

r_1, r_2 complex : $y(x) = c_1 |x|^{\lambda} \cos\left(\mu \ln|x|\right) + c_2 x^{\lambda} \sin\left(\mu \ln|x|\right),$

where $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

Shifted Equations

- The solutions to the Euler equation

$$(x-x_0)^2 y'' + \alpha (x-x_0) y' + \beta y = 0$$

are similar to the ones given in previous slide:

$$r_1 \neq r_2: \qquad y(x) = c_1 |x-x_0|^{r_1} + c_2 |x-x_0|^{r_2}$$

$$r_1 = r_2: \qquad y(x) = \left(c_1 + c_2 \ln|x-x_0|\right)|x-x_0|^{r_1}$$

r_1, r_2 complex:

$$y(x) = c_1 |x-x_0|^{\lambda} \cos\left(\mu \ln|x-x_0|\right) + c_2 x^{\lambda} \sin\left(\mu \ln|x-x_0|\right),$$

where $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

Solution Behavior and Singular Points

- If we attempt to use the methods of the preceding two sections to solve the differential equation in a neighborhood of a singular point x_0, we will find that these methods fail.
- This is because the solution may not be analytic at x_0, and hence will not have a Taylor series expansion about x_0. Instead, we must use a more general series expansion.
- A differential equation may only have a few singular points, but solution behavior near these singular points is important.
- For example, solutions often become unbounded or experience rapid changes in magnitude near a singular point.
- Also, geometric singularities in a physical problem, such as corners or sharp edges, may lead to singular points in the corresponding differential equation.

Numerical Methods and Singular Points

- As an alternative to analytical methods, we could consider using numerical methods, which are discussed in Chapter 8.
- However, numerical methods are not well suited for the study of solutions near singular points.
- When a numerical method is used, it helps to combine with it the analytical methods of this chapter in order to examine the behavior of solutions near singular points.

Solution Behavior Near Singular Points

- Thus without more information about Q/P and R/P in the neighborhood of a singular point x_0, it may be impossible to describe solution behavior near x_0.
- It may be that there are two linearly independent solutions that remain bounded as $x \to x_0$; or there may be only one, with the other becoming unbounded as $x \to x_0$; or they may both become unbounded as $x \to x_0$.
- If a solution becomes unbounded, then we may want to know if $y \to \infty$ in the same manner as $(x - x_0)^{-1}$ or $|x - x_0|^{-½}$, or in some other manner.

Classifying Singular Points

- Our goal is to extend the method already developed for solving

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

  near an ordinary point so that it applies to the neighborhood of a singular point x_0.
- To do so, we restrict ourselves to cases in which singularities in Q/P and R/P at x_0 are not too severe, that is, to what might be called "weak singularities."
- It turns out that the appropriate conditions to distinguish "weak singularities" are

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} \text{ is finite } \textit{and } \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} \text{ is finite.}$$

Regular Singular Points

- Consider the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- If P and Q are polynomials, then a **regular singular** point x_0 is singular point for which

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \text{ is finite } \textit{and } \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \text{ is finite.}$$

- For more general functions than polynomials, x_0 is a regular singular point if it is a singular point with

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \textit{and} \quad (x - x_0)^2 \frac{R(x)}{P(x)} \text{ are analytic at } x = x_0.$$

- Any other singular point x_0 is an **irregular singular** point.

Example 4: Legendre Equation

- Consider the Legendre equation

$$\left(1-x^2\right)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

- The point $x = 1$ is a regular singular point, since both of the following limits are finite:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 1}(x-1)\left(\frac{-2x}{1-x^2}\right) = \lim_{x\to 1}\left(\frac{2x}{x+1}\right) = 1,$$

$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} = \lim_{x\to 1}(x-1)^2\left(\frac{\alpha(\alpha+1)}{1-x^2}\right) = \lim_{x\to 1}(x-1)\left(\frac{\alpha(\alpha+1)}{x+1}\right) = 0$$

- Similarly, it can be shown that $x = -1$ is a regular singular point.

Example 5

- Consider the equation

$$2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$$

- The point $x = 0$ is a regular singular point:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\left(\frac{3x}{2x(x-2)^2}\right) = \lim_{x\to 0}\frac{3x}{2(x-2)^2} = 0 < \infty,$$

$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\left(\frac{x-2}{2x(x-2)^2}\right) = \lim_{x\to 0}\frac{x}{2(x-2)} = 0 < \infty$$

- The point $x = 2$, however, is an irregular singular point, since the following limit does not exist:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 2}(x-2)\left(\frac{3x}{2x(x-2)^2}\right) = \lim_{x\to 2}\frac{3x}{2x(x-2)}$$

Example 6: Nonpolynomial Coefficients (1 of 2)

- Consider the equation

$$\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)y' + (\sin x)y = 0$$

- Note that $x = \pi/2$ is the only singular point.
- We will demonstrate that $x = \pi/2$ is a regular singular point by showing that the following functions are analytic at $\pi/2$:

$$(x - \pi/2)\frac{\cos x}{(x - \pi/2)^2} = \frac{\cos x}{x - \pi/2}, \quad (x - \pi/2)^2 \frac{\sin x}{(x - \pi/2)^2} = \sin x$$

Example 6: Regular Singular Point (2 of 2)

- Using methods of calculus, we can show that the Taylor series of $\cos x$ about $\pi/2$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}$$

- Thus

$$\frac{\cos x}{x - \pi/2} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n},$$

  which converges for all x, and hence is analytic at $\pi/2$.
- Similarly, $\sin x$ is analytic at $\pi/2$, with Taylor series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$$

- Thus $\pi/2$ is a regular singular point of the differential equation.

Boyce/DiPrima/Meade 11th ed, Ch 5.5: Series Solutions Near a Regular Singular Point, I

- We now consider solving the general second order linear equation in the neighborhood of a regular singular point x_0. For convenience, we will take $x_0 = 0$.
- Recall that the point $x_0 = 0$ is a regular singular point of

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

iff

$$x\frac{Q(x)}{P(x)} = xp(x) \text{ and } x^2\frac{R(x)}{P(x)} = x^2q(x) \text{ are analytic at } x = 0$$

iff

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \text{ and } x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \text{ convergent on } |x| < \rho$$

Transforming the Differential Equation

- Our differential equation has the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- Dividing by $P(x)$ and multiplying by x^2, we obtain

$$x^2 y'' + x\left[xp(x)\right]y' + \left[x^2 q(x)\right]y = 0$$

- Substituting in the power series representations of p and q,

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

we obtain

$$x^2 y'' + x\left(p_0 + p_1 x + p_2 x^2 + \cdots\right)y' + \left(q_0 + q_1 x + q_2 x^2 + \cdots\right)y = 0$$

Comparison with Euler Equations

- Our differential equation now has the form

$$x^2y'' + x\left(p_0 + p_1x + p_2x^2 + \cdots\right)y' + \left(q_0 + q_1x + q_2x^2 + \cdots\right)y = 0$$

- Note that if

$$p_1 = p_2 = \cdots = q_1 = q_2 = \cdots = 0$$

  then our differential equation reduces to the Euler Equation

$$x^2y'' + p_0xy' + q_0y = 0$$

- In any case, our equation is similar to an Euler Equation but with power series coefficients.
- Thus our solution method: assume solutions have the form

$$y(x) = x^r\left(a_0 + a_1x + a_2x^2 + \cdots\right) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

Example 1: Regular Singular Point (1 of 13)

- Consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0$$

- This equation can be rewritten as

$$x^2y'' - \frac{x}{2}y' + \frac{1+x}{2}y = 0$$

- Since the coefficients are polynomials, it follows that $x = 0$ is a regular singular point, since both limits below are finite:

$$\lim_{x\to 0} x\left(-\frac{x}{2x^2}\right) = -\frac{1}{2} < \infty \quad \text{and} \quad \lim_{x\to 0} x^2\left(\frac{1+x}{2x^2}\right) = \frac{1}{2} < \infty$$

$$2x^2y'' - xy' + (1+x)y = 0$$

Example 1: Euler Equation (2 of 13)

- Now $xp(x) = -1/2$ and $x^2q(x) = (1 + x)/2$, and thus for

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

it follows that

$$p_0 = -1/2,\ q_0 = 1/2,\ q_1 = 1/2,\ p_1 = p_2 = \cdots = q_2 = q_3 = \cdots = 0$$

- Thus the corresponding Euler Equation is

$$x^2y'' + p_0xy' + q_0y = 0 \iff 2x^2y'' - xy' + y = 0$$

- As in Section 5.5, we obtain

$$x^r\left[2r(r-1) - r + 1\right] = 0 \iff (2r-1)(r-1) = 0 \iff r = 1,\ r = 1/2$$

- We will refer to this result later.

$$2x^2y'' - xy' + (1+x)y = 0$$

Example 1: Differential Equation (3 of 13)

- For our differential equation, we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- By substitution, our differential equation becomes

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0$$

or

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

Example 1: Combining Series (4 of 13)

- Our equation

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0$$

  can next be written as

$$a_0[2r(r-1)-r+1]x^r + \sum_{n=1}^{\infty} \{ a_n [2(r+n)(r+n-1)-(r+n)+1] + a_{n-1} \} x^{r+n} = 0$$

- It follows that

$$a_0[2r(r-1)-r+1] = 0$$

  and

$$a_n[2(r+n)(r+n-1)-(r+n)+1] + a_{n-1} = 0, \quad n = 1, 2, \ldots$$

Example 1: Indicial Equation (5 of 13)

- From the previous slide, we have

$$a_0\left[2r(r-1)-r+1\right]x^r+\sum_{n=1}^{\infty}\left\{a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}\right\}x^{r+n}=0$$

- The equation

$$a_0\left[2r(r-1)-r+1\right]=0 \quad \overset{a_0\neq 0}{\Leftrightarrow} \quad 2r^2-3r+1=(2r-1)(r-1)=0$$

  is called the **indicial equation**, and was obtained earlier when we examined the corresponding Euler Equation.

- The roots $r_1 = 1$, $r_2 = ½$, of the indicial equation are called the **exponents of the singularity,** for regular singular point $x = 0$.
- The exponents of the singularity determine the qualitative behavior of solution in neighborhood of regular singular point.

Example 1: Recursion Relation (6 of 13)

- Recall that

$$a_0\left[2r(r-1)-r+1\right]x^r+\sum_{n=1}^{\infty}\left\{a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}\right\}x^{r+n}=0$$

- We now work with the coefficient on x^{r+n}:

$$a_n\left[2(r+n)(r+n-1)-(r+n)+1\right]+a_{n-1}=0$$

- It follows that

$$a_n=-\frac{a_{n-1}}{2(r+n)(r+n-1)-(r+n)+1}$$

$$=-\frac{a_{n-1}}{2(r+n)^2-3(r+n)+1}$$

$$=-\frac{a_{n-1}}{\left[2(r+n)-1\right]\left[(r+n)-1\right]},\quad n\geq 1$$

Example 1: First Root (7 of 13)

- We have

$$a_n = -\frac{a_{n-1}}{[2(r+n)-1][(r+n)-1]}, \text{ for } n \geq 1,\ r_1 = 1 \text{ and } r_1 = 1/2$$

- Starting with $r_1 = 1$, this recursion becomes

$$a_n = -\frac{a_{n-1}}{[2(1+n)-1][(1+n)-1]} = -\frac{a_{n-1}}{(2n+1)n},\ n \geq 1$$

- Thus

$$a_1 = -\frac{a_0}{3 \cdot 1}$$

$$a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)}$$

$$a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}, \text{ etc}$$

$$a_n = \frac{(-1)^n a_0}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!},\ n \geq 1$$

Example 1: First Solution (8 of 13)

- Thus we have an expression for the n-th term:

$$a_n = \frac{(-1)^n a_0}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!}, \quad n \geq 1$$

- Hence for $x > 0$, one solution to our differential equation is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= a_0 x + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1}}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!}$$

$$= a_0 x \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!} \right]$$

Example 1: Radius of Convergence for First Solution (9 of 13)

- Thus if we omit a_0, one solution of our differential equation is

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3\cdot 5\cdot 7\cdots(2n+1))n!}\right], \quad x > 0$$

- To determine the radius of convergence, use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{(3\cdot 5\cdot 7\cdots(2n+1))n!(-1)^{n+1}x^{n+1}}{(3\cdot 5\cdot 7\cdots(2n+1)(2n+3))(n+1)!(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)} = 0 < 1$$

- Thus the radius of convergence is infinite, and hence the series converges for all x.

Example 1: Second Root (10 of 13)

- Recall that

$$a_n = -\frac{a_{n-1}}{[2(r+n)-1][(r+n)-1]}, \text{ for } n \geq 1,\ r_1 = 1 \text{ and } r_1 = 1/2$$

- When $r_1 = 1/2$, this recursion becomes

$$a_n = -\frac{a_{n-1}}{[2(1/2+n)-1][(1/2+n)-1]} = -\frac{a_{n-1}}{2n(n-1/2)} = -\frac{a_{n-1}}{n(2n-1)}, \quad n \geq 1$$

- Thus

$$a_1 = -\frac{a_0}{1 \cdot 1} \qquad a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)}, \text{ etc}$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)} \qquad a_n = \frac{(-1)^n a_0}{((1 \cdot 3 \cdot 5) \cdots (2n-1))n!}, \quad n \geq 1$$

Example 1: Second Solution (11 of 13)

- Thus we have an expression for the n-th term:

$$a_n = \frac{(-1)^n a_0}{(1\cdot 3\cdot 5\cdots(2n-1))n!}, \quad n \geq 1$$

- Hence for $x > 0$, a second solution to our equation is

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= a_0 x^{1/2} + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+1/2}}{(1\cdot 3\cdot 5\cdots(2n-1))n!}$$

$$= a_0 x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1\cdot 3\cdot 5\cdots(2n-1))n!}\right]$$

Example 1: Radius of Convergence for Second Solution (12 of 13)

- Thus if we omit a_0, the second solution is

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1\cdot 3\cdot 5\cdots(2n-1))n!}\right]$$

- To determine the radius of convergence for this series, we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{(1\cdot 3\cdot 5\cdots(2n-1))n!(-1)^{n+1}x^{n+1}}{(1\cdot 3\cdot 5\cdots(2n-1)(2n+1))(n+1)!(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{(2n+1)n} = 0 < 1$$

- Thus the radius of convergence is infinite, and hence the series converges for all x.

Example 1: General Solution (13 of 13)

- The two solutions to our differential equation are

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3\cdot 5\cdot 7\cdots(2n+1))n!}\right]$$

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1\cdot 3\cdot 5\cdots(2n-1))n!}\right]$$

- Since the leading terms of y_1 and y_2 are x and $x^{1/2}$, respectively, it follows that y_1 and y_2 are linearly independent, and hence form a fundamental set of solutions for differential equation.
- Therefore the general solution of the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \quad x > 0,$$

where y_1 and y_2 are as given above.

Shifted Expansions & Discussion

- For the analysis given in this section, we focused on $x = 0$ as the regular singular point. In the more general case of a singular point at $x = x_0$, our series solution will have the form

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- If the roots r_1, r_2 of the indicial equation are equal or differ by an integer, then the second solution y_2 normally has a more complicated structure. These cases are discussed in Section 5.7.
- If the roots of the indicial equation are complex, then there are always two solutions with the above form. These solutions are complex valued, but we can obtain real-valued solutions from the real and imaginary parts of the complex solutions.

Boyce/DiPrima/Meade 11th ed, Ch 5.6: Series Solutions Near a Regular Singular Point, Part II

- Recall from Section 5.5 (Part I): The point $x_0 = 0$ is a regular singular point of

$$x^2y'' + x\left[xp(x)\right]y' + \left[x^2q(x)\right]y = 0$$

with

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \;\; x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \text{ convergent on } |x| < \rho$$

and corresponding Euler Equation

$$x^2y'' + p_0xy' + q_0y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r,x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0, \; x > 0$$

$$x^2y'' + x\left[xp(x)\right]y' + \left[x^2q(x)\right]y = 0$$

Substitute Derivatives into ODE

- Taking derivatives, we have

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- Substituting these derivatives into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} +$$

$$\left[\sum_{n=0}^{\infty} p_n x^r\right]\left[\sum_{n=0}^{\infty} a_n (r+n) x^{r+n}\right] + \left[\sum_{n=0}^{\infty} q_n x^r\right]\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right] = 0$$

Multiplying Series

$$\left[\sum_{n=0}^{\infty} p_n x^r\right]\left[\sum_{n=0}^{\infty} a_n (r+n) x^{r+n}\right]+\left[\sum_{n=0}^{\infty} q_n x^r\right]\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right]$$

$$=\left[p_0 + p_1 x + \cdots + p_n x^n + \cdots\right]\left[a_0 r x^r + a_1 (r+1) x^{r+1} + \cdots + a_n (r+n) x^{r+n} + \cdots\right] +$$
$$\left[q_0 + q_1 x + \cdots + q_n x^n + \cdots\right]\left[a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots\right]$$

$$=\left[p_0 r + q_0 a_0\right] x^r + \left[p_0 a_1 (r+1) + p_1 a_0 r + q_0 a_1 + q_1 a_0\right] x^{r+1} + \cdots +$$
$$\left[p_0 a_n (r+n) + p_1 a_{n-1} (r+n-1) + \cdots + p_n a_0 r + q_0 a_n + q_1 a_{n-1} + \cdots + q_n a_0\right] x^{r+n} + \cdots$$

$$=\left[a_0 (p_0 r + q_0)\right] x^r + \left[a_0 (p_1 r + q_1) + a_1 (p_0 (r+1) + q_0)\right] x^{r+1} + \cdots +$$
$$\left[a_0 (p_n r + q_n) + \cdots + a_{n-1} (p_1 (r+n-1) + q_1) + a_n (p_0 (r+n) + q_0)\right] x^{r+n} + \cdots$$

Combining Terms in ODE

- Our equation then becomes

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n}$$

$$+\left[\sum_{n=0}^{\infty} p_n x^r\right]\left[\sum_{n=0}^{\infty} a_n (r+n)x^{r+n}\right]+\left[\sum_{n=0}^{\infty} q_n x^r\right]\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right]=0$$

$$=\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n}$$

$$+\left[a_0(p_0 r+q_0)\right]x^r+\left[a_0(p_1 r+q_1)+a_1(p_0(r+1)+q_0)\right]x^{r+1}+\cdots$$

$$+\left[a_0(p_n r+q_n)+\cdots+a_{n-1}(p_1(r+n-1)+q_1)+a_n(p_0(r+n)+q_0)\right]x^{r+n}+\cdots=0$$

$$=\left[a_0(r(r-1)+p_0 r+q_0)\right]x^r+\left[a_0(p_1 r+q_1)+a_1(r(r+1)+p_0(r+1)+q_0)\right]x^{r+1}+\cdots$$

$$+\left[a_0(p_n r+q_n)+\cdots+a_n((r+n)(r+n-1)+p_0(r+n)+q_0)\right]x^{r+n}+\cdots=0$$

Rewriting ODE

- Define $F(r)$ by

$$F(r) = r(r-1) + p_0 r + q_0$$

- We can then rewrite our equation

$$\left[a_0\left(r(r-1)+p_0 r+q_0\right)\right]x^r + \left[a_0\left(p_1 r+q_1\right)+a_1\left(r(r+1)+p_0\left(r+1\right)+q_0\right)\right]x^{r+1}$$
$$+\cdots+\left[a_0\left(p_n r+q_n\right)+\cdots+a_n\left(\left(r+n\right)\left(r+n-1\right)+p_0\left(r+n\right)+q_0\right)\right]x^{r+n}+\cdots=0$$

in more compact form:

$$a_0 F(r)x^r + \sum_{n=1}^{\infty}\left\{a_n F(r+n) + \sum_{k=0}^{n-1} a_k\left[(r+k)\,p_{n-k}+q_k\right]\right\}x^{r+n} = 0$$

Indicial Equation

- Thus our equation is

$$a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_k \right] \right\} x^{r+n} = 0$$

- Since $a_0 \neq 0$, we must have

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

- This **indicial equation** is the same one obtained when seeking solutions $y = x^r$ to the corresponding Euler Equation.
- Note that $F(r)$ is quadratic in r, and hence has two roots, r_1 and r_2. If r_1 and r_2 are real, then assume $r_1 \geq r_2$.
- These roots are called the **exponents at the singularity**, and they determine behavior of solution near singular point.

Recurrence Relation

- From our equation,

$$a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_k \right] \right\} x^{r+n} = 0$$

  the **recurrence relation** is

$$a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_k \right] = 0$$

- This recurrence relation shows that in general, a_n depends on r and the previous coefficients $a_0, a_1, \ldots, a_{n-1}$.
- Note that we must have $r = r_1$ or $r = r_2$.

Recurrence Relation & First Solution

- With the recurrence relation

$$a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_k \right] = 0,$$

  we can compute $a_1, \ldots, a_{n\text{-}1}$ in terms of a_0, p_m and q_m, provided $F(r+1), F(r+2), \ldots, F(r+n), \ldots$ are not zero.
- Recall $r = r_1$ or $r = r_2$, and these are the only roots of $F(r)$.
- Since $r_1 \geq r_2$, we have $r_1 + n \neq r_1$ and $r_1 + n \neq r_2$ for $n \geq 1$.
- Thus $F(r_1 + n) \neq 0$ for $n \geq 1$, and at least one solution exists:

$$y_1(x) = x^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right], \quad a_0 = 1, \quad x > 0$$

  where the notation $a_n(r_1)$ indicates that a_n has been determined using $r = r_1$.

Recurrence Relation & Second Solution

- Now consider $r = r_2$. Using the recurrence relation

$$a_n F(r+n) + \sum_{k=0}^{n-1} a_k \left[(r+k) p_{n-k} + q_k \right] = 0,$$

we compute $a_1, \ldots, a_{n-1}$ in terms of a_0, p_m and q_m, provided $F(r_2 + 1), F(r_2 + 2), \ldots, F(r_2 + n), \ldots$ are not zero.

- If $r_2 \neq r_1$, and $r_2 - r_1 \neq n$ for $n \geq 1$, then $r_2 + n \neq r_1$ for $n \geq 1$.
- Thus $F(r_2 + n) \neq 0$ for $n \geq 1$, and a second solution exists:

$$y_2(x) = x^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \right], \quad a_0 = 1, \quad x > 0$$

where the notation $a_n(r_2)$ indicates that a_n has been determined using $r = r_2$.

Convergence of Solutions

- If the restrictions on r_2 are satisfied, we have two solutions

$$y_1(x) = x^{r_1}\left[1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right], \quad y_2(x) = x^{r_2}\left[1+\sum_{n=1}^{\infty} a_n(r_2)x^n\right]$$

where $a_0 = 1$ and $x > 0$. The series converge for $|x| < r$, and

$$f(x) = 1+\sum_{n=1}^{\infty} a_n(r_1)x^n \text{ and } g(x) = 1+\sum_{n=1}^{\infty} a_n(r_2)x^n$$

define analytic functions within their radii of convergence.

- It follows that any singular behavior of solutions y_1 and y_2 is due to the factors x^{r1} and x^{r2}.
- To obtain solutions for $x < 0$, it can be shown that we need only replace x^{r1} and x^{r2} by $|x^{r1}|$ and $|x^{r2}|$ in y_1 and y_2 above.
- If r_1 and r_2 are complex, then they are conjugates and $r_2 - r_1 \neq N$ for $N \geq 1$, and real-valued series solutions can be found.

Example 1: Singular Points (1 of 5)

- Find all regular singular points, determine indicial equation and exponents of singularity for each regular singular point. Then discuss nature of solutions near singular points.

$$2x(1+x)y'' + (3+x)y' - xy = 0$$

- Solution: The equation can be rewritten as

$$y'' + \frac{3+x}{2x(1+x)}y' - \frac{x}{2x(1+x)}y = 0$$

- The singular points are $x = 0$ and $x = -1$.
- Then $x = 0$ is a regular singular point, since

$$p_0 = \lim_{x\to 0} x\frac{3+x}{2x(1+x)} = \frac{3}{2} < \infty, \text{ and } q_0 = \lim_{x\to 0} x^2\frac{-x}{2x(1+x)} = 0 < \infty$$

Example 1: Indicial Equation, $x = 0$ (2 of 5)

- The corresponding indicial equation is given by

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

or

$$r(r-1) + \frac{3}{2} r = 0$$

- The exponents at the singularity for $x = 0$ are found by solving the indicial equation:

$$2r(r-1) + 3r = 0$$

$$2r^2 + r = 0$$

$$r(2r+1) = 0$$

- Thus $r_1 = 0$ and $r_2 = -1/2$, for the regular singular point $x = 0$.

Example 1: Series Solutions, $x = 0$ (3 of 5)

- The solutions corresponding to $x = 0$ have the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n, \quad y_2(x) = x^{-1/2}\left[1 + \sum_{n=1}^{\infty} a_n\left(-\frac{1}{2}\right)x^n\right]$$

- The coefficients $a_n(0)$ and $a_n(-1/2)$ are determined by the corresponding recurrence relation.
- Both series converge for $|x| < r$, where r is the smaller radius of convergence for the series representations about $x = 0$ for

$$xp(x) = \frac{3+x}{2x(1+x)}, \quad x^2q(x) = \frac{x}{2x(1+x)}$$

- The smallest r can be is 1, which is the distance between the two singular points $x = 0$ and $x = -1$.
- Note y_1 is bounded as $x \to 0$, whereas y_2 unbounded as $x \to 0$.

Example 1: Indicial Equation, $x = -1$ (4 of 5)

- Next, $x = -1$ is a regular singular point, since

$$p_0 = \lim_{x \to -1} (x+1) \frac{3+x}{2x(1+x)} = -1 < \infty$$

and

$$q_0 = \lim_{x \to -1} (x+1)^2 \frac{-x}{2x(1+x)} = 0 < \infty$$

- The indicial equation is given by

$$r(r-1) - r = 0$$

and hence the exponents at the singularity for $x = -1$ are

$$r^2 - 2r = 0 \iff r(r-2) = 0 \iff r_1 = 2,\ r_2 = 0$$

- Note that r_1 and r_2 differ by a positive integer.

Example 1: Series Solutions, $x = -1$ (5 of 5)

- The first solution corresponding to $x = -1$ has the form

$$y_1(x) = (x+1)^2\left[1 + \sum_{n=1}^{\infty} a_n(2)(x+1)^n\right] \qquad \rightarrow$$

- This series converges for at least $|x + 1| < 1$, and y_1 is an analytic function there.
- Since the roots $r_1 = 2$ and $r_2 = 0$ differ by a positive integer, there may or may not be a second solution of the form

$$y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x+1)^n$$

Equal Roots

- Recall that the general indicial equation is given by

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

- In the case of equal roots, $F(r)$ simplifies to

$$F(r) = (r_1 - 1)^2$$

- It can be shown (see text) that the solutions are given by

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \quad y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a'_n(r_1)x^n$$

Roots Differing by an Integer

- If roots of the indicial equation differ by a positive integer, it can be shown that the ODE solutions are given by

$$y_1(x) = x^{r_1}\left[1+\sum_{n=1}^{\infty} a_n(r_1)x^n\right], \quad y_2(x) = a y_1(x)\ln x + x^{r_2}\left[1+\sum_{n=1}^{\infty} c_n(r_2)x^n\right]$$

where the $c_n(r_1)$ are found by substituting y_2 into the differential equation and solving, as usual. Alternatively,

$$c_n(r_1) = \frac{d}{dr}\left[(r-r_2)a_n(r)\right]_{r=r_2}, \quad n=1,2,\ldots$$

and

$$a = \lim_{r\to r_2}\left[(r-r_2)a_N(r)\right]_{r=r_2}, \quad \text{where } r_1 - r_2 = N$$

- See **Theorem 5.6.1** for a summary of results in this section.

Boyce/DiPrima/Meade 11th ed, Ch 5.7: Bessel's Equation

- Bessel Equation of order n:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0$$

- Note that $x = 0$ is a regular singular point.
- Friedrich Wilhelm Bessel (1784 – 1846) studied disturbances in planetary motion, which led him in 1824 to make the first systematic analysis of solutions of this equation. The solutions became known as Bessel functions.
- In this section, we study the following cases:
  - Bessel Equations of order zero: $n = 0$
  - Bessel Equations of order one-half: $n = \frac{1}{2}$
  - Bessel Equations of order one: $n = 1$

Bessel Equation of Order Zero (1 of 12)

- The Bessel Equation of order zero is

$$x^2 y'' + xy' + x^2 y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0, \; x > 0$$

- Taking derivatives,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \;\; y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

Indicial Equation (2 of 12)

- From the previous slide,

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

- Rewriting,

$$a_0\left[r(r-1)+r\right]x^r + a_1\left[(r+1)r+(r+1)\right]x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left\{a_n\left[(r+n)(r+n-1)+(r+n)\right]+a_{n-2}\right\}x^{r+n} = 0$$

- or

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n (r+n)^2 + a_{n-2}\right\}x^{r+n} = 0$$

- The indicial equation is $r^2 = 0$, and hence $r_1 = r_2 = 0$.

Recurrence Relation (3 of 12)

- From the previous slide,

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} \left\{ a_n (r+n)^2 + a_{n-2} \right\} x^{r+n} = 0$$

- Note that $a_1 = 0$; the recurrence relation is

$$a_n = -\frac{a_{n-2}}{(r+n)^2}, \; n = 2, 3, \ldots$$

- We conclude $a_1 = a_3 = a_5 = \ldots = 0$, and since $r = 0$,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2}, \; m = 1, 2, \ldots$$

- Note: Recall dependence of a_n on r, which is indicated by $a_n(r)$. Thus we may write $a_{2m}(0)$ here instead of a_{2m}.

First Solution (4 of 12)

- From the previous slide,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2},\ m = 1, 2, \ldots$$

- Thus

$$a_2 = -\frac{a_0}{2^2},\ a_4 = -\frac{a_2}{4^2} = \frac{a_0}{4^2 2^2} = \frac{a_0}{2^4 (2\cdot 1)^2},\ a_6 = -\frac{a_0}{2^6 (3\cdot 2\cdot 1)^2}, \ldots$$

and in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2},\ m = 1, 2, \ldots$$

- Thus

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right],\quad x > 0$$

Bessel Function of First Kind, Order Zero (5 of 12)

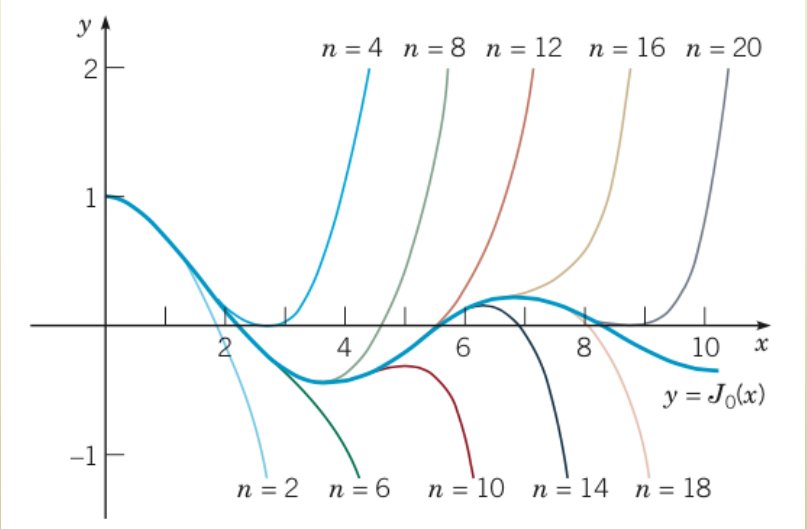
- Our first solution of Bessel's Equation of order zero is

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \quad x > 0$$

- The series converges for all x, and is called the **Bessel function of the first kind of order zero**, denoted by

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}, \quad x > 0$$

- The graphs of J_0 and several partial sum approximations are given here.



Second Solution: Odd Coefficients (6 of 12)

- Since indicial equation has repeated roots, recall that the coefficients in second solution can be found using

$$a_n'(r)\Big|_{r=0}$$

- Now

$$a_0(r)r^2x^r + a_1(r)(r+1)^2x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n(r)\left(r+n\right)^2 + a_{n-2}(r)\right\}x^{r+n} = 0$$

- Thus

$$a_1(r) = 0 \;\Rightarrow a_1'(0) = 0$$

- Also,

$$a_n(r) = -\frac{a_{n-2}(r)}{\left(r+n\right)^2},\;\; n = 2,3,\ldots$$

and hence

$$a_{2m+1}'(0) = 0,\;\; m = 1,2,\ldots$$

Second Solution: Even Coefficients (7 of 12)

- Thus we need only compute derivatives of the even coefficients, given by

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(r+2m)^2} \Rightarrow a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \cdots (r+2m)^2}, \quad m \geq 1$$

- It can be shown that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left[\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right]$$

and hence

$$a'_{2m}(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right] a_{2m}(0)$$

Second Solution: Series Representation (8 of 12)

- Thus

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \quad m = 1, 2, \ldots$$

where

$$H_m = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}$$

- Taking $a_0 = 1$ and using results of Section 5.7,

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \quad x > 0$$

Bessel Function of Second Kind, Order Zero (9 of 12)

- Instead of using y_2, the second solution is often taken to be a linear combination Y_0 of J_0 and y_2, known as the **Bessel function of second kind of order zero**. Here, we take

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + \left(\gamma - \ln 2\right)J_0(x)\right]$$

- The constant γ is the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(H_n - \ln n\right) \cong 0.5772$$

- Substituting the expression for y_2 from previous slide into equation for Y_0 above, we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}\left(m!\right)^2}x^{2m}\right], \quad x > 0$$

General Solution of Bessel's Equation, Order Zero (10 of 12)

- The general solution of Bessel's equation of order zero, $x > 0$, is given by

$$y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

where

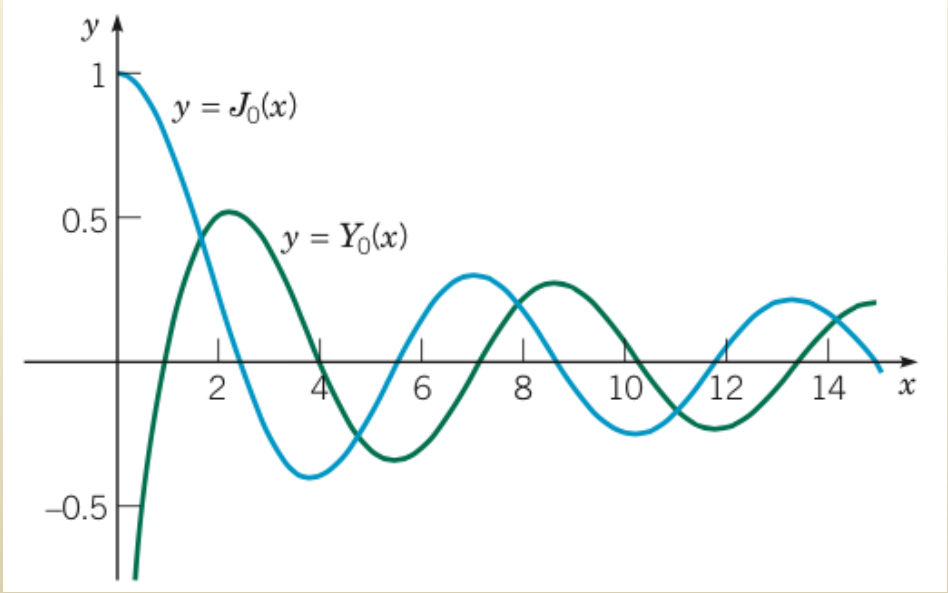
$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2},$$

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right]$$

- Note that $J_0 \to 1$ as $x \to 0$ while Y_0 has a logarithmic singularity at $x = 0$. If a solution which is bounded at the origin is desired, then Y_0 must be discarded.

Graphs of Bessel Functions, Order Zero (11 of 12)

- The graphs of J_0 and Y_0 are given below.
- Note that the behavior of J_0 and Y_0 appear to be similar to $\sin x$ and $\cos x$ for large x, except that oscillations of J_0 and Y_0 decay to zero.



Approximation of Bessel Functions, Order Zero (12 of 12)

- The fact that J_0 and Y_0 appear similar to sin x and cos x for large x may not be surprising, since ODE can be rewritten as

$$x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \iff y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

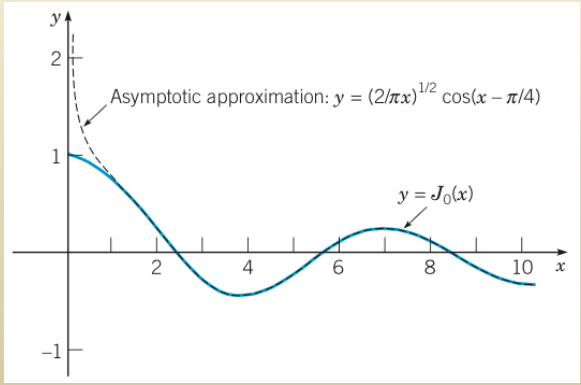
- Thus, for large x, our equation can be approximated by

$$y'' + y = 0,$$

whose solns are sin x and cos x. Indeed, it can be shown that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$



Bessel Equation of Order One-Half (1 of 8)

- The Bessel Equation of order one-half is

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

Recurrence Relation (2 of 8)

- Using the results of the previous slide, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-\frac{1}{4}\right]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-\frac{1}{4}\right)a_0 x^r+\left[(r+1)^2-\frac{1}{4}\right]a_1 x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of the indicial equation are $r_1 = ½$, $r_2 = -½$, and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1/4},\quad n=2,3,\ldots$$

First Solution: Coefficients (3 of 8)

- Consider first the case $r_1 = ½$. From the previous slide,

$$\left(r^2 - 1/4\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(r+n)^2 - \frac{1}{4}\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since $r_1 = ½$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have

$$a_{2m} = -\frac{a_{2m-2}}{\left(1/2 + 2m\right)^2 - 1/4} = -\frac{a_{2m-2}}{2m\left(2m+1\right)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_2 = -\frac{a_0}{3!}, \; a_4 = -\frac{a_2}{5 \cdot 4} = \frac{a_0}{5!}, \ldots$$

and

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, \ldots$$

Bessel Function of First Kind, Order One-Half (4 of 8)

- It follows that the first solution of our equation is, for $a_0 = 1$,

$$y_1(x) = x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}\right], \quad x > 0$$

$$= x^{-1/2}\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right], \quad x > 0$$

$$= x^{-1/2} \sin x, \quad x > 0$$

- The **Bessel function of the first kind of order one-half**, $J_{1/2}$, is defined as

$$J_{1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} y_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

Second Solution: Even Coefficients (5 of 8)

- Now consider the case $r_2 = -\frac{1}{2}$. We know that

$$\left(r^2 - 1/4\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(r+n)^2 - \frac{1}{4}\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since $r_2 = -\frac{1}{2}$, a_1 = arbitrary. For the even coefficients,

$$a_{2m} = -\frac{a_{2m-2}}{\left(-1/2 + 2m\right)^2 - 1/4} = -\frac{a_{2m-2}}{2m(2m-1)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_2 = -\frac{a_0}{2!}, \; a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \ldots$$

and

$$a_{2m} = \frac{(-1)^m a_0}{(2m)!}, \quad m = 1, 2, \ldots$$

Second Solution: Odd Coefficients (6 of 8)

- For the odd coefficients,

$$a_{2m+1} = -\frac{a_{2m-1}}{(-1/2+2m+1)^2 - 1/4} = -\frac{a_{2m-1}}{2m(2m+1)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_3 = -\frac{a_1}{3!}, \; a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \ldots$$

and

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!}, \quad m = 1, 2, \ldots$$

Second Solution (7 of 8)

- Therefore

$$y_2(x) = x^{-1/2}\left[a_0\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!} + a_1\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}\right], \quad x > 0$$

$$= x^{-1/2}\left[a_0\cos x + a_1\sin x\right], \quad x > 0$$

- The second solution is usually taken to be the function

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos x, \quad x > 0$$

where $a_0 = (2/\rho)^{1/2}$ and $a_1 = 0$.

- The general solution of Bessel's equation of order one-half is

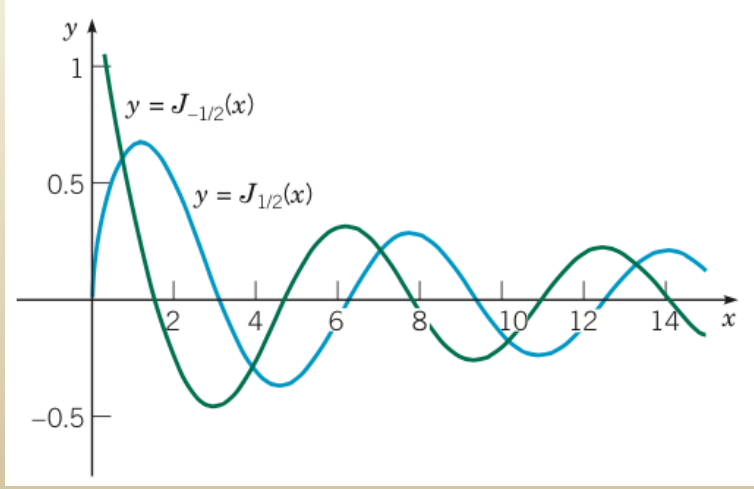
$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

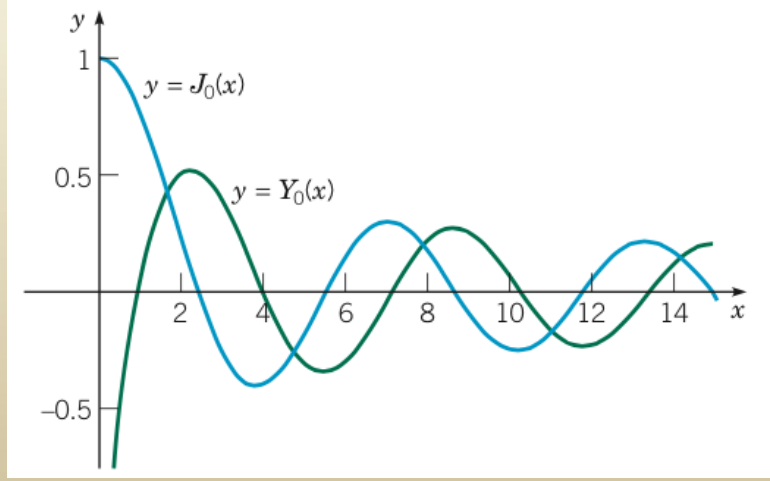
Graphs of Bessel Functions, Order One-Half (8 of 8)

- Graphs of $J_{1/2}$, $J_{-1/2}$ are given below. Note behavior of $J_{1/2}$, $J_{-1/2}$ similar to J_0, Y_0 for large x, with phase shift of $\pi/4$.

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right), \qquad Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$





Bessel Equation of Order One (1 of 6)

- The Bessel Equation of order one is

$$x^2 y'' + xy' + \left(x^2 - 1\right)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

Recurrence Relation (2 of 6)

- Using the results of the previous slide, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-1\right]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-1\right)a_0x^r+\left[(r+1)^2-1\right]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of indicial equation are $r_1 = 1$, $r_2 = -1$, and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1},\quad n=2,3,\ldots$$

First Solution: Coefficients (3 of 6)

- Consider first the case $r_1 = 1$. From previous slide,

$$\left(r^2-1\right)a_0x^r+\left[(r+1)^2-1\right]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- Since $r_1 = 1$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have

$$a_{2m}=-\frac{a_{2m-2}}{(1+2m)^2-1}=-\frac{a_{2m-2}}{2^2(m+1)m},\quad m=1,2,\ldots$$

- It follows that

$$a_2=-\frac{a_0}{2^2\cdot 2\cdot 1},\ a_4=-\frac{a_2}{2^2\cdot 3\cdot 2}=\frac{a_0}{2^4\,3!2!},\ldots$$

and

$$a_{2m}=\frac{(-1)^m a_0}{2^{2m}(m+1)!m!},\quad m=1,2,\ldots$$

Bessel Function of First Kind, Order One (4 of 6)

- It follows that the first solution of our differential equation is

$$y_1(x) = a_0 x\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^{2m}(m+1)!m!}x^{2m}\right],\ x>0$$

- Taking $a_0 = ½$, the **Bessel function of the first kind of order one**, J_1, is defined as

$$J_1(x) = \frac{x}{2}\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m}(m+1)!m!}x^{2m}\right],\ x>0$$

- The series converges for all x and hence J_1 is analytic everywhere.

Second Solution (5 of 6)

- For the case $r_1 = -1$, a solution of the form

$$y_2(x) = a\,J_1(x)\ln x + x^{-1}\left[1 + \sum_{n=1}^{\infty} c_n x^{2n}\right], \quad x > 0$$

  is guaranteed by Theorem 5.7.1.

- The coefficients c_n are determined by substituting y_2 into the ODE and obtaining a recurrence relation, etc. The result is:

$$y_2(x) = -J_1(x)\ln x + x^{-1}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m \left(H_m + H_{m-1}\right)}{2^{2m}\, m!(m-1)!} x^{2n}\right], \quad x > 0$$

  where H_k is as defined previously. See text for more details.

- Note that $J_1 \to 0$ as $x \to 0$ and is analytic at $x = 0$, while y_2 is unbounded as $x \to 0$ in the same manner as $1/x$.

Bessel Function of Second Kind, Order One (6 of 6)

- The second solution, the **Bessel function of the second kind of order one,** is usually taken to be the function

$$Y_1(x) = \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2)J_1(x)\right], \quad x > 0$$

where γ is the Euler-Mascheroni constant.

- The general solution of Bessel's equation of order one is

$$y(x) = c_1 J_1(x) + c_2 Y_1(x), \quad x > 0$$

- Note that J_1, Y_1 have same behavior at $x = 0$ as observed on previous slide for J_1 and y_2.