Boyce/DiPrima/Meade 11th ed, Ch 3.1: 2nd Order Linear Homogeneous Equations-Constant Coefficients

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- A **second order ordinary differential equation** has the general form
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$
where f is some given function.
- This equation is said to be **linear** if f is linear in y and y':
$$y'' + p(t)y' + q(t)y = g(t)$$
Otherwise the equation is said to be **nonlinear**.
- A second order linear equation often appears as
$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$
- If $g(t)$ or $G(t) = 0$ for all t, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.

Homogeneous Equations, Initial Values

- In Sections 3.5 and 3.6, we will see that once a solution to a homogeneous equation is found, then it is possible to solve the corresponding nonhomogeneous equation, or at least express the solution in terms of an integral.
- The focus of this chapter is thus on homogeneous equations; and in particular, those with constant coefficients:
$$ay'' + by' + cy = 0$$
We will examine the variable coefficient case in Chapter 5.
- Initial conditions typically take the form
$$y(t_0) = y_0,\; y'(t_0) = y'_0$$
- Thus solution passes through (t_0, y_0), and the slope of solution at (t_0, y_0) is equal to y_0'.

Example 1: Infinitely Many Solutions

- Consider the second order linear differential equation

$$y'' - y = 0$$

- Two solutions of this equation are

$$y_1(t) = e^t, \quad y_2(t) = e^{-t}$$

- Other solutions include

$$y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$$

- Based on these observations, we see that there are infinitely many solutions of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- It will be shown in Section 3.2 that all solutions of the differential equation above can be expressed in this form.

Example 1: Initial Conditions (2 of 3)

- Now consider the following initial value problem for our equation:

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

- We have found a general solution of the form

$$y(t) = c_1 e^t + c_2 e^{-t}$$

- Using the initial equations,

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow c_1 = \frac{1}{2},\ c_2 = \frac{3}{2}$$

- Thus

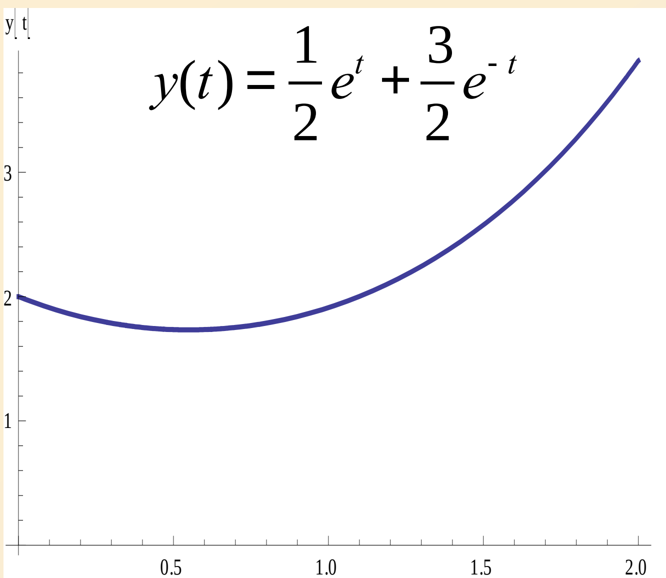
$$y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$$

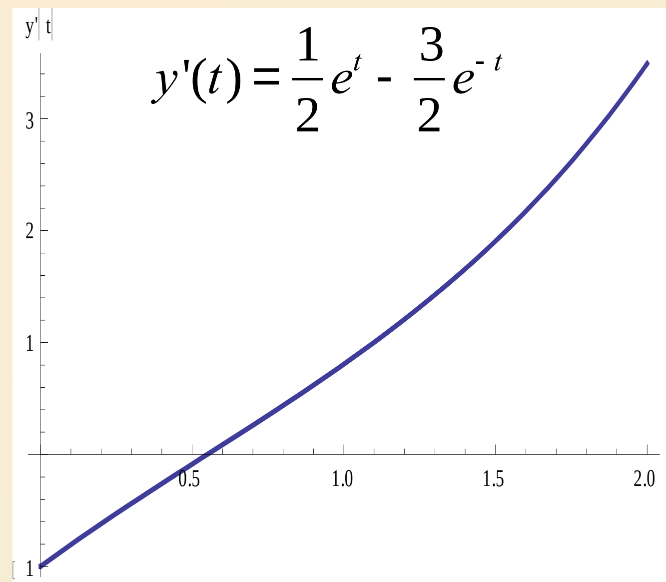
Example 1: Solution Graphs (3 of 3)

- Our initial value problem and solution are

$$y'' - y = 0, \quad y(0) = 2, \; y'(0) = -1 \;\Rightarrow\; y(t) = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$$

- Graphs of both y(t) and y′(t) are given below. Observe that both initial conditions are satisfied.





Characteristic Equation

- To solve the 2nd order equation with constant coefficients,
$$ay'' + by' + cy = 0,$$
we begin by assuming a solution of the form $y = e^{rt}$.
- Substituting this into the differential equation, we obtain
$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
- Simplifying,
$$e^{rt}(ar^2 + br + c) = 0$$
and hence
$$ar^2 + br + c = 0$$
- This last equation is called the **characteristic equation** of the differential equation.
- We then solve for r by factoring or using quadratic formula.

General Solution

- Using the quadratic formula on the characteristic equation
$$ar^2 + br + c = 0,$$
we obtain two solutions, r_1 and r_2.
- There are three possible results:
  - The roots r_1, r_2 are real and $r_1 \neq r_2$.
  - The roots r_1, r_2 are real and $r_1 = r_2$.
  - The roots r_1, r_2 are complex.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- In this section, we will assume r_1, r_2 are real and $r_1 \neq r_2$.
- In this case, the general solution has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Initial Conditions

- For the initial value problem

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

  we use the general solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  together with the initial conditions to find c_1 and c_2. That is,

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0 \end{aligned}\right\} \Rightarrow c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}$$

- Since we are assuming $r_1 \neq r_2$, it follows that a solution of the form $y = e^{rt}$ to the above initial value problem will always exist, for any set of initial conditions.

Example 2 (General Solution)

- Consider the linear differential equation

$$y'' + 5y' + 6y = 0$$

- Assuming an exponential solution leads to the characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 6 = 0 \Leftrightarrow (r+2)(r+3) = 0$$

- Factoring the characteristic equation yields two solutions: $r_1 = -2$ and $r_2 = -3$
- Therefore, the general solution to this differential equation has the form

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

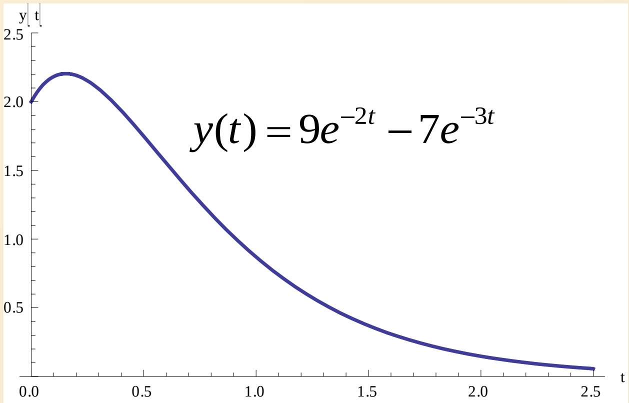
Example 3 (Particular Solution)

- Consider the initial value problem

$$y''+5y'+6y=0,\ y(0)=2,\ y'(0)=3$$

- From the preceding example, we know the general solution has the form: $y(t)=c_1e^{-2t}+c_2e^{-3t}$
- With derivative: $y'(t)=-2c_1e^{-2t}-3c_2e^{-3t}$
- Using the initial conditions:

$$\left.\begin{aligned} c_1+c_2&=1 \\ -2c_1-3c_2&=3 \end{aligned}\right\}\Rightarrow c_1=9,\ c_2=-7$$



- Thus $y(t)=9e^{-2t}-7e^{-3t}$

Example 4: Initial Value Problem

- Consider the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \; y'(0) = \frac{1}{2}$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad 4r^2 - 8r + 3 = 0 \Leftrightarrow (2r-3)(2r-1) = 0$$

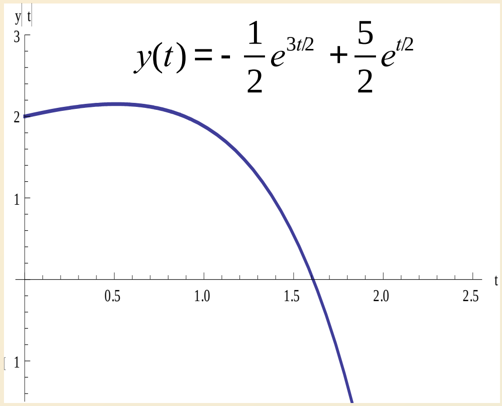
- Factoring yields two solutions, $r_1 = \frac{3}{2}$ and $r_2 = \frac{1}{2}$
- The general solution has the form

$$y(t) = c_1 e^{3t/2} + c_2 e^{t/2}$$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 &= \frac{1}{2} \end{aligned}\right\} \Rightarrow c_1 = -\frac{1}{2}, \; c_2 = \frac{5}{2}$$

- Thus $y(t) = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}$



Example 5: Find Maximum Value

- For the initial value problem in Example 3, to find the maximum value attained by the solution, we set $y'(t) = 0$ and solve for t:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'(t) = -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0$$

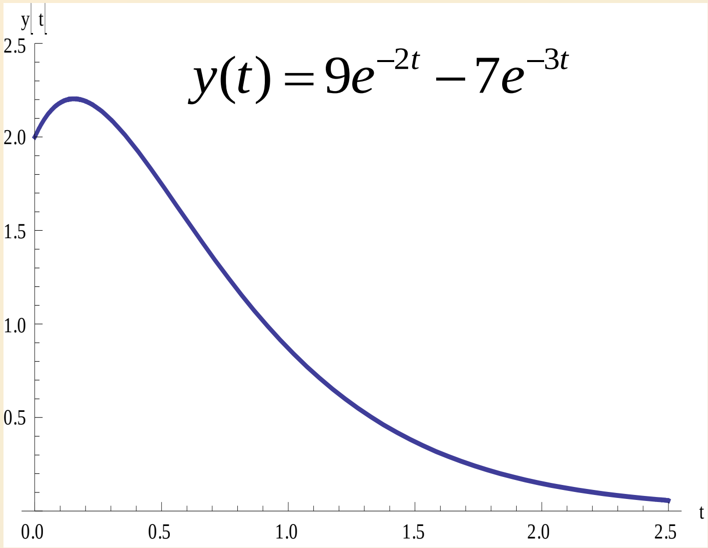
$$6e^{-2t} = 7e^{-3t}$$

$$e^{t} = 7/6$$

$$t = \ln(7/6)$$

$$t \approx 0.1542$$

$$y \approx 2.204$$



Boyce/DiPrima/Meade 11th ed, Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

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- Let *p*, *q* be continuous functions on an interval $I = (a, b)$ which could be infinite. For any function *y* that is twice differentiable on *I*, define the differential operator *L* by

$$L[y] = y'' + py' + qy$$

- Note that $L[y]$ is a function on *I*, with output value

$$L[y](t) = y''(t) + p(t)y'(t) + q(t)y(t)$$

- For example,

$$p(t) = t^2,\ q(t) = e^{2t}, y(t) = \sin(t),\ \ I = (0, 2\pi)$$
$$L[y](t) = -\sin(t) + t^2\cos(t) + 2e^{2t}\sin(t)$$

Differential Operator Notation

- In this section we will discuss the second order linear homogeneous equation $L[y](t) = 0$, along with initial conditions as indicated below:

$$L\left[y\right] = y'' + p(t)\,y' + q(t)\,y = 0$$
$$y(t_0) = y_0,\; y'(t_0) = y_1$$

- We would like to know if there are solutions to this initial value problem, and if so, are they unique.
- Also, we would like to know what can be said about the form and structure of solutions that might be helpful in finding solutions to particular problems.
- These questions are addressed in the theorems of this section.

Theorem 3.2.1 (Existence and Uniqueness)

- Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

- where p, q, and g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = f(t)$ on I.

- Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equations.

Example 1

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y_1$$

- Consider the second order linear initial value problem
$$(t^2 - 3t)y'' + ty' - (t+3)y = 0,\ y(1) = 2,\ y'(1) = 1$$
- Writing the differential equation in the form :
$$y'' + p(t)y' + q(t)y = g(t)$$
$$p(t) = \frac{1}{t-3},\ q(t) = -\frac{t+3}{t(t-3)} \text{ and } g(t) = 0$$
- The only points of discontinuity for these coefficients are $t = 0$ and $t = 3$. So the longest open interval containing the initial point $t = 1$ in which all the coefficients are continuous is $0 < t < 3$
- Therefore, the longest interval in which Theorem 3.2.1 guarantees the existence of the solution is $0 < t < 3$

Example 2

- Consider the second order linear initial value problem
$$y'' + p(t)y' + q(t)y = 0,\ y(0) = 0,\ y'(0) = 0$$
where p, q are continuous on an open interval I containing t_0.

- In light of the initial conditions, note that $y = 0$ is a solution to this homogeneous initial value problem.

- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that $y = 0$ is the only solution of this problem.

Theorem 3.2.2 (Principle of Superposition)

- If y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

  then the linear combination $c_1y_1 + y_2c_2$ is also a solution, for all constants c_1 and c_2.

- To prove this theorem, substitute $c_1y_1 + y_2c_2$ in for y in the equation above, and use the fact that y_1 and y_2 are solutions.
- Thus for any two solutions y_1 and y_2, we can construct an infinite family of solutions, each of the form $y = c_1y_1 + c_2 y_2$.
- Can all solutions can be written this way, or do some solutions have a different form altogether? To answer this question, we use the Wronskian determinant.

The Wronskian Determinant (1 of 3)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

- From Theorem 3.2.2, we know that $y = c_1y_1 + c_2y_2$ is a solution to this equation.
- Next, find coefficients such that $y = c_1y_1 + c_2y_2$ satisfies the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

- To do so, we need to solve the following equations:

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0 \\ c_1y'_1(t_0) + c_2y'_2(t_0) &= y'_0 \end{aligned}$$

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

The Wronskian Determinant (2 of 3)

- Solving the equations, we obtain

$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$
$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}$$

- In terms of determinants:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

The Wronskian Determinant (3 of 3)

- In order for these formulas to be valid, the determinant W in the denominator cannot be zero:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$

- W is called the **Wronskian determinant**, or more simply, the Wronskian of the solutions y_1 and y_2. We will sometimes use the notation

$$W(y_1, y_2)(t_0)$$

Theorem 3.2.3

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

with the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0'$$

Then it is always possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and initial conditions if and ony if the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

is not zero at the point t_0

Example 3

- In Example 2 of Section 3.1, we found that

$$y_1(t) = e^{-2t} \text{ and } y_2(t) = e^{-3t}$$

were solutions to the differential equation

$$y'' + 5y' + 6y = 0$$

- The Wronskian of these two functions is

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

- Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the differential equation with initial conditions at any value of t

Theorem 3.2.4 (Fundamental Solutions)

- Suppose y_1 and y_2 are solutions to the equation
$$L[y] = y'' + p(t)y' + q(t)y = 0.$$
Then the family of solutions
$$y = c_1 y_1 + c_2 y_2$$
with arbitrary coefficients c_1, c_2 includes every solution to the differential equation if an only if there is a point t_0 such that $W(y_1,y_2)(t_0) \neq 0$, .

- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.

Example 4

- Consider the general second order linear equation below, with the two solutions indicated:

$$y'' + p(t)\, y' + q(t)\, y = 0$$

- Suppose the functions below are solutions to this equation:

$$y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \quad r_1 \neq r_2$$

- The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0 \text{ for all } t.$$

- Thus y_1 and y_2 form a fundamental set of solutions to the equation, and can be used to construct all of its solutions.
- The general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Example 5: Solutions (1 of 2)

- Consider the following differential equation:

$$2t^2 y'' + 3t y' - y = 0, \; t > 0$$

- Show that the functions below are fundamental solutions:

$$y_1 = t^{1/2}, y_2 = t^{-1}$$

- To show this, first substitute y_1 into the equation:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0$$

- Thus y_1 is a indeed a solution of the differential equation.
- Similarly, y_2 is also a solution:

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$$

Example 5: Fundamental Solutions (2 of 2)

- Recall that

$$y_1 = t^{1/2}, y_2 = t^{-1}$$

- To show that y_1 and y_2 form a fundamental set of solutions, we evaluate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}$$

- Since $W \neq 0$ for $t > 0$, y_1 and y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\, y' - y = 0, \; t > 0$$

Theorem 3.2.5: Existence of Fundamental Set of Solutions

- Consider the differential equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

- Let t_0 be a point in I, and y_1 and y_2 solutions of the equation with y_1 satisfying initial conditions

$$y_1(t_0) = 1,\ y'_1(t_0) = 0$$

and y_2 satisfying initial conditions

$$y_2(t_0) = 0,\ y'_2(t_0) = 1$$

- Then y_1 and y_2 form a fundamental set of solutions to the given differential equation.

Example 6: Apply Theorem 3.2.5 (1 of 3)

- Find the fundamental set specified by Theorem 3.2.5 for the differential equation and initial point

$$y'' - y = 0, \quad t_0 = 0$$

- In Section 3.1, we found two solutions of this equation:

$$y_1 = e^t, y_2 = e^{-t}$$

  The Wronskian of these solutions is $W(y_1, y_2)(t_0) = -2 \neq 0$ so they form a fundamental set of solutions.
- But these two solutions do not satisfy the initial conditions stated in Theorem 3.2.5, and thus they do not form the fundamental set of solutions mentioned in that theorem.
- Let y_3 and y_4 be the fundamental solutions of Thm 3.2.5.

$$y_3(0) = 1, \; y_3'(0) = 0; \quad y_4(0) = 0, \; y_4'(0) = 1$$

Example 6: General Solution (2 of 3)

- Since y_1 and y_2 form a fundamental set of solutions,

$$y_3 = c_1 e^t + c_2 e^{-t}, \quad y_3(0) = 1, y_3'(0) = 0$$

$$y_4 = d_1 e^t + d_2 e^{-t}, \quad y_4(0) = 0, y_4'(0) = 1$$

- Solving each equation, we obtain

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh(t), \quad y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

- The Wronskian of y_3 and y_4 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0$$

- Thus y_3, y_4 form the fundamental set of solutions indicated in Theorem 3.2.5, with general solution in this case

$$y(t) = k_1 \cosh(t) + k_2 \sinh(t)$$

Example 6:
Many Fundamental Solution Sets (3 of 3)

- Thus

$$S_1 = \left\{ e^t, e^{-t} \right\}, \quad S_2 = \left\{ \cosh t, \sinh t \right\}$$

  both form fundamental solution sets to the differential equation and initial point

$$y'' - y = 0, \quad t_0 = 0$$

- In general, a differential equation will have infinitely many different fundamental solution sets. Typically, we pick the one that is most convenient or useful.

Theorem 3.2.6

Consider again the equation (2):

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where *p* and *q* are continuous real-valued functions.

If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part *u* and its imaginary part *v* are also solutions of this equation.

Theorem 3.2.7 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on some open interval I. Then the $W[y_1,y_2](t)$ is given by

$$W[y_1, y_2](t) = ce^{-\int p(t)dt}$$

where c is a constant that depends on y_1 and y_2 but not on t.

- Note that $W[y_1,y_2](t)$ is either zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Example 7 Apply Abel's Theorem

- Recall the following differential equation and its solutions:

$$2t^2 y'' + 3t\,y' - y = 0,\ t > 0 \quad \text{with solutions} \quad y_1 = t^{1/2}, y_2 = t^{-1}$$

- We computed the Wronskian for these solutions to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Writing the differential equation in the standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,\ t > 0$$

- So $p(t) = \dfrac{3}{2t}$ and the Wronskian given by Thm.3.2.6 is

$$W[y_1, y_2](t) = ce^{-\int \frac{3}{2t}dt} = ce^{-\frac{3}{2}\ln t} = ct^{-3/2}$$

- This is the Wronskian for any pair of fundamental solutions. For the solutions given above, we must let $c = -3/2$

Summary

- To find a general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad a < t < b$$

  we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W[y_1, y_2](t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.

Boyce/DiPrima/Meade 11th ed, Ch 3.3: Complex Roots of Characteristic Equation

- Recall our discussion of the equation

$$ay'' + by' + cy = 0$$

where a, b and c are constants.

- Assuming an exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then complex roots: $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

Thus

$$y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}$$

Euler's Formula; Complex Valued Solutions

- Substituting *it* into Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

- Generalizing Euler's formula, we obtain

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

- Then

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left[\cos \mu t + i \sin \mu t\right] = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

- Therefore

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

$$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

Real Valued Solutions

- Our two solutions thus far are complex-valued functions:

$$y_1(t) = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$
$$y_2(t) = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

- We would prefer to have real-valued solutions, since our differential equation has real coefficients.
- To achieve this, recall that linear combinations of solutions are themselves solutions:

$$y_1(t) + y_2(t) = 2e^{\lambda t} \cos \mu t$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t} \sin \mu t$$

- Ignoring constants, we obtain the two solutions

$$y_3(t) = e^{\lambda t} \cos \mu t, \;\; y_4(t) = e^{\lambda t} \sin \mu t$$

Real Valued Solutions: The Wronskian

- Thus we have the following real-valued functions:

$$y_3(t) = e^{\lambda t}\cos\mu t, \;\; y_4(t) = e^{\lambda t}\sin\mu t$$

- Checking the Wronskian, we obtain

$$W = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}\left(\lambda\cos\mu t - \mu\sin\mu t\right) & e^{\lambda t}\left(\lambda\sin\mu t + \mu\cos\mu t\right) \end{vmatrix}$$

$$= \mu e^{2\lambda t} \neq 0$$

- Thus y_3 and y_4 form a fundamental solution set for our ODE, and the general solution can be expressed as

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$

Example 1 (1 of 2)

- Consider the differential equation

$$y'' + y' + 9.25y = 0$$

- For an exponential solution, the characteristic equation is

$$y(t) = e^{rt} \;\Rightarrow\; r^2 + r + 1 = 0 \;\Leftrightarrow\; r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

- Therefore, separating the real and imaginary components,

$$l = -\frac{1}{2},\; m = 3$$

and thus the general solution is

$$y(t) = c_1 e^{-t/2}\cos(3t) + c_2 e^{-t/2}\sin(3t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

Example 1 (2 of 2)

- Using the general solution just determined

$$y(t) = e^{-t/2}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

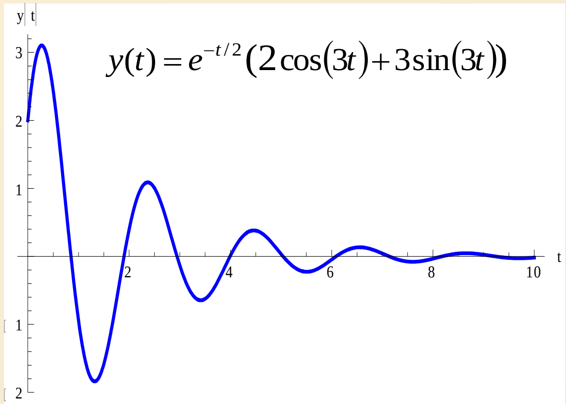
- We can determine the particular solution that satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$
- So

$$\left.\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= -\tfrac{1}{2}c_1 + 3c_2 = 8 \end{aligned}\right\} \Rightarrow c_1 = 2,\ c_2 = 3$$

- Thus the solution of this IVP is

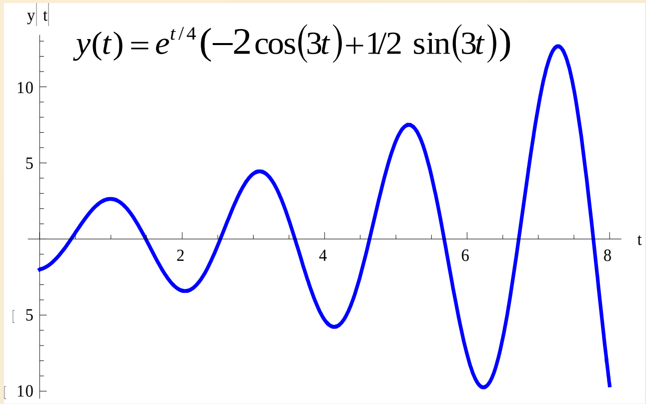
$$y(t) = e^{-t/2}\left(2\cos(3t) + 3\sin(3t)\right)$$

- The solution is a decaying oscillation



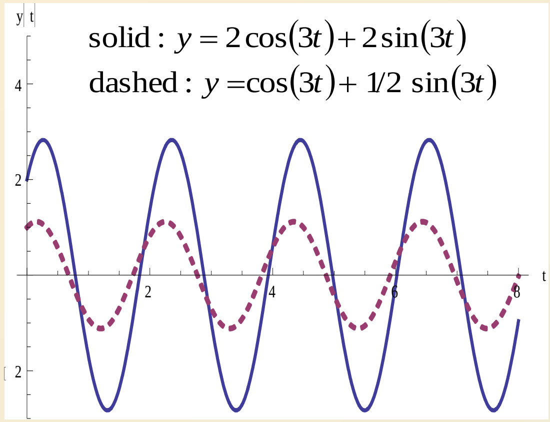
Example 2

- Consider the initial value problem
$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1$$
- Then $y(t) = e^{rt} \Rightarrow 16r^2 - 8r + 145 = 0 \Leftrightarrow r = \frac{1}{4} \pm 3i$
- Thus the general solution is $y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t)$
- And
$$\left.\begin{aligned} y(0) &= c_1 = -2 \\ y'(0) &= \frac{1}{4}c_1 + 3c_2 = 1 \end{aligned}\right\} \Rightarrow c_1 = -2,\ c_2 = \frac{1}{2}$$
- The solution of the IVP is
$$y(t) = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t)$$
- The solution is displays a growing oscillation



Example 3

- Consider the equation
$$y'' + 9y = 0$$
- Then $y(t) = e^{rt} \;\Rightarrow\; r^2 + 9 = 0 \;\Leftrightarrow\; r = \pm 3i$
- Therefore $\lambda = 0,\ \mu = 3$
- and thus the general solution is
$$y(t) = c_1 \cos(3t) + c_2 \sin(3t)$$
- Because $\lambda = 0$, there is no exponential factor in the solution, so the amplitude of each oscillation remains constant. The figure shows the graph of two typical solutions



Boyce/DiPrima/Meade 11th ed, Ch 3.4: Repeated Roots; Reduction of Order

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade ©2017 by John Wiley & Sons, Inc.

- Recall our 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

- where a, b and c are constants.
- Assuming an exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/(2a)$, since method only gives one solution:

$$y_1(t) = ce^{-bt/(2a)}$$

Second Solution: Multiplying Factor *v*(*t*)

- We know that

$$y_1(t) \text{ a solution} \Rightarrow y_2(t) = cy_1(t) \text{ a solution}$$

- Since y_1 and y_2 are linearly dependent, we generalize this approach and multiply by a function *v*, and determine conditions for which y_2 is a solution:

$$y_1(t) = e^{-bt/(2a)} \text{ a solution } \Rightarrow \quad \text{try } \; y_2(t) = v(t)e^{-bt/(2a)}$$

- Then

$$y_2(t) = v(t)e^{-bt/(2a)}$$

$$y_2'(t) = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)}$$

$$y_2''(t) = v''(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} - \frac{b}{2a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/(2a)}$$

$$ay'' + by' + cy = 0$$

Finding Multiplying Factor *v*(*t*)

- Substituting derivatives into ODE, we seek a formula for *v*:

$$e^{-bt/(2a)}\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\} = 0$$

$$av''(t) - bv'(t) + \frac{b^2}{4a}v(t) + bv'(t) - \frac{b^2}{2a}v(t) + cv(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0 \iff av''(t) + \left(\frac{-b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0$$

$$av''(t) - \left(\frac{b^2 - 4ac}{4a}\right)v(t) = 0$$

$$v''(t) = 0 \quad \Rightarrow \quad v(t) = k_3 t + k_4$$

General Solution

- To find our general solution, we have:

$$\begin{aligned} y(t) &= k_1 e^{-bt/(2a)} + k_2 v(t) e^{-bt/(2a)} \\ &= k_1 e^{-bt/(2a)} + \left(k_3 t + k_4\right) e^{-bt/(2a)} \\ &= c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)} \end{aligned}$$

- Thus the general solution for repeated roots is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

Wronskian

- The general solution is

$$y(t) = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

- Thus every solution is a linear combination of

$$y_1(t) = e^{-bt/(2a)}, \; y_2(t) = t e^{-bt/(2a)}$$

- The Wronskian of the two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/(2a)} & t e^{-bt/(2a)} \\ -\dfrac{b}{2a} e^{-bt/(2a)} & \left(1 - \dfrac{bt}{2a}\right) e^{-bt/(2a)} \end{vmatrix}$$

$$= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right)$$

$$= e^{-bt/a} \neq 0 \quad \text{for all } t$$

- Thus y_1 and y_2 form a fundamental solution set for equation.

Example 1 (1 of 2)

- Consider the initial value problem

$$y'' + 4y' + 4y = 0$$

- Assuming exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Leftrightarrow r = -2$$

- So one solution is $y_1(t) = e^{-2t}$ and a second solution is found:

$$y_2(t) = v(t)e^{-2t}$$

$$y_2'(t) = v'(t)e^{-2t} - 2v(t)e^{-2t}$$

$$y_2''(t) = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}$$

- Substituting these into the differential equation and simplifying yields $v''(t) = 0,\ v'(t) = k_1,\ v(t) = k_1 t + k_2$ where c_1 and c_2 are arbitrary constants.

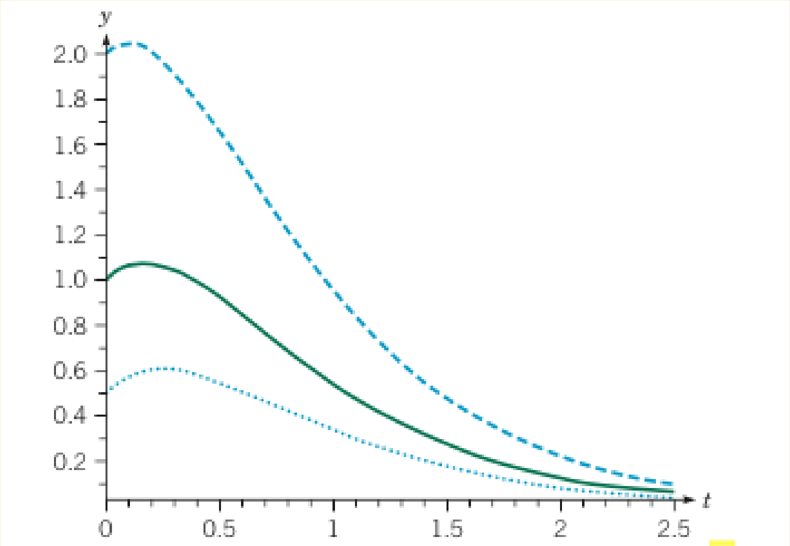
Example 1 (2 of 2)

- Letting $k_1 = 1$ and $k_2 = 0$, $v(t) = t$ and $y_2(t) = te^{-2t}$
- So the general solution is

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t}$$

- Note that both y_1 and y_2 tend to 0 as $t \to \infty$ regardless of the values of c_1 and c_2

- Here are three solutions of this equation with different sets of initial conditions.
- $y(0) = 2$, $y'(0) = 1$ (top)
- $y(0) = 1$, $y'(0) = 1$ (middle)
- $y(0) = ½$, $y'(0) = 1$ (bottom)



Example 2 (1 of 2)

- Consider the initial value problem

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \;\Rightarrow\; r^2 - r + \frac{1}{4} = 0 \;\Leftrightarrow\; (r - \frac{1}{2})^2 = 0 \;\Leftrightarrow\; r = \frac{1}{2}$$

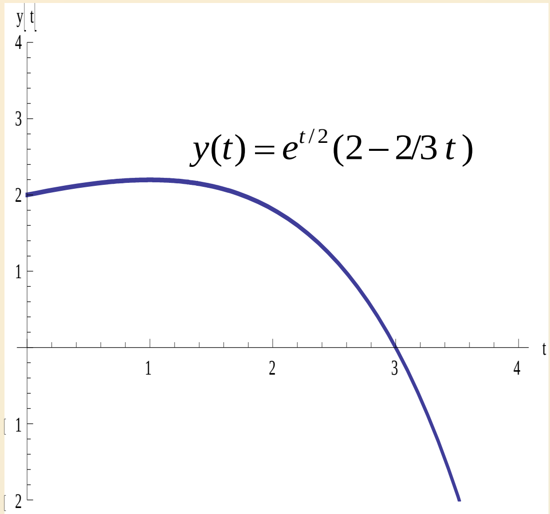
- Thus the general solution is

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

- Using the initial conditions:

$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{3} \end{array}\right\} \Rightarrow c_1 = 2,\; c_2 = -\frac{2}{3}$$

- Thus $\quad y(t) = 2e^{t/2} - \dfrac{2}{3}te^{t/2}$



Example 2 (2 of 2)

- Suppose that the initial slope in the previous problem was increased

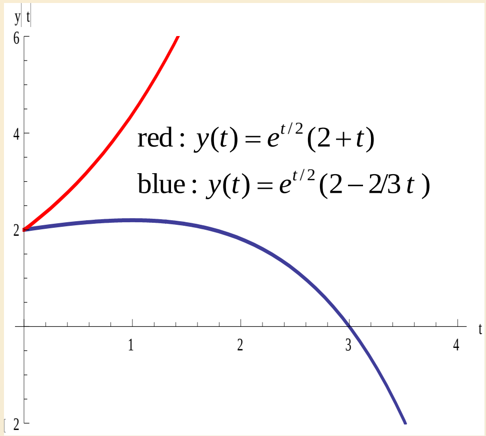
$$y(0) = 2, \;\; y'(0) = 2$$

- The solution of this modified problem is

$$y(t) = 2e^{t/2} + te^{t/2}$$

- Notice that the coefficient of the second term is now positive. This makes a big difference in the graph, since the exponential function is raised to a positive power:

$$/ = \frac{1}{2} > 0$$



Reduction of Order

- The method used so far in this section also works for equations with nonconstant coefficients:

$$y'' + p(t)y' + q(t)y = 0$$

- That is, given that y_1 is solution, try $y_2 = v(t)y_1$:

$$y_2(t) = v(t)y_1(t)$$
$$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t)$$
$$y_2''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

- Substituting these into ODE and collecting terms,

$$y_1v'' + \left(2y_1' + py_1\right)v' + \left(y_1'' + py_1' + qy_1\right)v = 0$$

- Since y_1 is a solution to the differential equation, this last equation reduces to a first order equation in v :

$$y_1v'' + \left(2y_1' + py_1\right)v' = 0$$

Example 3: Reduction of Order

- Given the variable coefficient equation and solution y_1,

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

use reduction of order method to find a second solution:

$$y_2(t) = v(t)\, t^{-1}$$

$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$

$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

- Substituting these into the ODE and collecting terms,

$$2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} = 0$$

$$\Leftrightarrow 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$

$$\Leftrightarrow \; 2tv'' - v' = 0$$

$$\Leftrightarrow \; 2tu' - u = 0, \; \text{ where } u(t) = v'(t)$$

Example 3: Finding $v(t)$

- To solve

$$2tu' - u = 0, \quad u(t) = v'(t)$$

for u, we can use the separation of variables method:

$$2t\frac{du}{dt} - u = 0 \quad \Leftrightarrow \quad \int \frac{du}{u} = \int \frac{1}{2t}\,dt \quad \Leftrightarrow \quad \ln|u| = 1/2\ \ln|t| + C$$

$$\Leftrightarrow \quad |u| = |t|^{1/2} e^{C} \quad \Leftrightarrow \quad u = ct^{1/2}, \quad \text{since } t > 0.$$

- Thus

$$v' = ct^{1/2}$$

and hence

$$v(t) = \frac{2}{3}c\,t^{3/2} + k$$

Example 3: General Solution (3 of 3)

- Since $v(t) = \frac{2}{3} c\, t^{3/2} + k$

$$y_2(t) = \left(\frac{2}{3} c\, t^{3/2} + k \right) t^{-1} = \frac{2}{3} c\, t^{1/2} + k\, t^{-1}$$

- Recall $y_1(t) = t^{-1}$
- So we can neglect the second term of y_2 to obtain

$$y_2(t) = t^{1/2}$$

- The Wronskian of $y_1(t)$ and $y_2(t)$ can be computed

$$W[y_1, y_2](t) = \frac{3}{2} t^{-3/2} \neq 0, \quad t > 0$$

- Hence the general solution to the differential equation is

$$y(t) = c_1 t^{-1} + c_2 t^{1/2}$$

Boyce/DiPrima/Meade 11th ed, Ch 3.5: Nonhomogeneous Equations; Method of Undetermined Coefficients

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, , and Doug Meade ©2017 by John Wiley & Sons, Inc.

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section we will learn the method of undetermined coefficients to solve the nonhomogeneous equation, which relies on knowing solutions to the homogeneous equation.

Theorem 3.5.1

- If Y_1 and Y_2 are solutions of the nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t)$$
then $Y_1 - Y_2$ is a solution of the homogeneous equation
$$y'' + p(t)y' + q(t)y = 0$$
- If, in addition, $\{y_1, y_2\}$ forms a fundamental solution set of the homogeneous equation, then there exist constants c_1 and c_2 such that
$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 3.5.2 (General Solution)

- To solve the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

we need to do three things:

1. Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation. This is called the **complementary solution** and may be denoted by $y_c(t)$.
2. Find any solution $Y(t)$ of the nonhomogeneous equation. This is often referred to as a **particular solution**.
3. Form the sum of the functions found in steps 1 and 2.

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

  with general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

- In this section we use the method of **undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.
- The method of undetermined coefficients is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Example 1: Exponential $g(t)$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}$$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t}, \; Y''(t) = 4Ae^{2t}$$

- Substituting these derivatives into the differential equation,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Leftrightarrow \quad -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow \quad A = -1/2$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = -\frac{1}{2}e^{2t}$$

Example 2: Sine $g(t)$, First Attempt

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin t$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

- Substituting these derivatives into the differential equation,

$$\begin{aligned} & -A\sin t - 3A\cos t - 4A\sin t = 2\sin t \\ \Leftrightarrow\ & (2 + 5A)\sin t + 3A\cos t = 0 \\ \Leftrightarrow\ & c_1 \sin t + c_2 \cos t = 0 \end{aligned}$$

- Since sin(x) and cos(x) are not multiples of each other, we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y'' - 3y' - 4y = 2\sin t$$

Example 2: Sine $g(t)$, Particular Solution (2 of 2)

- Our next attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$
$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Substituting these derivatives into ODE, we obtain

$$\left(-A\sin t - B\cos t\right) - 3\left(A\cos t - B\sin t\right) - 4\left(A\sin t + B\cos t\right) = 2\sin t$$
$$\Leftrightarrow \left(-5A + 3B\right)\sin t + \left(-3A - 5B\right)\cos t = 2\sin t$$
$$\Leftrightarrow -5A + 3B = 2,\ -3A - 5B = 0$$
$$\Leftrightarrow A = -\frac{5}{17},\ B = \frac{3}{17}$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

Example 3: Product $g(t)$

- Consider the nonhomogeneous equation

$$y'' - 3y' - 4y = -8e^{t}\cos(2t)$$

- We seek Y satisfying this equation, as follows:

$$Y(t) = Ae^{t}\cos(2t) + Be^{t}\sin(2t)$$

$$\begin{aligned} Y'(t) &= Ae^{t}\cos(2t) - 2Ae^{t}\sin(2t) + Be^{t}\sin(2t) + 2Be^{t}\cos(2t) \\ &= \left(A + 2B\right)e^{t}\cos(2t) + \left(-2A + B\right)e^{t}\sin(2t) \end{aligned}$$

$$\begin{aligned} Y''(t) &= \left(A + 2B\right)e^{t}\cos(2t) - 2\left(A + 2B\right)e^{t}\sin(2t) + \left(-2A + B\right)e^{t}\sin(2t) \\ &\quad + 2\left(-2A + B\right)e^{t}\cos(2t) \\ &= \left(-3A + 4B\right)e^{t}\cos(2t) + \left(-4A - 3B\right)e^{t}\sin(2t) \end{aligned}$$

- Substituting these into the ODE and solving for A and B:

$$A = \frac{10}{13},\; B = \frac{2}{13} \;\Rightarrow\; Y(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t)$$

Discussion: Sum $g(t)$

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

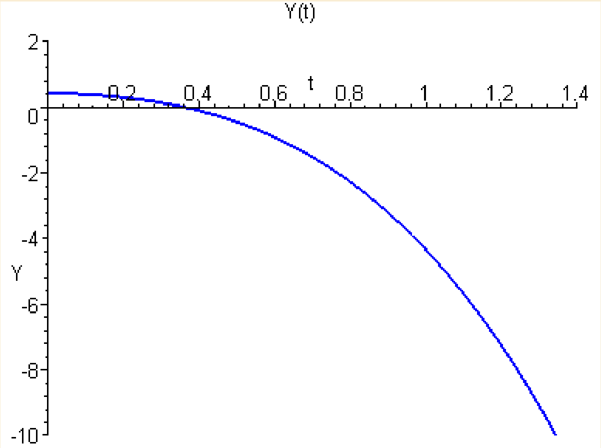
Example 4: Sum $g(t)$

- Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

- Our equations to solve individually are

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$



- Our particular solution is then

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

Example 5: First Attempt (1 of 3)

- Consider the nonhomogeneous equation
$$y'' - 3y' - 4y = 2e^{-t}$$
- We seek Y satisfying this equation. We begin with
$$Y(t) = Ae^{-t} \Rightarrow Y'(t) = -Ae^{-t},\ Y''(t) = Ae^{-t}$$
- Substituting these derivatives into differential equation,
$$(A + 3A - 4A)e^{-t} = 2e^{-t}$$
- Since the left side of the above equation is always 0, no value of A can be found to make $Y(t) = Ae^{-t}$ a solution to the nonhomogeneous equation.
- To understand why this happens, we will look at the solution of the corresponding homogeneous differential equation

Example 5: Homogeneous Solution (2 of 3)

- To solve the corresponding homogeneous equation:

$$y'' - 3y' - 4y = 0$$

- We use the techniques from Section 3.1 and get

$$y_1(t) = e^{-t} \text{and } y_2(t) = e^{4t}$$

- Thus our assumed particular solution $Y(t) = Ae^{-t}$ solves the homogeneous equation instead of the nonhomogeneous equation.
- So we need another form for $Y(t)$ to arrive at the general solution of the form:

$$y(t) = c_1 e^{-t} + c_2 e^{4t} + Y(t)$$

$$y'' - 3y' - 4y = 2e^{-t}$$

Example 5: Particular Solution (3 of 3)

- Our next attempt at finding a $Y(t)$ is:

$$Y(t) = Ate^{-t}$$

$$Y'(t) = Ae^{-t} - Ate^{-t}$$

$$Y''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t} = Ate^{-t} - 2Ae^{-t}$$

- Substituting these into the ODE,

e^{-t}

$$Ate^{-t} - 2Ae^{-t} - 3Ae^{-t} + 3Ate^{-t} - 4Ate^{-t} = 2e^{-t}$$

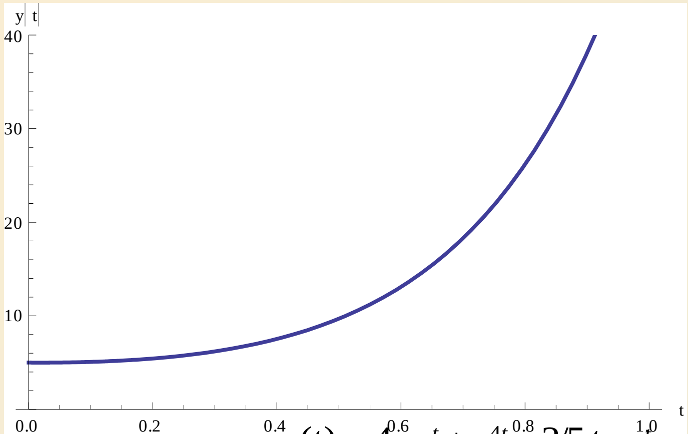
$$0 \cdot Ate^{-t} - 5Ate^{-t} = -5Ate^{-t} = 2e^{-t}$$

$$\Rightarrow A = -2/5$$

$$\Rightarrow Y(t) = -\frac{2}{5}te^{-t}$$

- So the general solution to the IVP is

$$y(t) = c_1 e^{-t} + c_2 e^{4t} - \frac{2}{5} t e^{-t}$$



$$y(t) = 4e^{-t} + e^{4t} - 2/5\, t e^{-t}$$

Summary – Undetermined Coefficients

- For the differential equation
$$ay'' + by' + cy = g(t)$$
where a, b, and c are constants, if $g(t)$ belongs to the class of functions discussed in this section (involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of these), the method of undetermined coefficients may be used to find a particular solution to the nonhomogeneous equation.
- The first step is to find the general solution for the corresponding homogeneous equation with $g(t) = 0$.
$$y_C(t) = c_1 y_1(t) + c_2 y_2(t)$$

Summary – Undetermined Coefficients (2 of 2)

- The second step is to select an appropriate form for the particular solution, $Y(t)$, to the nonhomogeneous equation and determine the derivatives of that function.
- After substituting $Y(t)$, $Y'(t)$, and $Y''(t)$ into the nonhomogeneous differential equation, if the form for $Y(t)$ is correct, all the coefficients in $Y(t)$ can be determined.
- Finally, the general solution to the nonhomogeneous differential equation can be written as

$$y_{gen}(t) = y_C(t) + Y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Boyce/DiPrima/Meade 11th ed, Ch 3.6:
Variation of Parameters

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade

- Recall the nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t)$$
where *p*, *q*, *g* are continuous functions on an open interval *I*.
- The associated homogeneous equation is
$$y'' + p(t)y' + q(t)y = 0$$
- In this section we will learn the **variation of parameters** method to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to the homogeneous equation.
- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.

Example 1: Variation of Parameters (1 of 6)

- We seek a particular solution to the equation below.

$$y'' + 4y = 8\tan t, \quad -\pi/2 < t < \pi/2$$

- We cannot use the undetermined coefficients method since $g(t)$ is a quotient of $\sin t$ or $\cos t$, instead of a sum or product.
- Recall that the solution to the homogeneous equation is

$$y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- To find a particular solution to the nonhomogeneous equation, we begin with the form

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t)$$

- Then

$$y'(t) = u_1'(t)\cos(2t) - 2u_1(t)\sin(2t) + u_2'(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- or

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

Example 1: Derivatives, 2$^{\text{nd}}$ Equation (2 of 6)

- From the previous slide,

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t)$$

- Note that we need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

- Then

$$y'(t) = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t)$$

- Next,

$$y''(t) = -2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$

Example 1: Two Equations (3 of 6)

- Recall that our differential equation is

$$y'' + 4y = 8\tan t$$

- Substituting y'' and y into this equation, we obtain

$$-2u_1'(t)\sin(2t) - 4u_1(t)\cos(2t) + 2u_2'(t)\cos(2t) - 4u_2(t)\sin(2t)$$
$$+4\left(u_1(t)\cos(2t) + u_2(t)\sin(2t)\right) = 8\tan t$$

- This equation simplifies to

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$

- Thus, to solve for u_1 and u_2, we have the two equations:

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$
$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

Example 1: Solve for u_1' (4 of 6)

- To find u_1 and u_2 , we first need to solve for u_1' and u_2'

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t$$

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0$$

- From second equation,

$$u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Substituting this into the first equation,

$$-2u_1'(t)\sin(2t) + 2\left[-u_1'(t)\frac{\cos(2t)}{\sin(2t)}\right]\cos(2t) = 8\tan t$$

$$-2u_1'(t)\sin^2(2t) - 2u_1'(t)\cos^2(2t) = 8\tan t\sin(2t)$$

$$-2u_1'(t)\left[\sin^2(2t) + \cos^2(2t)\right] = 8\left[\frac{2\sin^2 t\cos t}{\cos t}\right]$$

$$u_1'(t) = -8\sin^2 t$$

Example 1: Solve for u_1 and u_2 (5 of 6)

- From the previous slide,

$$u_1'(t) = -8\sin^2 t, \quad u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$$

- Then

$$u_2'(t) = 8\sin^2 t\,\frac{\cos(2t)}{\sin(2t)} = 4\,\frac{\sin t(2\cos^2 t - 1)}{\cos t} = 4\sin t\left(2\cos t - \frac{1}{\cos t}\right)$$

- Thus

$$u_1(t) = \int u_1'(t)\,dt = 4\sin t\cos t - 4t + c_1$$

$$u_2(t) = \int u_2'(t)\,dt = 4\ln(\cos t) - 4\cos^2 t + c_2$$

Example 1: General Solution (6 of 6)

- Recall our equation and homogeneous solution y_C:

$$y'' + 4y = 8\tan t, \quad y_C(t) = c_1\cos(2t) + c_2\sin(2t)$$

- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is

$$\begin{aligned} y(t) &= u_1(t)\cos 2t + u_2(t)\sin 2t + y_C(t) \\ &= (4\sin t\cos t)\cos(2t) + (4\ln(\cos t) - 4\cos^2 t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \\ &= -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t) \end{aligned}$$

$$y'' + p(t)y' + q(t)y = g(t)$$

Summary

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on y'' is 1.
- To find u_1 and u_2, we need to solve the equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- Doing so, and using the Wronskian, we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

- Thus

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\,dt + c_2$$

Theorem 3.6.1

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions p, q and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W\left(y_1, y_2\right)(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W\left(y_1, y_2\right)(t)}\,dt$$

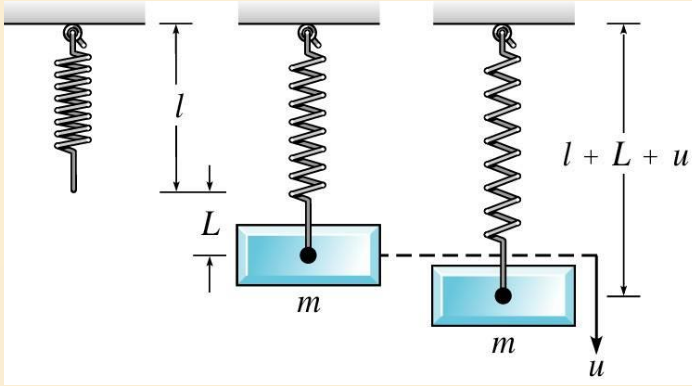
and the general solution is

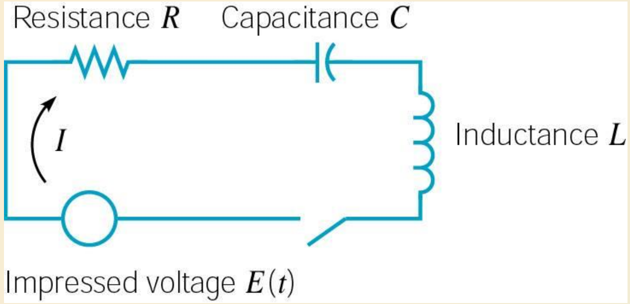
$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Boyce/DiPrima/Meade 11th ed, Ch 3.7: Mechanical & Electrical Vibrations

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade

- Two important areas of application for second order linear equations with constant coefficients are in modeling mechanical and electrical oscillations.
- We will study the motion of a mass on a spring in detail.
- An understanding of the behavior of this simple system is the first step in investigation of more complex vibrating systems.

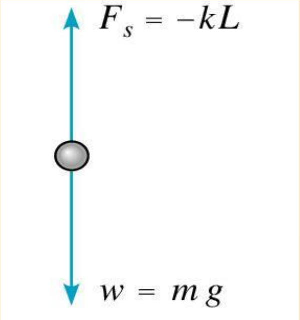


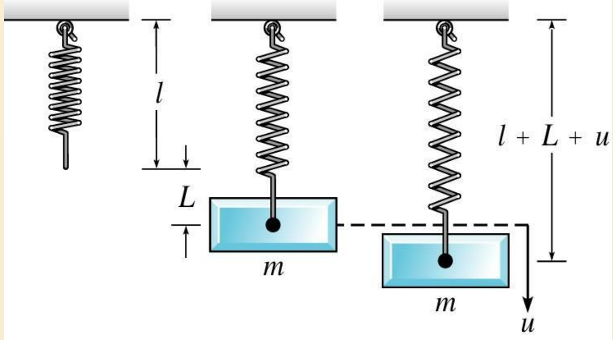


Spring – Mass System

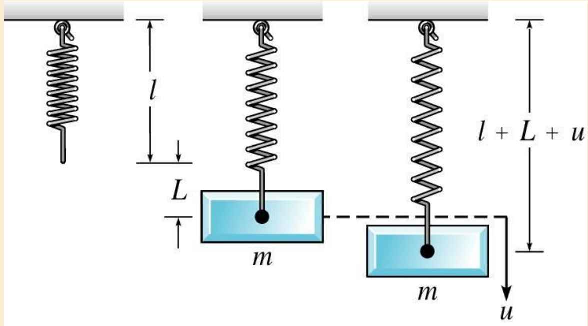
- Suppose a mass m hangs from a vertical spring of original length l. The mass causes an elongation L of the spring.
- The force F_G of gravity pulls the mass down. This force has magnitude mg, where g is acceleration due to gravity.
- The force F_S of the spring stiffness pulls the mass up. For small elongations L, this force is proportional to L.

  That is, $F_s = kL$ (Hooke's Law).
- When the mass is in equilibrium, the forces balance each other: $mg = kL$



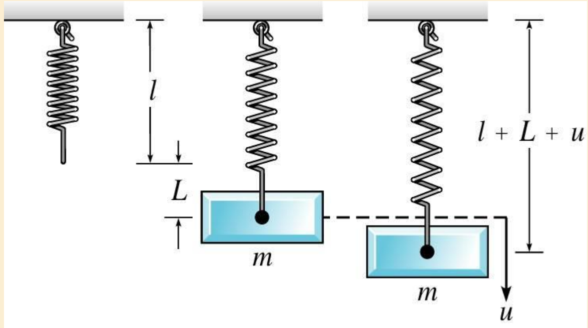


Spring Model



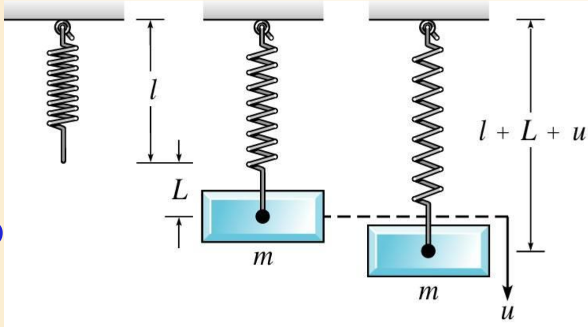
- We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced.
- Let $u(t)$ denote the displacement of the mass from its equilibrium position at time t, measured downward.
- Let f be the net force acting on the mass. We will use Newton's 2nd Law: $mu''(t) = f(t)$
- In determining f, there are four separate forces to consider:
  - Weight: $w = mg$ (downward force)
  - Spring force: $F_s = -k(L + u)$ (up or down force, see next slide)
  - Damping force: $F_d(t) = -\gamma u'(t)$ (up or down, see following slide)
  - External force: $F(t)$ (up or down force, see text)

Spring Model: Spring Force Details



- The spring force F_s acts to restore a spring to the natural position, and is proportional to $L + u$. If $L + u > 0$, then the spring is extended and the spring force acts upward. In this case
$$F_s = -k(L+u)$$
- If $L + u < 0$, then spring is compressed a distance of $|L + u|$, and the spring force acts downward. In this case
$$F_s = k|L+u| = k[-(L+u)] = -k(L+u)$$
- In either case,
$$F_s = -k(L+u)$$

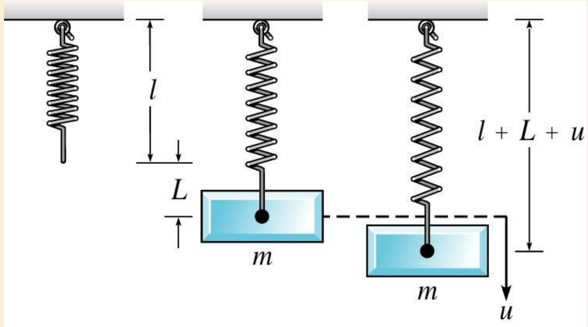
Spring Model: Damping Force Details



- The damping or resistive force F_d acts in the opposite direction as the motion of the mass. This can be complicated to model. F_d may be due to air resistance, internal energy dissipation due to action of spring, friction between the mass and guides, or a mechanical device (dashpot) imparting a resistive force to the mass.
- We simplify this and assume F_d is proportional to the velocity.
- In particular, we find that
  - If $u' > 0$, then u is increasing, so the mass is moving downward. Thus F_d acts upward and hence $F_d(t) = -\gamma u'(t)$.
  - If $u' < 0$, then u is decreasing, so the mass is moving upward. Thus F_d acts downward and hence $F_d(t) = -\gamma u'(t)$
  - In either case,

$$F_d(t) = -\gamma u'(t), \quad \gamma > 0$$

Spring Model: Differential Equation



- Taking into account these forces, Newton's Law becomes:

$$mu''(t) = mg + F_s(t) + F_d(t) + F(t)$$
$$= mg - k[L + u(t)] - \gamma u'(t) + F(t)$$

- Recalling that $mg = kL$, this equation reduces to

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

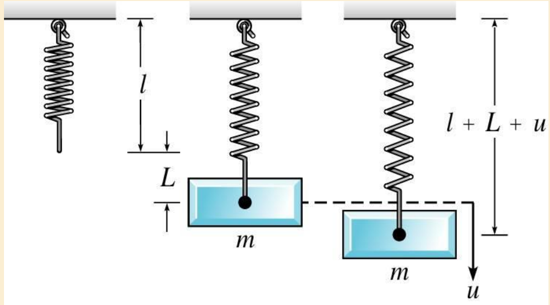
where the constants m, γ, and k are positive.

- We can prescribe initial conditions also:

$$u(0) = u_0,\ \ u'(0) = v_0$$

- It follows from Theorem 3.2.1 that there is a unique solution to this initial value problem. Physically, if the mass is set in motion with a given initial displacement and velocity, then its position is uniquely determined at all future times.

Example 1: Find Coefficients (1 of 2)



- A 4 lb mass stretches a spring 2". The mass is displaced an additional 6" and then released; and is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/sec. Formulate the IVP that governs the motion of this mass:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),\ \ u(0) = u_0,\ \ u'(0) = v_0$$

- Find m:

$$w = mg \Rightarrow m = \frac{w}{g} \Rightarrow m = \frac{4\,\text{lb}}{32\text{ft}/\sec^2} \Rightarrow m = \frac{1}{8}\frac{\text{lb}\sec^2}{\text{ft}}$$

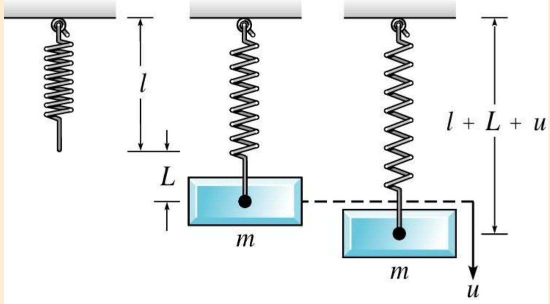
- Find γ:

$$\gamma u' = 6\,\text{lb} \Rightarrow \gamma = \frac{6\,\text{lb}}{3\text{ft}/\sec} \Rightarrow \gamma = 2\frac{\text{lb}\sec}{\text{ft}}$$

- Find k:

$$F_s = -k\,L \Rightarrow k = \frac{4\,\text{lb}}{2\,\text{in}} \Rightarrow\ \ k = \frac{4\,\text{lb}}{1/6\,\text{ft}} \Rightarrow k = 24\frac{\text{lb}}{\text{ft}}$$

Example 1: Find IVP (2 of 2)



- Thus our differential equation becomes

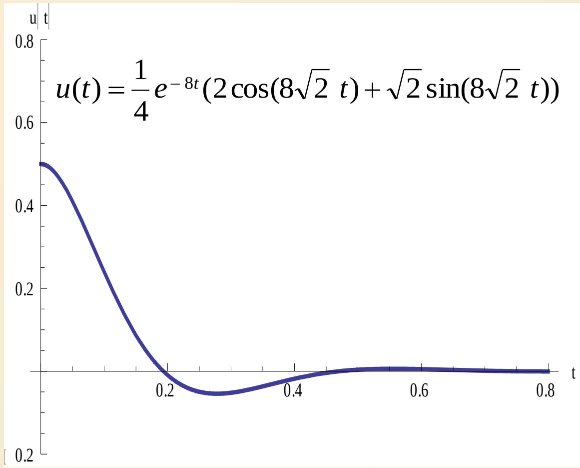
$$\frac{1}{8}u''(t)+2u'(t)+24u(t)=0$$

and hence the initial value problem can be written as

$$u''(t)+16u'(t)+192u(t)=0$$
$$u(0)=\frac{1}{2}, \quad u'(0)=0$$

- This problem can be solved using the methods of Chapter 3.3 and yields the solution

$$u(t)=\frac{1}{4}e^{-8t}\left(2\cos(8\sqrt{2}\,t)+\sqrt{2}\sin(8\sqrt{2}\,t)\right)$$



Spring Model: Undamped Free Vibrations (1 of 4)

- Recall our differential equation for spring motion:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

- Suppose there is no external driving force and no damping. Then $F(t) = 0$ and $\gamma = 0$, and our equation becomes

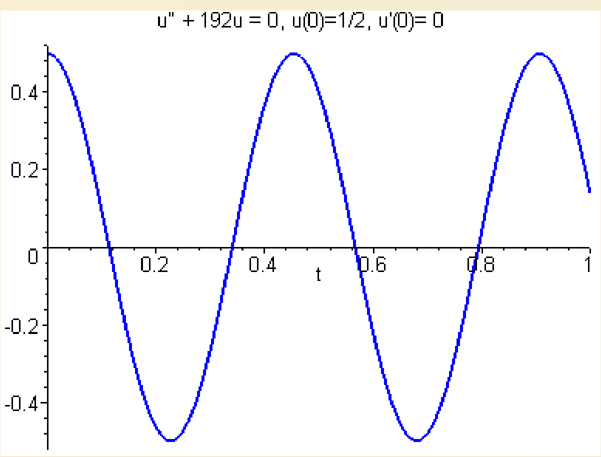
$$mu''(t) + ku(t) = 0$$

- The general solution to this equation is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where

$$\omega_0^2 = k/m$$



Spring Model: Undamped Free Vibrations (2 of 4)

- Using trigonometric identities, the solution

$$u(t) = A\cos w_0 t + B\sin w_0 t, \quad w_0^2 = \frac{k}{m}$$

can be rewritten as follows:

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad \Leftrightarrow \quad u(t) = R\cos(\omega_0 t - \delta)$$

$$\Leftrightarrow u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t,$$

where

$$A = R\cos\delta, \;\; B = R\sin\delta \quad \Rightarrow \quad R = \sqrt{A^2 + B^2}, \quad \tan\delta = \frac{B}{A}$$

- Note that in finding d, we must be careful to choose the correct quadrant. This is done using the signs of cos d and sin d.

Spring Model: Undamped Free Vibrations (3 of 4)

- Thus our solution is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad = R\cos\left(\omega_0 t - \delta\right)$$

where

$$\omega_0 = \sqrt{k/m}$$

- The solution is a shifted cosine (or sine) curve, that describes simple harmonic motion, with period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

- The circular frequency ω_0 (radians/time) is the **natural frequency** of the vibration, *R* is the **amplitude** of the maximum displacement of mass from equilibrium, and δ is the **phase** or phase angle (dimensionless).

Spring Model: Undamped Free Vibrations (4 of 4)

- Note that our solution

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \ = R\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{k/m}$$

is a shifted cosine (or sine) curve with period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Initial conditions determine *A* & *B*, hence also the amplitude *R*.
- The system always vibrates with the same frequency w_0, regardless of the initial conditions.
- The period *T* increases as *m* increases, so larger masses vibrate more slowly. However, *T* decreases as *k* increases, so stiffer springs cause a system to vibrate more rapidly.

Example 2: Find IVP (1 of 3)

- A 10 lb mass stretches a spring 2". The mass is displaced an additional 2" and then set in motion with an initial upward velocity of 1 ft/sec. Determine the position of the mass at any later time, and find the period, amplitude, and phase of the motion.

$$mu''(t) + ku(t) = 0,\ \ u(0) = u_0,\ \ u'(0) = v_0$$

- Find *m*:

$$w = mg \Rightarrow m = \frac{w}{g} \Rightarrow m = \frac{10\,\text{lb}}{32\text{ft}/\sec^2} \Rightarrow m = \frac{5}{16}\,\frac{\text{lb}\sec^2}{\text{ft}}$$

- Find *k*:

$$F_s = -k\,L \Rightarrow k = \frac{10\,\text{lb}}{2\,\text{in}} \Rightarrow\ k = \frac{10\,\text{lb}}{1/6\,\text{ft}} \Rightarrow k = 60\,\frac{\text{lb}}{\text{ft}}$$

- Thus our IVP is

$$\frac{5}{16}u''(t) + 60u(t) = 0,\ \ u(0) = \frac{1}{6},\ \ u'(t) = -1$$

Example 2: Find Solution **(2 of 3)**

- Simplifying, we obtain

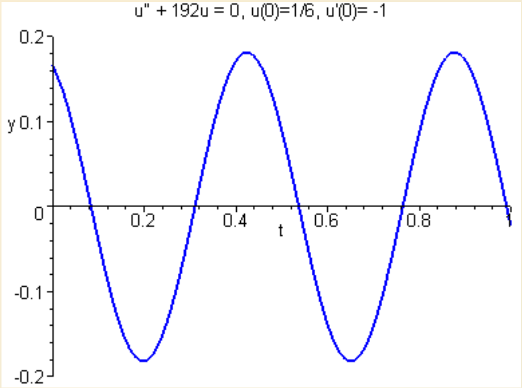
$$u''(t) + 192u(t) = 0, \quad u(0) = 1/6, \quad u'(0) = -1$$

- To solve, use methods of Ch 3.3 to obtain

$$u(t) = \frac{1}{6}\cos\sqrt{192}\, t - \frac{1}{\sqrt{192}}\sin\sqrt{192}\, t$$

or

$$u(t) = \frac{1}{6}\cos 8\sqrt{3}\, t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}\, t$$



Example 2: $u(t) = \frac{1}{6}\cos 8\sqrt{3}t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$

Find Period, Amplitude, Phase (3 of 3)

- The natural frequency is

$$\omega_0 = \sqrt{k/m} = \sqrt{192} = 8\sqrt{3} \cong 13.856 \text{ rad/sec}$$

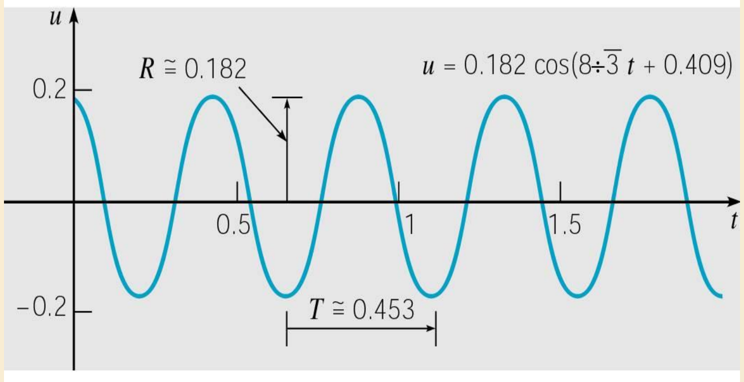
- The period is

$$T = 2\pi/\omega_0 \cong 0.45345 \text{ sec}$$

- The amplitude is

$$R = \sqrt{A^2 + B^2} \cong 0.18162 \text{ ft}$$

- Next, determine the phase *d*:



$$A = R\cos\delta, \;\; B = R\sin\delta, \;\; \tan\delta = B/A$$

$$\tan\delta = \frac{B}{A} \Rightarrow \tan\delta = \frac{-\sqrt{3}}{4} \Rightarrow \delta = \tan^{-1}\left(\frac{-\sqrt{3}}{4}\right) \cong -0.40864 \text{ rad}$$

Thus $u(t) = 0.182\cos\left(8\sqrt{3}t + 0.409\right)$

Spring Model: Damped Free Vibrations (1 of 8)

- Suppose there is damping but no external driving force $F(t)$:

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

- What is effect of the damping coefficient g on the system?
- The characteristic equation is

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{\gamma}{2m}\left[-1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right]$$

- Three cases for the solution:

$$\gamma^2 - 4mk > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \ \text{where } r_1 < 0, \ r_2 < 0;$$

$$\gamma^2 - 4mk = 0: \quad u(t) = (A + Bt)e^{-\gamma t/2m}, \ \text{where } \gamma / 2m > 0;$$

$$\gamma^2 - 4mk < 0: \quad u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \ \mu = \frac{\sqrt{4mk - \gamma^2}}{2m} > 0.$$

Note: In all three cases, $\lim_{t\to\infty} u(t) = 0$, as expected from the damping term.

Damped Free Vibrations: Small Damping (2 of 8)

- Of the cases for solution form, the last is most important, which occurs when the damping is small:

$$\gamma^2 - 4mk > 0: \; u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \, r_2 < 0$$

$$\gamma^2 - 4mk = 0: \; u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma / 2m > 0$$

$$\gamma^2 - 4mk < 0: \; u(t) = e^{-\gamma t/2m}(A\cos \mu t + B\sin \mu t), \; \mu > 0$$

- We examine this last case. Recall

$$A = R\cos\delta, \; B = R\sin\delta$$

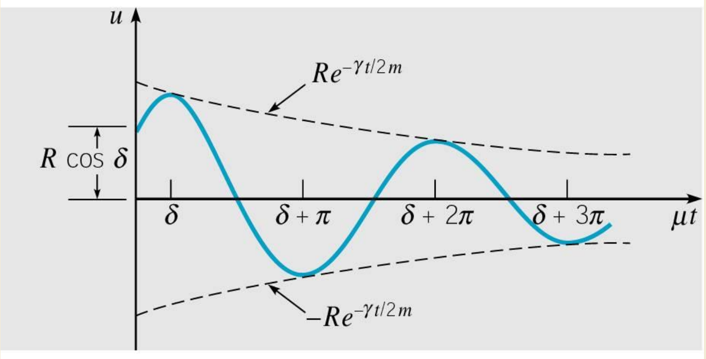
- Then

$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

and hence

$$|u(t)| \le Re^{-\gamma t/2m}$$

(damped oscillation)



Damped Free Vibrations: Quasi Frequency (3 of 8)

- Thus we have damped oscillations:

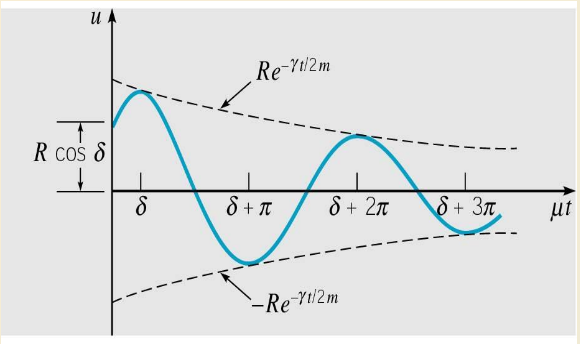
$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta) \quad \Rightarrow \quad |u(t)| \le Re^{-\gamma t/2m}$$

- The amplitude *R* depends on the initial conditions, since

$$u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \; A = R\cos\delta, \; B = R\sin\delta$$

- Although the motion is not periodic, the parameter μ determines the mass oscillation frequency.
- Thus μ is called the **quasi frequency**.
- Recall

$$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m}$$



Damped Free Vibrations: Quasi Period

- Compare n with w_0, the frequency of undamped motion:

$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km-\gamma^2}}{2m\sqrt{k/m}} = \frac{\sqrt{4km-\gamma^2}}{\sqrt{4m^2}\sqrt{k/m}} = \frac{\sqrt{4km-\gamma^2}}{\sqrt{4km}} = \sqrt{1-\frac{\gamma^2}{4km}}$$

For small $\frac{g^2}{4km}$

$$\cong \sqrt{1-\frac{\gamma^2}{4km}+\frac{\gamma^4}{64k^2m^2}} = \sqrt{\left(1-\frac{\gamma^2}{8km}\right)^2} = 1-\frac{\gamma^2}{8km}$$

- Thus, small damping reduces oscillation frequency slightly.
- Similarly, the **quasi period** is defined as $T_d = 2p / n$. Then

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \left(1-\frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1-\frac{\gamma^2}{8km}\right)^{-1} \cong 1+\frac{\gamma^2}{8km}$$

- Thus, small damping increases quasi period.

Damped Free Vibrations: Neglecting Damping for Small $\dfrac{\gamma^2}{4km}$ (5 of 8)

- Consider again the comparisons between damped and undamped frequency and period:

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- Thus it turns out that a small γ is not as telling as a small ratio $\dfrac{\gamma^2}{4km}$.
- For small $\dfrac{\gamma^2}{4km}$, we can neglect the effect of damping when calculating the quasi frequency and quasi period of motion. But if we want a detailed description of the motion of the mass, then we cannot neglect the damping force, no matter how small it is.

Damped Free Vibrations: Frequency, Period (6 of 8)

- Ratios of damped and undamped frequency, period:

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2}$$

- Thus

$$\lim_{\gamma \to 2\sqrt{km}} \mu = 0 \ \text{ and } \ \lim_{\gamma \to 2\sqrt{km}} T_d = \infty$$

- The importance of the relationship between g^2 and $4km$ is supported by our previous equations:

$$\gamma^2 - 4mk > 0: \ \ u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \ r_2 < 0$$

$$\gamma^2 - 4mk = 0: \ \ u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0$$

$$\gamma^2 - 4mk < 0: \ \ u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \ \ \mu > 0$$

Damped Free Vibrations: Critical Damping Value (7 of 8)

- Thus the nature of the solution changes as γ passes through the value $2\sqrt{km}$.
- This value of $\gamma = 2\sqrt{km}$ is known as the **critical damping** value, and for larger values of γ the motion is said to be **overdamped**.
- Thus for the solutions given by these cases,

$$\gamma^2 - 4mk > 0: \;\; u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0,\, r_2 < 0 \qquad (1)$$

$$\gamma^2 - 4mk = 0: \;\; u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma/2m > 0 \qquad (2)$$

$$\gamma^2 - 4mk < 0: \;\; u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t), \;\; \mu > 0 \qquad (3)$$

  we see that the mass creeps back to its equilibrium position for solutions (1) and (2), but does not oscillate about it, as it does for small γ in solution (3).
- Solution (1) is overdamped and (2) is critically damped.

Damped Free Vibrations: Characterization of Vibration (8 of 8)

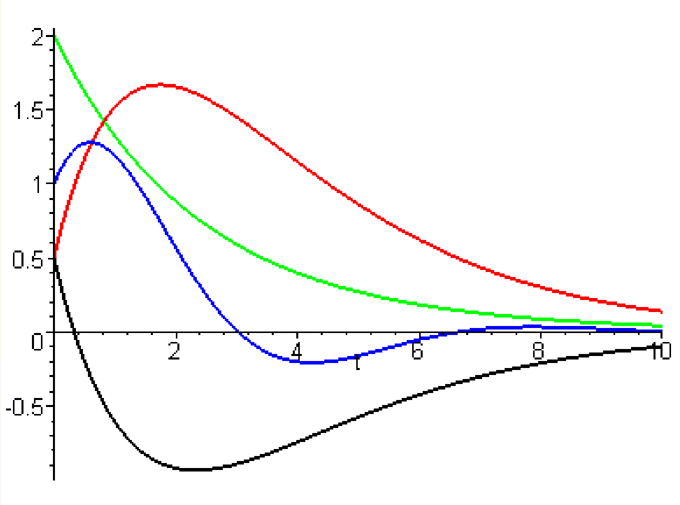
- The mass creeps back to the equilibrium position for solutions (1) & (2), but does not oscillate about it, as it does for small γ in solution (3).

$$\gamma^2 - 4mk > 0: \;\; u(t) = Ae^{r_1 t} + Be^{r_2 t}, \;\; r_1 < 0, \; r_2 < 0 \quad \text{(Green)} \qquad (1)$$

$$\gamma^2 - 4mk = 0: \;\; u(t) = (A + Bt)e^{-\gamma t/2m}, \;\; \gamma/2m > 0 \quad \text{(Red, Black)} \qquad (2)$$

$$\gamma^2 - 4mk < 0: \;\; u(t) = e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t) \quad \text{(Blue)} \qquad (3)$$

- Solution (1) is overdamped and
- Solution (2) is critically damped.
- Solution (3) is underdamped



Example 3: Initial Value Problem

- Suppose that the motion of a spring-mass system is governed by the initial value problem

$$u'' + \frac{1}{8}u' + u = 0, \;\; u(0) = 2, \;\; u'(0) = 0$$

- Find the following:

  (a) quasi frequency and quasi period;

  (b) time at which mass passes through equilibrium position;

  (c) time τ such that $|u(t)| < 0.1$ for all $t > \tau$.

- For Part (a), using methods of this chapter we obtain:

$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

where

$$\tan\delta = \frac{1}{\sqrt{255}} \Rightarrow \delta \cong 0.06254 \quad (\text{recall } A = R\cos\delta, \; B = R\sin\delta)$$

Example 3: Quasi Frequency & Period

- The solution to the initial value problem is:

$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) = \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

- The graph of this solution, along with solution to the corresponding undamped problem, is given below.
- The quasi frequency is

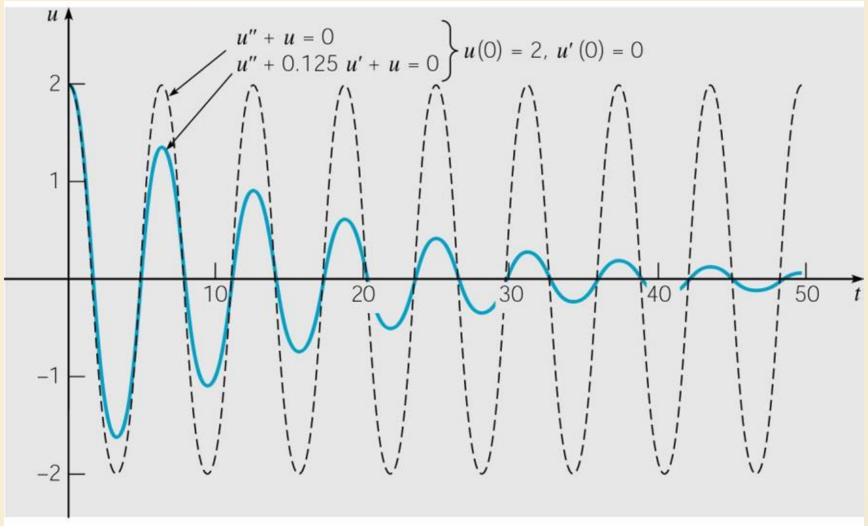
$$\mu = \sqrt{255}/16 \cong 0.998$$

and quasi period is

$$T_d = 2\pi/\mu \cong 6.295$$

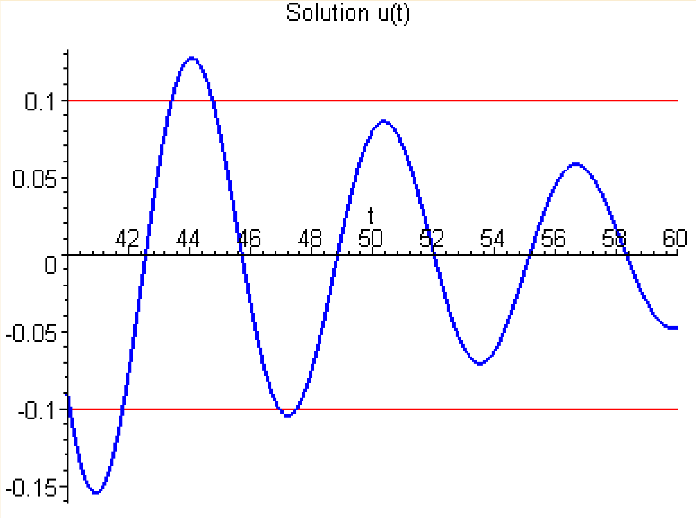
- For the undamped case:

$$\omega_0 = 1,\ T = 2\pi \cong 6.283$$



Example 3: Quasi Frequency & Period (3 of 4)

- The damping coefficient is $\gamma = 0.125 = 1/8$, and this is 1/16 of the critical value $2\sqrt{km} = 2$
- Thus damping is small relative to mass and spring stiffness. Nevertheless the oscillation amplitude diminishes quickly.
- Using a solver, we find that $|u(t)| < 0.1$ for $t > \tau \cong 47.5149$ sec



Example 3: Quasi Frequency & Period (4 of 4)

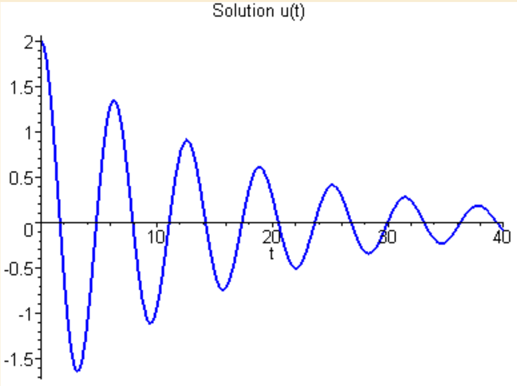
- To find the time at which the mass first passes through the equilibrium position, we must solve

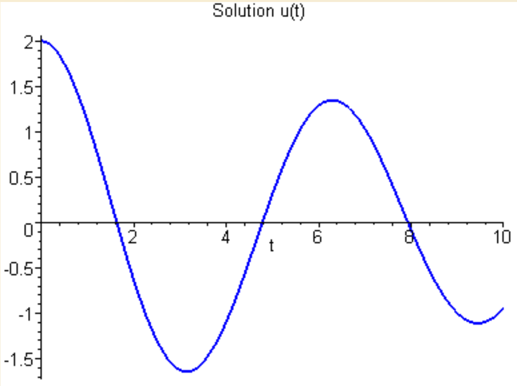
$$u(t) = \frac{32}{\sqrt{255}} e^{-t/16} \cos\left(\frac{\sqrt{255}}{16} t - \delta\right) = 0$$

- Or more simply, solve

$$\frac{\sqrt{255}}{16} t - \delta = \frac{\pi}{2}$$

$$\Rightarrow t = \frac{16}{\sqrt{255}}\left(\frac{\pi}{2} + \delta\right) \cong 1.637 \text{ sec}$$





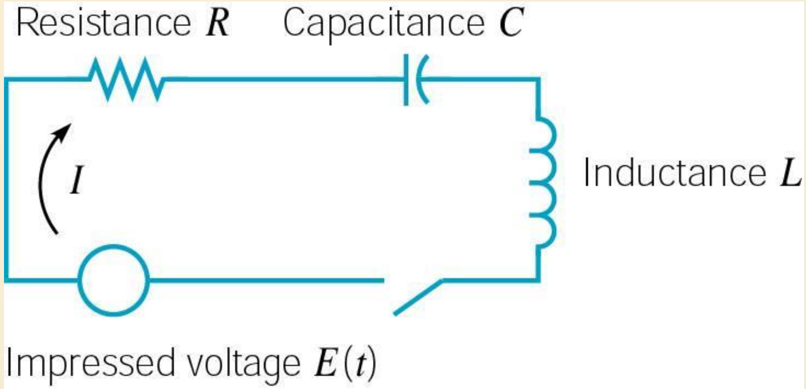
Electric Circuits

- The flow of current in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:

$$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = E'(t)$$

$$I(0) = I_0, \;\; I'(0) = I_0'$$

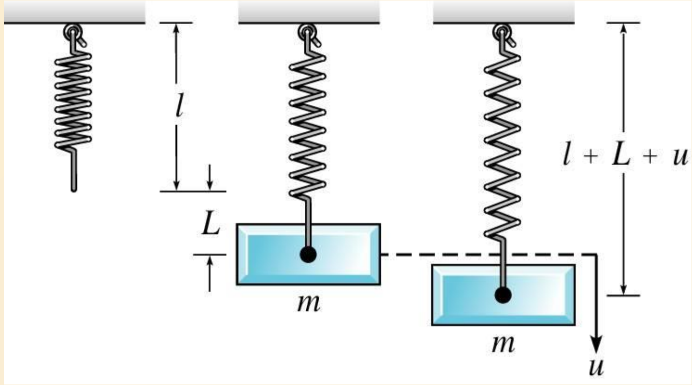
- It is interesting that the flow of current in this circuit is mathematically equivalent to motion of spring-mass system.
- For more details, see text.



Boyce/DiPrima/Meade 11th ed, Ch 3.8: Forced Periodic Vibrations

- We continue the discussion of the last section, and now consider the presence of a periodic external force:

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$



Forced Vibrations with Damping

- Consider the equation below for damped motion and external forcing function $F_0 \cos(\omega t)$.

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

- The general solution of this equation has the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_C(t) + U(t)$$

where the general solution of the homogeneous equation is

$$u_C(t) = c_1 u_1(t) + c_2 u_2(t)$$

and the particular solution of the nonhomogeneous equation is

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

Homogeneous Solution

- The homogeneous solutions u_1 and u_2 depend on the roots r_1 and r_2 of the characteristic equation:

$$mr^2 + \gamma r + kr = 0 \quad \Rightarrow \quad r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- Since m, g , and k are are all positive constants, it follows that r_1 and r_2 are either real and negative, or complex conjugates with negative real part. In the first case,

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0,$$

while in the second case

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t\right) = 0.$$

- Thus in either case,

$$\lim_{t\to\infty} u_C(t) = 0$$

Transient and Steady-State Solutions

- Thus for the following equation and its general solution,

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$
$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)},$$

we have

$$\lim_{t \to \infty} u_C(t) = \lim_{t \to \infty} \left(c_1 u_1(t) + c_2 u_2(t) \right) = 0$$

- Thus $u_C(t)$ is called the **transient solution**. Note however that

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

is a steady oscillation with same frequency as forcing function.

- For this reason, $U(t)$ is called the **steady-state solution**, or **forced response**.

Transient Solution and Initial Conditions

- For the following equation and its general solution,

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)}$$

  the transient solution $u_C(t)$ enables us to satisfy whatever initial conditions might be imposed.
- With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the response $U(t)$ of the system to the external force $F_0 \cos(\omega t)$.
- Without damping, the effect of the initial conditions would persist for all time.

Example 1 (1 of 2)

- Consider a spring-mass system satisfying the differential equation and initial condition

$$u'' + u' + \frac{5}{4}u = 3\cos t, \quad u(0) = 2, \; u'(0) = 3$$

- Begin by finding the solution to the homogeneous equation
- The methods of Chapter 3.3 yield the solution

$$u_C(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t$$

- A particular solution to the nonhomogeneous equation will have the form $U(t)$ = A cos t + B sin t and substitution gives $A = \frac{12}{17}$ and $B = \frac{48}{17}$. So

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

$$u'' + u' + 1.25u = 0$$

Example 1 (2 of 2)

$$u(0) = 2, \; u'(0) = 3$$

- The general solution for the nonhomogeneous equation is

$$u(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

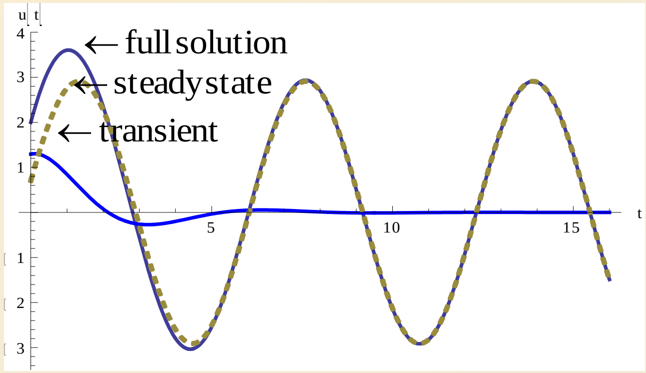
- Applying the initial conditions yields

$$\left.\begin{aligned} u(0) &= c_1 + \frac{12}{17} = 2 \\ u'(0) &= -\tfrac{1}{2}c_1 + c_2 + \frac{48}{17} = 3 \end{aligned}\right\} \Rightarrow c_1 = \frac{22}{17},\; c_2 = \frac{14}{17}$$

- Therefore, the solution to the IVP is

$$u(t) = \frac{22}{17}e^{-t/2}\cos t + \frac{14}{17}e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

- The graph breaks the solution into its steady state ($U(t)$) and transient ($u_C(t)$) components



Rewriting Forced Response

- Using trigonometric identities, it can be shown that

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

can be rewritten as

$$U(t) = R\cos(\omega t - \delta)$$

- It can also be shown that

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

where $w_0^2 = \dfrac{k}{m}$

Amplitude Analysis of Forced Response

- The amplitude R of the steady state solution

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

depends on the driving frequency ω. For low-frequency excitation we have

$$\lim_{\omega \to 0} R = \lim_{\omega \to 0} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

where we recall $(\omega_0)^2 = k/m$. Note that F_0/k is the static displacement of the spring produced by force F_0.

- For high frequency excitation,

$$\lim_{\omega \to \infty} R = \lim_{\omega \to \infty} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = 0$$

Maximum Amplitude of Forced Response

- Thus

$$\lim_{\omega \to 0} R = F_0 / k, \quad \lim_{\omega \to \infty} R = 0$$

- At an intermediate value of ω, the amplitude R may have a maximum value. To find this frequency ω, differentiate R and set the result equal to zero. Solving for $\omega_{\max}$, we obtain

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk} \right)$$

where $(\omega_0)^2 = k/m$. Note $\omega_{\max} < \omega_0$, and $\omega_{\max}$ is close to ω_0 for small γ. The maximum value of R is

$$R_{\max} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - (\gamma^2 / 4mk)}}$$

Maximum Amplitude for Imaginary $\omega_{\max}$

- We have

$$\omega_{\max}^2 = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right)$$

and

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

where the last expression is an approximation for small γ. If $\gamma^2/(mk) > 2$, then $\omega_{\max}$ is imaginary.

- In this case the maximum value of R occurs for $\omega = 0$, and R is a monotone decreasing function of ω.
- Recall that critical damping occurs when $\dfrac{\gamma^2}{mk} = 4$.

Resonance

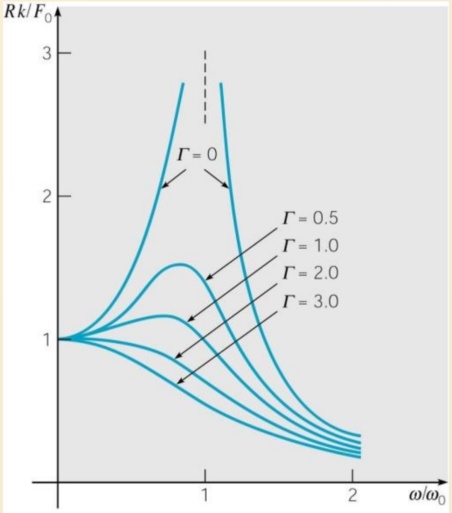
- From the expression

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

  we see that $R_{\max} \cong \frac{F_0}{\gamma\omega_0}$ for small γ.
- Thus for lightly damped systems, the amplitude R of the forced response is large for near ω_0. ω
- This is true even for relatively small external forces, and the smaller the γ the greater the effect.
- This phenomena is known as **resonance**. Resonance can be either good or bad, depending on circumstances; for example, when building bridges or designing seismographs.

Graphical Analysis of Quantities

- To get a better understanding of the quantities we have been examining, we graph the ratios $\frac{Rk}{F_0}$ versus $\frac{\omega}{\omega_0}$ for several values of $\Gamma = \frac{\gamma^2}{mk}$, as shown below.
- Note that the peaks tend to get higher as damping decreases.
- As damping decreases to zero, the values of Rk/F_0 become asymptotic to $\omega = \omega_0$.
- The graph corresponding to $\Gamma = 0.015625$ is included because it appears in the next example.



Analysis of Phase Angle

- Recall that the phase angle δ given in the forced response

$$U(t) = R\cos(\omega t - \delta)$$

is characterized by the equations

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \sin\delta = \frac{\gamma\,\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

- For ω near zero, $\cos\delta \cong 1$ and $\sin\delta \cong 0$, and they rise and fall together. Assuming their maxima and minima nearly together.
- For $\omega = \omega_0$, $\cos\delta = 0$ and $\sin\delta = 1$, so $\delta = \dfrac{\pi}{2}$ and response lags behind the excitation.
- For very large ω, $\delta \cong \pi$, and the response is out of phase. That is the response is a minimum when excitation is a maximum.

Example 2:
Forced Vibrations with Damping (1 of 4)

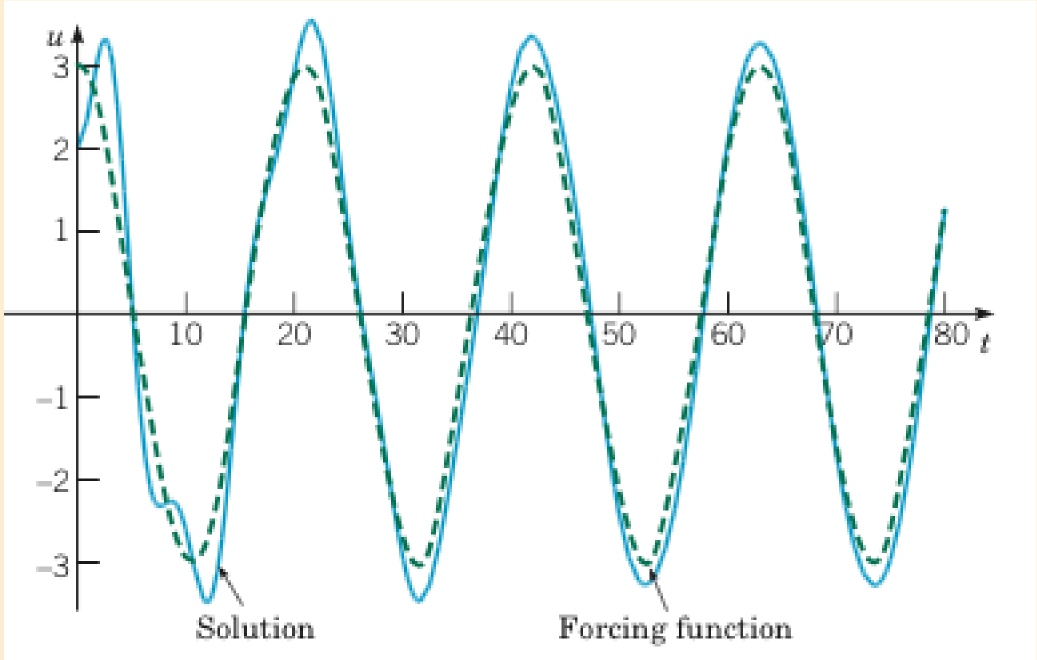
- Consider the initial value problem

$$u''(t) + \frac{1}{8}u'(t) + u(t) = 3\cos\omega t,\ u(0) = 2,\ u'(0) = 0$$

- Then $\omega_0 = 1$, $F_0/k = 3$, and $\Gamma = 1/64 = 0.015625$
- The unforced motion of this system was discussed in Ch 3.7, with the graph of the solution on the next slide, along with the graph of the ratios Rk/F vs. ω/ω_0 for different values of ω.

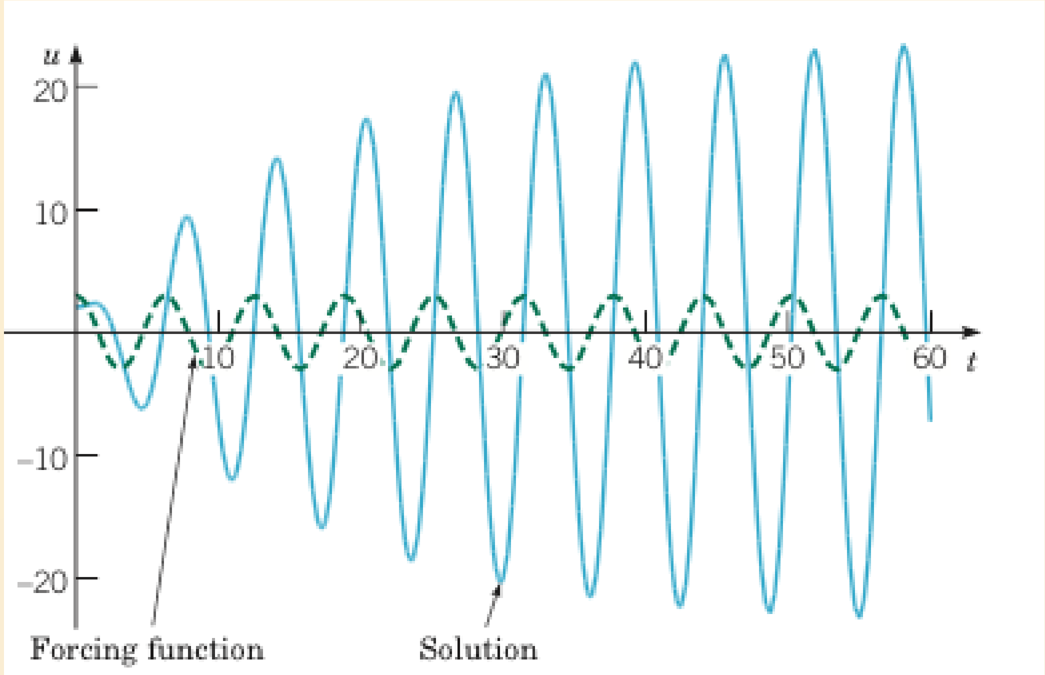
Example 2: Forced Vibrations with Damping

- Graphs of the solution, along with the graph of the ratios Rk/F vs. ω/ω_0 for $\omega = 0.3$



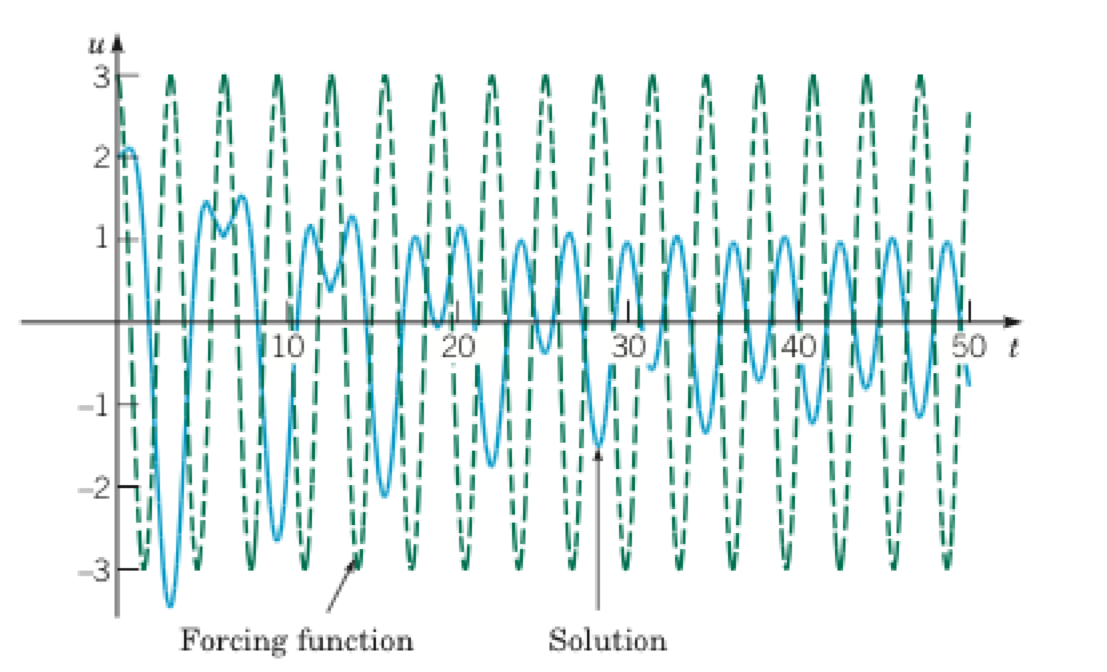
Example 2: Forced Vibrations with Damping

- Graphs of the solution, along with the graph of the ratios Rk/F vs. ω / ω_0 for $\omega = 1$



Example 2: Forced Vibrations with Damping

- Graphs of the solution, along with the graph of the ratios *Rk*/*F* vs. ω / ω_0 for $\omega = 2$



Undamped Equation: General Solution for the Case $\gamma = 0$

- Suppose there is no damping term. Then our equation is

$$mu''(t) + ku(t) = F_0 \cos(\omega t)$$

- Assuming $\omega \neq \omega_0$, then the method of undetermined coefficients can be use to show that the general solution is

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Undamped Equation: Mass Initially at Rest (1 of 3)

- If the mass is initially at rest, then the corresponding initial value problem is

$$mu''(t) + ku(t) = F_0 \cos \omega t, \;\; u(0) = 0, \; u'(0) = 0$$

- Recall that the general solution to the differential equation is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

- Using the initial conditions to solve for c_1 and c_2, we obtain

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0$$

- Hence

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t \right)$$

Undamped Equation: Solution to Initial Value Problem (2 of 3)

- Thus our solution is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos \omega t - \cos \omega_0 t\right)$$

- To simplify the solution even further, let $A = \frac{1}{2}(\omega_0 + \omega)t$ and $B = \frac{1}{2}(\omega_0 - \omega)t$. Then $A + B = \omega_0 t$ and $A - B = \omega t$. Using the trigonometric identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

it follows that

$$\cos \omega t = \cos A \cos B + \sin A \sin B$$

$$\cos \omega_0 t = \cos A \cos B - \sin A \sin B$$

and hence

$$\cos \omega t - \cos \omega_0 t = 2 \sin A \sin B$$

Undamped Equation: Beats (3 of 3)

- Using the results of the previous slide, it follows that

$$u(t)=\left[\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{(\omega_0-\omega)t}{2}\right]\sin\frac{(\omega_0+\omega)t}{2}$$

- When

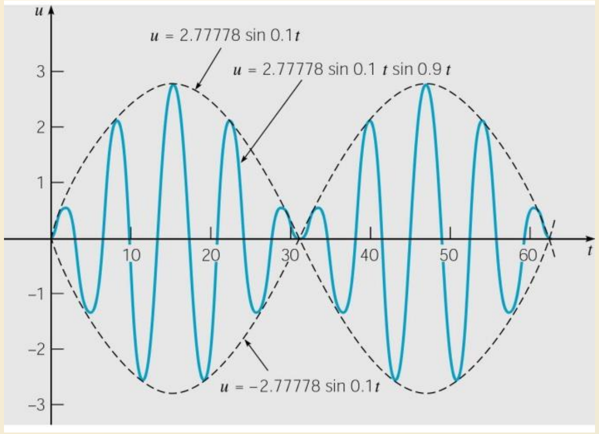
$|\omega_0-\omega|\cong 0$, then $\omega_0+\omega$ is much greater than $|\omega_0+\omega|$.

So $\sin(\frac{1}{2}(\omega_0+\omega)t)$ is oscillating more rapidly than $\sin(\frac{1}{2}(\omega_0-\omega)t)$.

- Thus motion is a rapid oscillation with frequency $\dfrac{\omega_0+\omega}{2}$, but with slowly varying sinusoidal amplitude given by

$$\frac{2F_0}{m\left|\omega_0^2-\omega^2\right|}\left|\sin\frac{(\omega_0-\omega)t}{2}\right|$$

- This phenomena is called a **beat**.
- Beats occur with two tuning forks of nearly equal frequency.



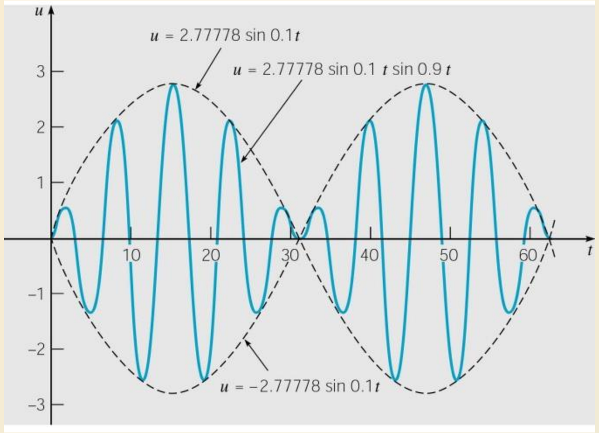
Example 3: Undamped Equation, Mass Initially at Rest (1 of 2)

- Consider the initial value problem

$$u''(t) + u(t) = 0.5\cos 0.8t, \quad u(0) = 0, \; u'(0) = 0$$

- Then $\omega_0 = 1, \; \omega = 0.8, \text{ and } F_0 = \dfrac{1}{2}$, and hence the solution is

- The displacement of the spring–mass system oscillates with a frequency of 0.9, slightly less than natural frequency $\omega_0 = 1$.
- The amplitude variation has a slow frequency of 0.1 and period of 20π.
- A half-period of 10π corresponds to a single cycle of increasing and then decreasing amplitude.

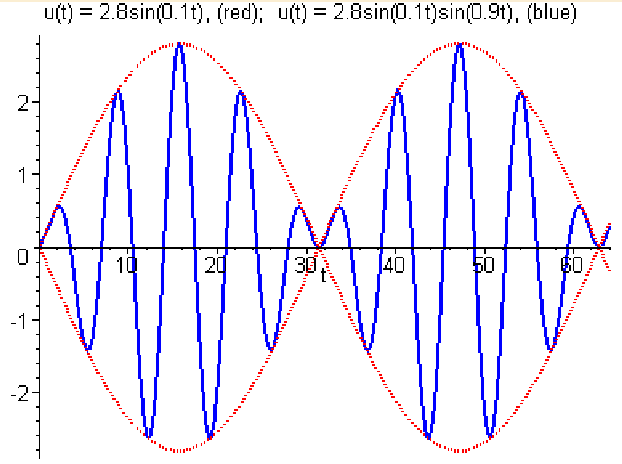


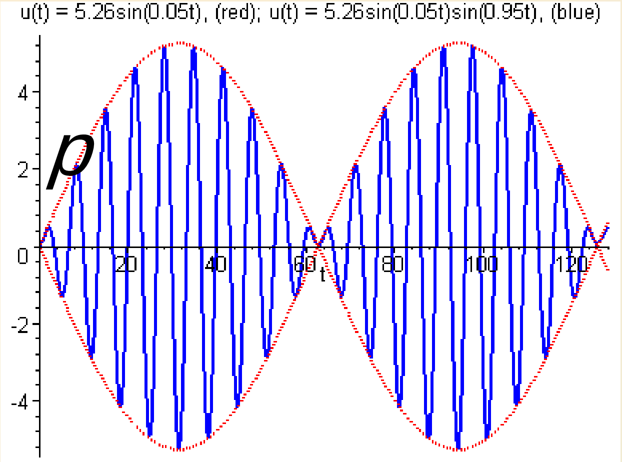
Example 3: Increased Frequency (2 of 2)

- Recall our initial value problem

$$u''(t) + u(t) = 0.5\cos 0.8t, \;\; u(0) = 0, \; u'(0) = 0$$

- If driving frequency ω is increased to 0.9, then the slow frequency is halved to 0.05 with half-period doubled to 20π.
- The multiplier 2.77778 is increased to 5.2632, and the fast frequency only marginally increased, to 0.095.





Undamped Equation: General Solution for the Case $\omega_0 = \omega$ (1 of 2)

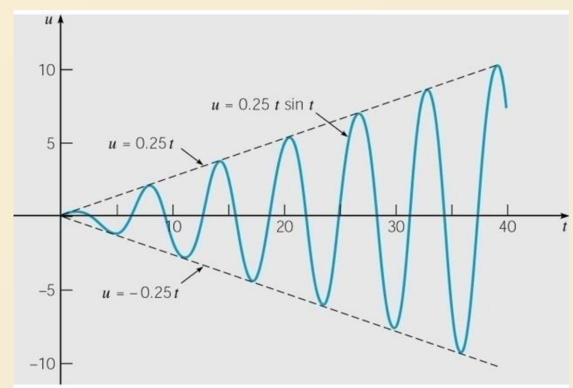
- Recall our equation for the undamped case:

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

- If forcing frequency equals natural frequency of system, i.e., $\omega = \omega_0$, then nonhomogeneous term $F_0 \cos \omega t$ is a solution of homogeneous equation. It can then be shown that

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- Thus solution u becomes unbounded.
- Note: Model invalid when u gets large, since we assume small oscillations u.

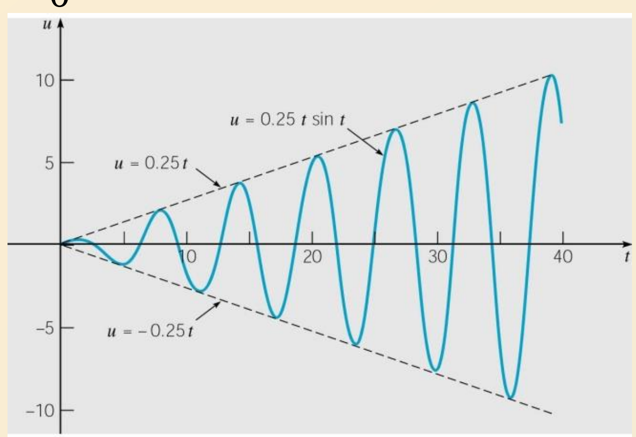


Undamped Equation: Resonance (2 of 2)

- If forcing frequency equals natural frequency of system, i.e., $\omega = \omega_0$, then our solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- Motion *u* remains bounded if damping present. However, response *u* to input $F_0 \cos \omega t$ may be large if damping is small and $\omega \cong \omega_0$, in which case we have resonance.



Example 4

- Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos t,\ u(0) = 0,\ u'(0) = 0$$

And plot the graph of the solution.

The general solution of the differential equation is

$$u = c_1 \cos t + c_2 \sin t + \frac{1}{4} t \sin t$$

And the initial conditions require that $c_1 = c_2 = 0$. Thus the solution of the given initial value problem is

$$u = \frac{t}{4}\sin t$$