

# Boyce/DiPrima/Meade 11<sup>th</sup> ed, Ch 4.1: Higher Order Linear ODEs: General Theory

Elementary Differential Equations and Boundary Value Problems, 11<sup>th</sup> edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade ©2017 by John Wiley & Sons, Inc.

- An ***n*th order ODE** has the general form

$$P_0(t) \frac{d^n y}{dt^n} + P_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t) \frac{dy}{dt} + P_n(t) y = G(t)$$

- We assume that  $P_0, \dots, P_n$ , and  $G$  are continuous real-valued functions on some interval  $I = (a, b)$ , and that  $P_0$  is nowhere zero on  $I$ .
- Dividing by  $P_0$ , the ODE becomes

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t)$$

- For an *n*th order ODE, there are typically *n* initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

## Theorem 4.1.1

- Consider the  $n$ th order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t)$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

- If the functions  $p_1, \dots, p_n$ , and  $g$  are continuous on an open interval  $I$ , then there exists exactly one solution  $y = f(t)$  that satisfies the initial value problem. This solution exists throughout the interval  $I$ .

# Homogeneous Equations

- As with 2<sup>nd</sup> order case, we begin with homogeneous ODE:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = 0$$

- If  $y_1, \dots, y_n$  are solns to ODE, then so is linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Every soln can be expressed in this form, with coefficients determined by initial conditions, iff we can solve:

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0$$

$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0'$$

⋮

$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

# Homogeneous Equations & Wronskian

- The system of equations on the previous slide has a unique solution iff its determinant, or Wronskian, is nonzero at  $t_0$ :

$$W(y_1, y_2, \dots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

- Since  $t_0$  can be any point in the interval  $I$ , the Wronskian determinant needs to be nonzero at every point in  $I$ .
- As before, it turns out that the Wronskian is either zero for every point in  $I$ , or it is never zero on  $I$ .

## Theorem 4.1.2

- Consider the  $n$ th order initial value problem

$$\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = 0$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \dots, \quad y^{(n-1)}(t_0) = y^{(n-1)}_0$$

- If the functions  $p_1, \dots, p_n$  are continuous on an open interval  $I$ , and if  $y_1, \dots, y_n$  are solutions with  $W(y_1, \dots, y_n)(t) \neq 0$  for at least one  $t$  in  $I$ , then every solution  $y$  of the ODE can be expressed as a linear combination of  $y_1, \dots, y_n$ :

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

# Linear Dependence and Independence

- Two functions  $f$  and  $g$  are **linearly dependent** if there exist constants  $c_1$  and  $c_2$ , not both zero, such that

$$c_1 f(t) + c_2 g(t) = 0$$

for all  $t$  in  $I$ . Note that this reduces to determining whether  $f$  and  $g$  are multiples of each other.

- If the only solution to this equation is  $c_1 = c_2 = 0$ , then  $f$  and  $g$  are **linearly independent**.
- For example, let  $f(x) = \sin 2x$  and  $g(x) = \sin x \cos x$ , and consider the linear combination

$$c_1 \sin 2x + c_2 \sin x \cos x = 0$$

This equation is satisfied if we choose  $c_1 = 1$ ,  $c_2 = -2$ , and hence  $f$  and  $g$  are linearly dependent.

# Example 1

- Are the following functions linearly independent or dependent on the interval I:  $0 < t < \infty$

$$f_1(t) = 1, f_2(t) = t, f_3(t) = t^2$$

- Form the linear combination and set it equal to zero

$$k_1 + k_2 t + k_3 t^2 = 0$$

- Evaluating this at  $t = 0$ ,  $t = 1$ , and  $t = -1$ , we get

$$k_1 = 0$$

$$k_1 + k_2 + k_3 = 0$$

$$k_1 - k_2 + k_3 = 0$$

- The only solution to this system is  $k_1 = k_2 = k_3 = 0$
- Therefore, the given functions are linearly independent

## Example 2

- Are the following functions linearly independent or dependent on any interval I:

$$f_1(t) = 1, f_2(t) = 2 + t, f_3(t) = 3 - t^2, f_4(t) = 4t + t^2$$

- Form the linear combination and set it equal to zero

$$k_1 + k_2(2 + t) + k_3(3 - t^2) + k_4(4t + t^2) = 0$$

- Evaluating this at  $t = 0$ ,  $t = 1$ , and  $t = -1$ , we get

$$k_1 + 2k_2 + k_3 = 0$$

$$k_2 + 4k_4 = 0$$

$$-k_3 + k_4 = 0$$

- There are many nonzero solutions to this system of equations
- Therefore, the given functions are linearly dependent



## Theorem 4.1.3

- If  $\{y_1, \dots, y_n\}$  is a fundamental set of solutions of

$$L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval  $I$ , then  $\{y_1, \dots, y_n\}$  are linearly independent on that interval.

- Conversely, if  $\{y_1, \dots, y_n\}$  are linearly independent solutions to the above differential equation, then they form a fundamental set of solutions on the interval  $I$

# Fundamental Solutions & Linear Independence

- Consider the  $n$ th order ODE:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

- A set  $\{y_1, \dots, y_n\}$  of solutions with  $W(y_1, \dots, y_n) \neq 0$  on  $I$  is called a **fundamental set of solutions**.
- Since all solutions can be expressed as a linear combination of the fundamental set of solutions, the **general solution** is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- If  $y_1, \dots, y_n$  are fundamental solutions, then  $W(y_1, \dots, y_n) \neq 0$  on  $I$ . It can be shown that this is equivalent to saying that  $y_1, \dots, y_n$  are **linearly independent**:

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0 \text{ iff } c_1 = c_2 = \cdots = c_n = 0$$

# Nonhomogeneous Equations

- Consider the nonhomogeneous equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t)$$

- If  $Y_1, Y_2$  are solns to nonhomogeneous equation, then  $Y_1 - Y_2$  is a solution to the homogeneous equation:

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$$

- Then there exist coefficients  $c_1, \dots, c_n$  such that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

- Thus the general solution to the nonhomogeneous ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)$$

where  $Y$  is any particular solution to nonhomogeneous ODE.

# Boyce/DiPrima/Meade 11<sup>th</sup> ed, Ch 4.2: Homogeneous Differential Equations with Constant Coefficients

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- Consider the  $n$ th order linear homogeneous differential equation with constant, real coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- As with second order linear equations with constant coefficients,  $y = e^{rt}$  is a solution for values of  $r$  that make characteristic polynomial  $Z(r)$  zero:

$$L[e^{rt}] = e^{rt} \underbrace{\left[ a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \right]}_{\text{characteristic polynomial } Z(r)} = 0$$

- By the fundamental theorem of algebra, a polynomial of degree  $n$  has  $n$  roots  $r_1, r_2, \dots, r_n$ , and hence

$$Z(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_n)$$

# Real and Unequal Roots

- If roots of characteristic polynomial  $Z(r)$  are real and unequal, then there are  $n$  distinct solutions of the differential equation:

$$e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t}$$

- If these functions are linearly independent, then general solution of differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \dots + c_n e^{r_n t}$$

- The Wronskian can be used to determine linear independence of solutions.

## Example 1: Distinct Real Roots (1 of 3)

- Consider the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1$$

- Assuming exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \Rightarrow r^4 + r^3 - 7r^2 - r + 6 = 0$$

$$\Leftrightarrow (r-1)(r+1)(r-2)(r+3) = 0$$

- Thus the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$$

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$$

## Example 1: Solution (2 of 3)

- The initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = -1$$

yield

$$c_1 + c_2 + c_3 + c_4 = 1$$

$$c_1 - c_2 + 2c_3 - 3c_4 = 0$$

$$c_1 + c_2 + 4c_3 + 9c_4 = -2$$

$$c_1 - c_2 + 8c_3 - 27c_4 = -1$$

- Solving,

$$c_1 = \frac{11}{8}, c_2 = \frac{5}{12}, c_3 = -\frac{2}{3}, c_4 = -\frac{1}{8}$$

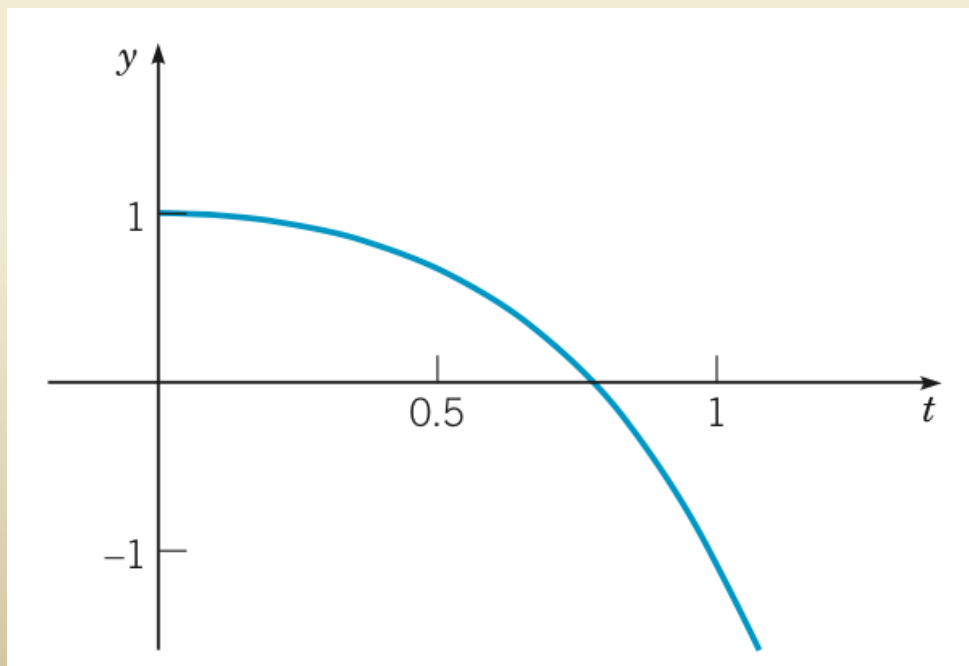
- Hence

$$y(t) = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}$$

## Example 1: Graph of Solution (3 of 3)

- The graph of the solution is given below. Note the effect of the largest root of the characteristic equation.

$$y(t) = \frac{11}{8}e^t + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$





# Complex Roots

- If the characteristic polynomial  $Z(r)$  has complex roots, then they must occur in conjugate pairs,  $\lambda \pm im$
- Note that not all the roots need be complex.
- Solutions corresponding to complex roots have the form

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cos \mu t + ie^{\lambda t} \sin \mu t$$

$$e^{(\lambda-i\mu)t} = e^{\lambda t} \cos \mu t - ie^{\lambda t} \sin \mu t$$

- As in Chapter 3.4, we use the real-valued solutions

$$e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t$$

## Example 2: Complex Roots (1 of 2)

- Consider the initial value problem

$$y^{(4)} - y = 0, \quad y(0) = 7/2, \quad y'(0) = -4, \quad y''(0) = 5/2, \quad y'''(0) = -2$$

- Then

$$y(t) = e^{rt} \Rightarrow r^4 - 1 = 0 \Leftrightarrow (r^2 - 1)(r^2 + 1) = 0$$

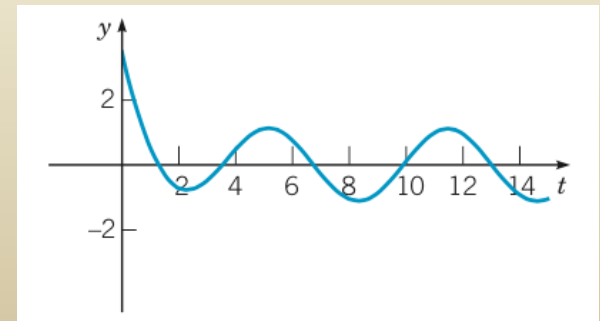
- The roots are 1, -1,  $i$ ,  $-i$ . Thus the general solution is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

- Using the initial conditions, we obtain

$$y(t) = 0e^t + 3e^{-t} + \frac{1}{2} \cos(t) - \sin(t)$$

- The graph of solution is given on right.



$$y(t) = 0e^t + 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

## Example 2:

### Small Change in an Initial Condition (2 of 2)

- Note that if one initial condition is slightly modified, then the solution can change significantly. For example, replace

$$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -2$$

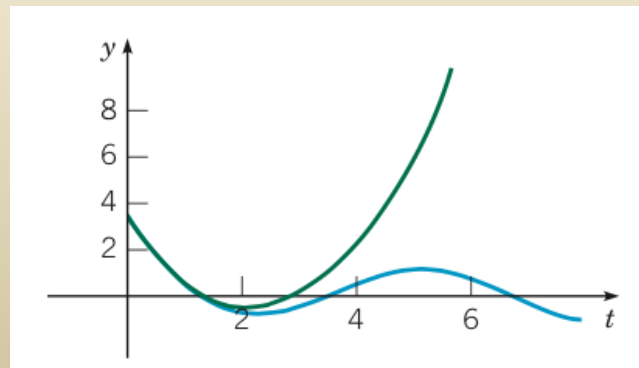
with

$$y(0) = 7/2, y'(0) = -4, y''(0) = 5/2, y'''(0) = -15/8$$

then

$$y(t) = \frac{1}{32}e^t + \frac{95}{32}e^{-t} + \frac{1}{2}\cos(t) - \frac{17}{16}\sin(t)$$

- The graph of this solution and original are given below.



# Repeated Roots

- Suppose a root  $r_k$  of characteristic polynomial  $Z(r)$  is a repeated root with multiplicity  $s$ . Then linearly independent solutions corresponding to this repeated root have the form

$$e^{r_k t}, te^{r_k t}, t^2 e^{r_k t}, \dots, t^{s-1} e^{r_k t}$$

- If a complex root  $\lambda + im$  is repeated  $s$  times, then so is its conjugate  $\lambda - im$ . There are  $2s$  corresponding linearly independent solns, derived from real and imaginary parts of

$$e^{(\lambda+iu)t}, te^{(\lambda+iu)t}, t^2 e^{(\lambda+iu)t}, \dots, t^{s-1} e^{(\lambda+iu)t}$$

or

$$e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t, te^{\lambda t} \cos \mu t, te^{\lambda t} \sin \mu t, \dots, \\ t^{s-1} e^{r_k t} \cos \mu t, t^{s-1} e^{r_k t} e^{\lambda t} \sin \mu t,$$

## Example 3: Repeated Roots

- Consider the equation

$$y^{(4)} + 2y'' + y = 0$$

- Then

$$y(t) = e^{rt} \Rightarrow r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2 + 1)(r^2 + 1) = 0$$

- The roots are  $i, i, -i, -i$ . Thus the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)$$

## Example 4: Complex Roots of $-1$ (1 of 2)

- For the general solution of  $y^{(4)} + y = 0$ , the characteristic equation is  $r^4 + 1 = 0$ .
- To solve this equation, we need to use Euler's equation to find the four 4<sup>th</sup> roots of  $-1$ :

$$-1 = \cos \rho + i \sin \rho = e^{i\rho} \text{ or}$$

$$-1 = \cos(\rho + 2m\rho) + i \sin(\rho + 2m\rho) = e^{i(\rho + 2m\rho)} \text{ for any integer } m$$

$$(-1)^{1/4} = e^{i(\rho + 2m\rho)/4} = \cos\left(\frac{\rho}{4} + \frac{m\rho}{2}\right) + i \sin\left(\frac{\rho}{4} + \frac{m\rho}{2}\right)$$

- Letting  $m = 0, 1, 2,$  and  $3,$  we get the roots:

$$\frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \text{ respectively.}$$

$$r = \left\{ \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}} \right\}$$

## Example 4: Complex Roots of $-1$ (2 of 2)

- Given the four complex roots, extending the ideas from Chapter 4, we can form four linearly independent real solutions.

- For the complex conjugate pair  $\frac{1 \pm i}{\sqrt{2}}$ , we get the solutions

$$y_1 = e^{t/\sqrt{2}} \cos(t/\sqrt{2}), \quad y_2 = e^{t/\sqrt{2}} \sin(t/\sqrt{2})$$

- For the complex conjugate pair  $\frac{-1 \pm i}{\sqrt{2}}$ , we get the solutions

$$y_3 = e^{-t/\sqrt{2}} \cos(t/\sqrt{2}), \quad y_4 = e^{-t/\sqrt{2}} \sin(t/\sqrt{2})$$

- So the general solution can be written as

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$

# Boyce/DiPrima/Meade 11<sup>th</sup> ed, Ch 4.3: The Method of Undetermined Coefficients

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- The method of undetermined coefficients can be used to find a particular solution  $Y$  of an  $n$ th order linear, constant coefficient, nonhomogeneous ODE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

provided  $g$  is of an appropriate form.

- As with 2<sup>nd</sup> order equations, the method of undetermined coefficients is typically used when  $g$  is a sum or product of polynomial, exponential, and sine or cosine functions.
- Section 4.4 discusses the more general variation of parameters method.



## Example 1

- Consider the differential equation

$$y''' - 3y'' + 3y' - y = 4e^t$$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow (r-1)^3 = 0$$

- Thus the general solution of homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

- For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with

$$Y(t) = At^3 e^{2t}$$

- As in Chapter 3, it can be shown that

$$Y(t) = \frac{2}{3} t^3 e^{2t} \Rightarrow y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^{2t}$$

## Example 2

- Consider the equation  $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^4 + 2r^2 + 1 = 0 \Leftrightarrow (r^2 + 1)(r^2 + 1) = 0$$

- Thus the general solution of the homogeneous equation is

$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)$$

- For the nonhomogeneous case, because of the form of the solution for the homogeneous equation, we need

$$Y(t) = t^2(A \sin t + B \cos t)$$

- As in Chapter 3, it can be shown that  $Y(t) = -\frac{3}{8} \sin t + \frac{5}{8} \cos t$

- Thus, the general solution for the nonhomogeneous equation is

$$y(t) = y_c(t) + Y(t)$$

## Example 3

- Consider the equation

$$y''' - 4y' = t + 3\cos t + e^{-2t}$$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 4r = 0 \Leftrightarrow r(r^2 - 4) \Leftrightarrow r(r-2)(r+2) = 0$$

- Thus the general solution of homogeneous equation is

$$y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$$

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases:

$$Y_1(t) = (A + Bt)t, Y_2(t) = C \cos t + D \sin t, Y_3(t) = Ete^{2t},$$

- As in Chapter 3, can be shown that  $Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}$
- The general solution is  $y(t) = y_c(t) + Y(t)$

# Boyce/DiPrima/Meade 11<sup>th</sup> ed, Ch 4.4: The Method of Variation of Parameters

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- The variation of parameters method can be used to find a particular solution of the nonhomogeneous  $n$ th order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t),$$

provided  $g$  is continuous.

- As with 2<sup>nd</sup> order equations, begin by assuming  $y_1, y_2, \dots, y_n$  are fundamental solutions to homogeneous equation.
- Next, assume the particular solution  $Y$  has the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)$$

where  $u_1, u_2, \dots, u_n$  are functions to be solved for.

- In order to find these  $n$  functions, we need  $n$  equations.

## Variation of Parameters Derivation (2 of 5)

- First, consider the derivatives of  $Y$ :

$$Y' = (u'_1 y_1 + u'_2 y_2 + \cdots + u'_n y_n) + (u_1 y'_1 + u_2 y'_2 + \cdots + u_n y'_n)$$

- If we require

$$u'_1 y_1 + u'_2 y_2 + \cdots + u'_n y_n = 0$$

then

$$Y'' = (u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n) + (u_1 y''_1 + u_2 y''_2 + \cdots + u_n y''_n)$$

- Thus we next require

$$u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n = 0$$

- Continuing in this way, we require

$$u'_1 y_1^{(m)} + u'_2 y_2^{(m)} + \cdots + u'_n y_n^{(m)} = 0, \quad m = 1, \dots, n-2$$

and hence

$$Y^{(m)} = u_1 y_1^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 2, 3, \dots, n-1$$

## Variation of Parameters Derivation (3 of 5)

- From the previous slide,

$$Y^{(m)} = u_1 y_1^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 2, 3, \dots, n-1$$

- Finally,

$$Y^{(n)} = \left( u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \right) + \left( u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} \right)$$

- Next, substitute these derivatives into our equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

- Recalling that  $y_1, y_2, \dots, y_n$  are solutions to homogeneous equation, and after rearranging terms, we obtain

$$u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

## Variation of Parameters Derivation (4 of 5)

- The  $n$  equations needed in order to find the  $n$  functions  $u_1, u_2, \dots, u_n$  are

$$u_1' y_1 + \cdots + u_n' y_1 = 0$$

$$u_1' y_1' + \cdots + u_n' y_n' = 0$$

$$\vdots$$

$$u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

- Using Cramer's Rule, for each  $m = 1, \dots, n$ ,

$$u_m(t) = \frac{g(t)W_m(t)}{W(t)}, \text{ where } W(t) = W(y_1, \dots, y_n)(t)$$

and  $W_m$  is determinant obtained by replacing  $m$ th column of  $W$  with  $(0, 0, \dots, 1)$ .

# Variation of Parameters Derivation (5 of 5)

- From the previous slide,

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, \quad m = 1, \dots, n$$

- Integrate to obtain  $u_1, u_2, \dots, u_n$ :

$$u_m(t) = \int_{t_0}^t \frac{g(s)W_m(s)}{W(s)} ds, \quad m = 1, \dots, n$$

- Thus, a particular solution  $Y$  is given by

$$Y(t) = \sum_{m=1}^n \left[ \int_{t_0}^t \frac{g(s)W_m(s)}{W(s)} ds \right] y_m(t)$$

where  $t_0$  is arbitrary.



## Example 1 (1 of 3)

- Consider the equation below, along with the given solutions of corresponding homogeneous solutions  $y_1, y_2, y_3$ :

$$y''' - y'' - y' + y = g(t), \quad y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = e^{-t}$$

- Then a particular solution of this ODE is given by

$$Y(t) = \sum_{m=1}^3 \left[ \int_{t_0}^t \frac{e^{2s} W_m(s)}{W(s)} ds \right] y_m(t)$$

- It can be shown that

$$W(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = 4e^t$$

## Example 1 (2 of 3)

- Also,

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix} = -2t - 1$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = 2$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^t$$

## Example 1 (3 of 3)

- Thus a particular solution in integral form is

$$\begin{aligned} Y(t) &= \sum_{m=1}^3 \left[ \int_{t_0}^t \frac{g(s)W_m(s)}{W(s)} ds \right] y_m(t) \\ &= e^t \int_{t_0}^t \frac{g(s)(-2s-1)}{4e^s} ds + te^t \int_{t_0}^t \frac{g(s)2}{4e^s} ds + e^{-t} \int_{t_0}^t \frac{g(s)e^{2s}}{4e^s} ds \\ &= \frac{1}{4} \int_{t_0}^t \left[ e^{t-s}(-1+2(t-s)) + e^{-(t-s)} \right] g(s) ds \end{aligned}$$