Boyce/DiPrima/Meade 11th ed, Ch 2.1: Linear Equations; Method of Integrating Factors

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce and Richard C. DiPrima, ©2017 by John Wiley & Sons, Inc.

- A linear first order ODE has the general form

$$\frac{dy}{dt} = f(t, y)$$

where *f* is linear in *y*. Examples include equations with constant coefficients, such as those in Chapter 1,

$$y' = -ay + b$$

or equations with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Constant Coefficient Case

- For a first order linear equation with constant coefficients,

$$\frac{dy}{dt} = -ay + b,$$

recall that we can use methods of calculus to solve:

$$\frac{dy/dt}{y - b/a} = -a$$

$$\int \frac{dy}{y - b/a} = -\int a\,dt$$

$$\ln\left|y - b/a\right| = -a\,t + C$$

$$y = b/a + ke^{at},\;\; k = \pm e^{C}$$

Variable Coefficient Case: Method of Integrating Factors

- We next consider linear first order ODEs with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

- The method of integrating factors involves multiplying this equation by a function $m(t)$, chosen so that the resulting equation is easily integrated.

Example 2: Integrating Factor (1 of 2)

- Consider the following equation:

$$\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3}$$

- Multiplying both sides by $m(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t) y = \frac{1}{2}\mu(t) e^{t/3}$$

- We will choose $m(t)$ so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}\left(m(t)y\right) = m(t)\frac{dy}{dt} + \frac{d\,m(t)}{dt} y$$

- Choose $m(t)$ so that

$$\mu'(t) = \frac{1}{2}\mu(t) \quad \Rightarrow \quad \mu(t) = e^{t/2}$$

Example 2: General Solution (2 of 2)

- With $m(t) = e^{t/2}$, we solve the original equation as follows:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}$$

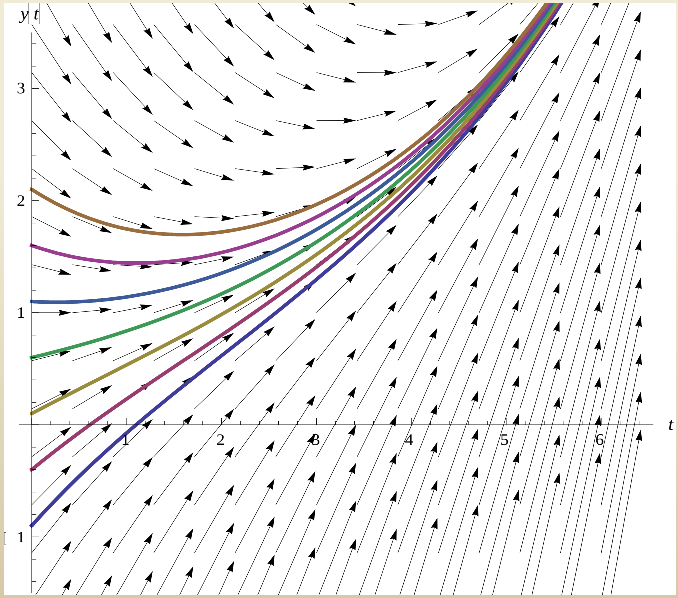
$$\frac{d}{dt}\left(e^{t/2}y\right) = \frac{1}{2}e^{5t/6}$$

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + c$$

general solution:

$$y = \frac{3}{5}e^{t/3} + ce^{-t/2}$$

Sample Solutions : $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$



Method of Integrating Factors: Variable Right Side

- In general, for variable right side $g(t)$, the solution can be found by choosing $\mu(t) = e^{at}$:

$$\frac{dy}{dt} + ay = g(t)$$

$$\mu(t)\frac{dy}{dt} + a\mu(t)y = \mu(t)g(t)$$

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}g(t)$$

$$e^{at}y = \int e^{at}g(t)\,dt + c$$

$$y = e^{-at}\int e^{at}g(t)\,dt + ce^{-at}$$

Example 3: General Solution (1 of 2)

- We can solve the following equation

$$\frac{dy}{dt} - 2y = 4 - t$$

by multiplying by the integrating factor $\mu(t) = e^{-2t}$:
giving us $\frac{d}{dt}(e^{-2t}y) = 4e^{-2t} - te^{-2t}$ which we can integrate on both sides.

- Integrating by parts, $e^{-2t}y = \int 4e^{-2t} - te^{-2t}\,dt$

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c$$

$$e^{-2t}y = -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + c$$

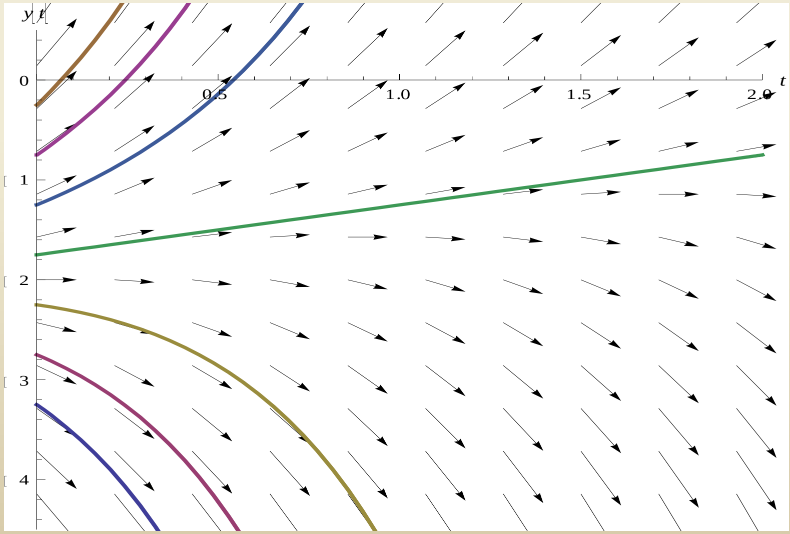
- Thus $y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}$

$$\frac{dy}{dt} - 2y = 4 - t$$

Example 3: Graphs of Solutions (2 of 2)

- The graph shows the direction field along with several integral curves. If we set $c = 0$, the exponential term drops out and you should notice how the solution in that case, through the point (0, -7/4), separates the solutions into those that grow exponentially in the positive direction from those that grow exponentially in the negative direction..

$$y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}$$



Method of Integrating Factors for General First Order Linear Equation

- Next, we consider the general first order linear equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

- Multiplying both sides by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t)$$

- Next, we want $\mu(t)$ such that $\frac{d\mu(t)}{dt} = p(t)\mu(t)$, from which it will follow that

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$$

Integrating Factor for General First Order Linear Equation

- Assuming $m(t) > 0$, it follows that

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \;\; \Rightarrow \;\; \ln \mu(t) = \int p(t)dt + k$$

- Choosing $k = 0$, we then have

$$\mu(t) = e^{\int p(t)dt},$$

and note $m(t) > 0$ as desired.

Solution for
General First Order Linear Equation

- Thus we have the following:

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

- Then

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(t)g(t)\,dt + c$$

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right)$$

where t_0 is some convenient lower limit of integration.

Example 4: General Solution (1 of 2)

- To solve the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

first put into standard form:

$$y' + \frac{2}{t}y = 4t, \quad \text{for } t \neq 0$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln(t^2)} = t^2$$

and hence

$$t^2y' + 2ty = (t^2y)' = 4t^3 \text{ Þ } t^2y = t^4 + c \text{ Þ } y = t^2 + \frac{c}{t^2}$$

Giving us the solution $y = t^2 + \dfrac{x}{t^2}$

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

Example 4: Particular Solution (2 of 2)

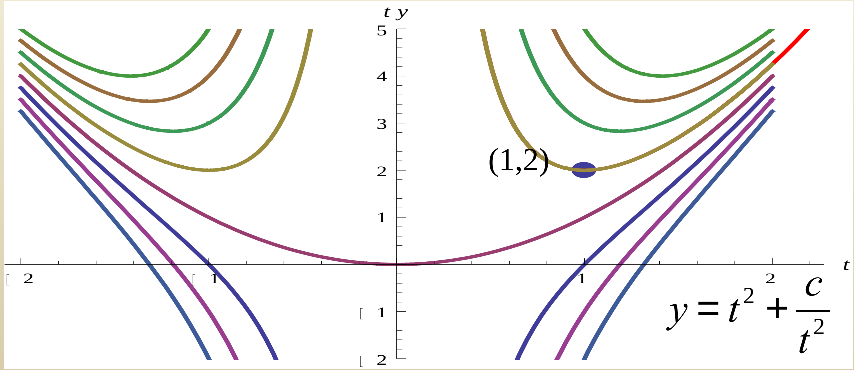
- Using the initial condition $y(1) = 2$ and general solution

$$y = t^2 + \frac{c}{t^2}, \quad 2 = 1 + c \Rightarrow c = 1$$

it follows that

$$y = t^2 + \frac{1}{t^2}, \quad t > 0$$

- The graphs below show solution curves for the differential equation, including a particular solution whose graph contains the initial point (1,2).
- Notice that when c=0, we get the parabolic solution $y = t^2$ and that solution separates the solutions into those that are asymptotic to the positive versus negative y-axis.



Example 5: A Solution in Integral Form (1 of 2)

- To solve the initial value problem

$$2y' + ty = 2, \quad y(0) = 1,$$

first put into standard form:

$$y' + \frac{t}{2}y = 1$$

- Then

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{t}{2}dt} = e^{\frac{t^2}{4}}$$

and hence

$$y = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds + c\right) = e^{-t^2/4}\left(\int_0^t e^{s^2/4}ds\right) + ce^{-t^2/4}$$

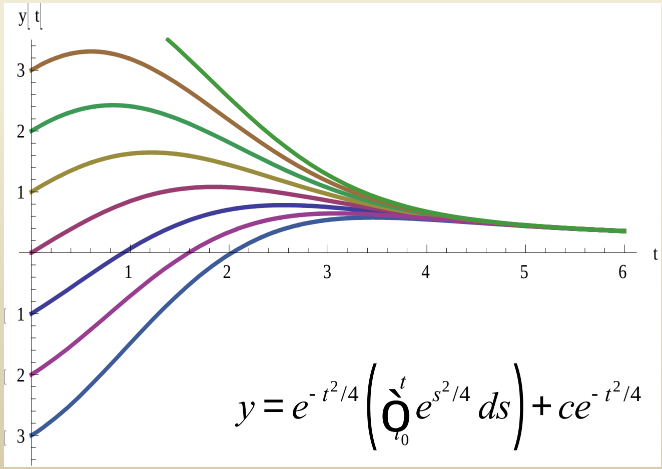
$$2y' + ty = 2, \quad y(0) = 1,$$

Example 5: A Solution in Integral Form (2 of 2)

- Notice that this solution must be left in the form of an integral, since there is no closed form for the integral.

$$y = e^{-t^2/4}\left(\int_{t_0}^{t} e^{s^2/4}\,ds\right) + ce^{-t^2/4}$$

- Using software such as *Mathematica* or Maple, we can approximate the solution for the given initial conditions as well as for other initial conditions.
- Several solution curves are shown.



Boyce/DiPrima/Meade 11th ed, Ch 2.2: Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,

$$M(x, y)dx + N(x, y)dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)dx + N(y)dy = 0$$

- In this case, the equation is called **separable**.

Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

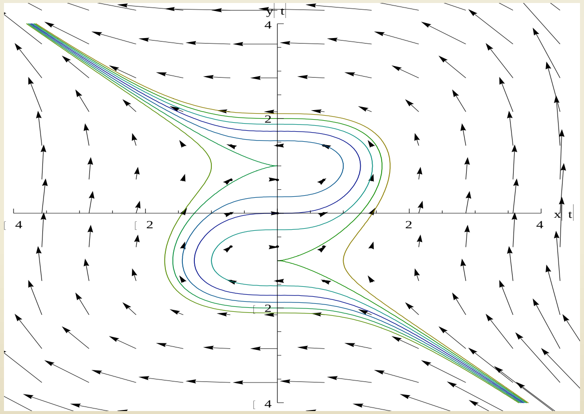
- Separating variables, and using calculus, we obtain

$$\left(1-y^2\right)dy = \left(x^2\right)dx$$

$$\int\left(1-y^2\right)dy = \int\left(x^2\right)dx$$

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + c$$

$$3y - y^3 = x^3 + c$$



- The equation above defines the solution *y* implicitly. A graph showing the direction field and implicit plots of several solution curves for the differential equation is given above.

Example 2:
Implicit and Explicit Solutions (1 of 4)

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

- Separating variables and using calculus, we obtain

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx$$

$$2\int (y-1)\,dy = \int (3x^2 + 4x + 2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

- The equation above defines the solution *y* implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - (x^3 + 2x^2 + 2x + c) = 0 \Rightarrow \quad y = \frac{2 \pm \sqrt{4 + 4(x^3 + 2x^2 + 2x + c)}}{2}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Example 2: Initial Value Problem (2 of 4)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = C \quad \Rightarrow \quad C = 3$$

- Thus the implicit equation defining y is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

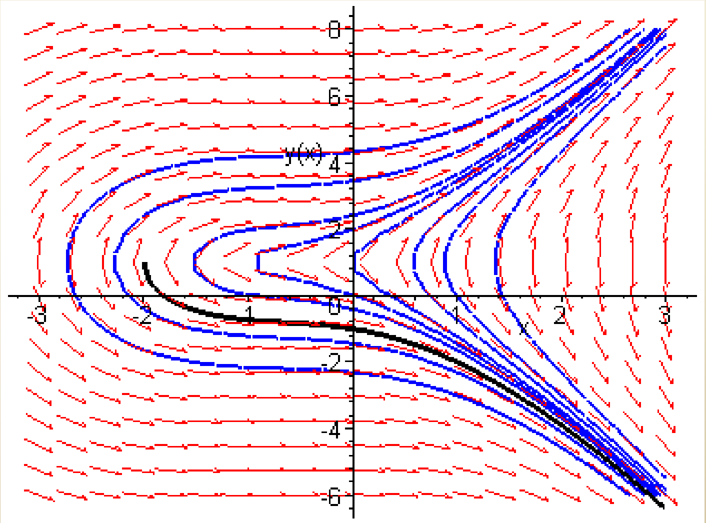
- Using an explicit expression of y,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

$$-1 = 1 \pm \sqrt{C} \quad \Rightarrow \quad C = 4$$

- It follows that

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$



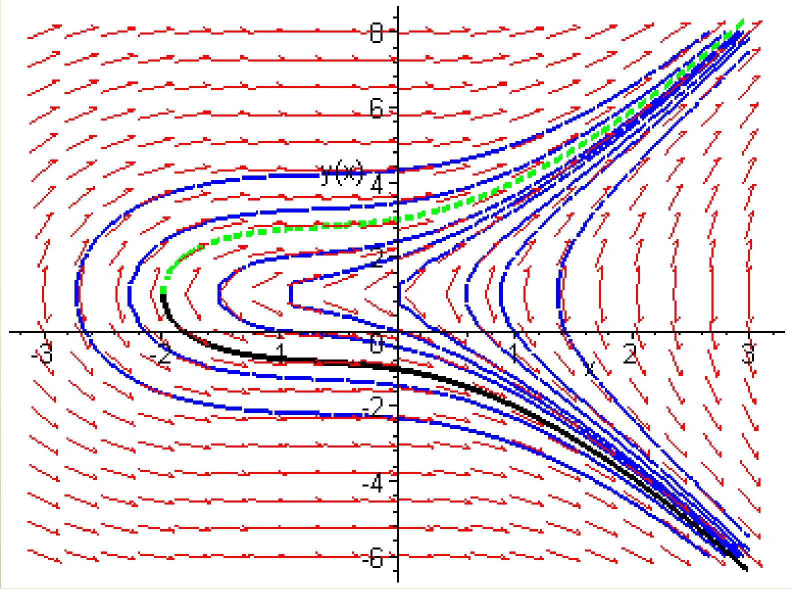
$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Example 2: Initial Condition $y(0) = 3$

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on the square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

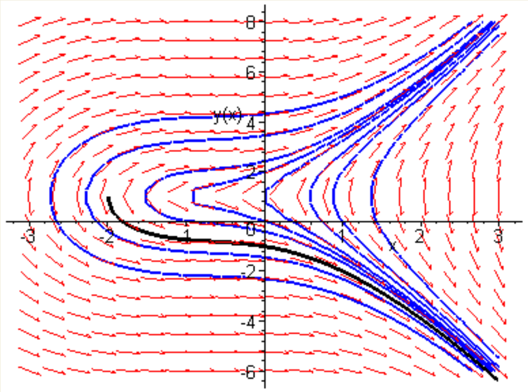
- This is indicated on the graph in green.



Example 2: Domain (4 of 4)

- Thus the solutions to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$



are given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \quad \text{(implicit)}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{(explicit)}$$

- From explicit representation of *y*, it follows that

$$y = 1 - \sqrt{x^2(x+2) + 2(x+2)} = 1 - \sqrt{(x+2)(x^2+2)}$$

and hence the domain of *y* is (–2, ¥). Note *x* = –2 yields *y* = 1, which makes the denominator of *dy*/*dx* zero (vertical tangent).

- Conversely, the domain of *y* can be estimated by locating vertical tangents on the graph (useful for implicitly defined solutions).

Example 3: Implicit Solution of an Initial Value Problem (1 of 2)

- Consider the following initial value problem:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1$$

- Separating variables and using calculus, we obtain

$$(4 + y^3)dy = (4x - x^3)dx$$

$$\int \left(4 + y^3\right)dy = \int (4x - x^3)dx$$

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + c$$

$$y^4 + 16y + x^4 - 8x^2 = C \text{ where } C = 4c$$

- Using the initial condition, $y(0)=1$, it follows that $C = 17$.

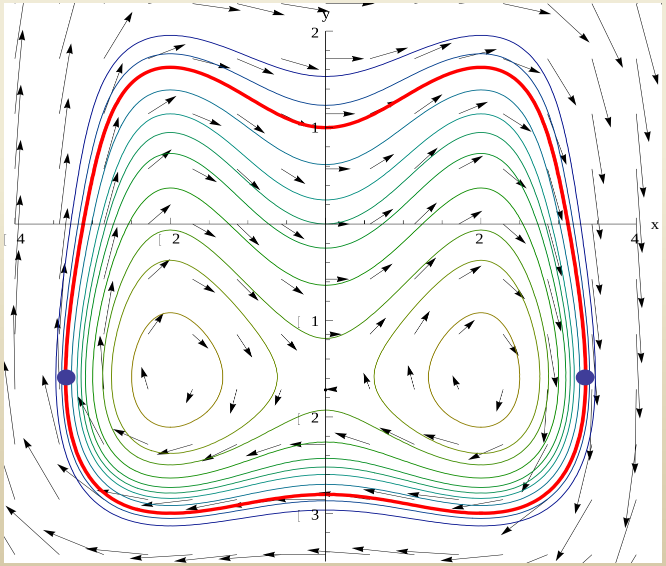
$$y^4 + 16y + x^4 - 8x^2 = 17$$

$$y' = \frac{4x - x^3}{4 + y^3}, \quad y(0) = 1$$

Example 3: Graph of Solutions (2 of 2)

- Thus the general solution is $y^4 + 16y + x^4 - 8x^2 = C$

  and the solution through (0,2) is $y^4 + 16y + x^4 - 8x^2 = 17$
- The graph of this particular solution through (0, 2) is shown in red along with the graphs of the direction field and several other solution curves for this differential equation, are shown:
- The points identified with blue dots correspond to the points on the red curve where the tangent line is vertical: $y = \sqrt[3]{-4} \approx -1.5874$ $x \approx \pm 3.3488$ on the red curve, but at all points where the line connecting the blue points intersects solution curves the tangent line is vertical.



Parametric Equations

- The differential equation: $$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}$$

  is sometimes easier to solve if x and y are thought of as dependent variables of the independent variable t and rewriting the single differential equation as the system of differential equations:

$$\frac{dy}{dt} = F(x, y) \text{ and } \frac{dx}{dt} = G(x, y)$$

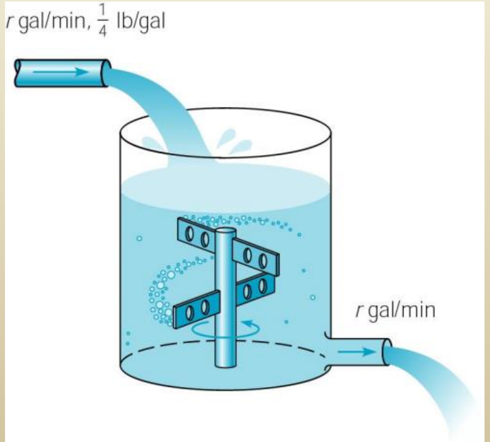
  Chapter 9 is devoted to the solution of systems such as these.

Boyce/DiPrima/Meade 11th ed, Ch 2.3: Modeling with First Order Equations

- Mathematical models characterize physical systems, often using differential equations.
- **Model Construction**: Translating physical situation into mathematical terms. Clearly state physical principles believed to govern process. Differential equation is a mathematical model of process, typically an approximation.
- **Analysis of Model**: Solving equations or obtaining qualitative understanding of solution. May simplify model, as long as physical essentials are preserved.
- **Comparison with Experiment or Observation**: Verifies solution or suggests refinement of model.

Example 1: Salt Solution   (1 of 7)

- At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing ¼ lb of salt/gal is entering tank at rate of r gal/min, and leaves at same rate.
  - (a) Set up IVP that describes this salt solution flow process.
  - (b) Find amount of salt $Q(t)$ in tank at any given time t.
  - (c) Find limiting amount Q_L of salt $Q(t)$ in tank after a very long time.
  - (d) If $r = 3$ & $Q_0 = 2Q_L$, find time T after which salt is within 2% of Q_L .
  - (e) Find flow rate r required if T is not to exceed 45 min.



Example 1: (a) Initial Value Problem (2 of 7)

- At time $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume water containing ¼ lb of salt/gal enters tank at rate of r gal/min, and leaves at same rate.
- Assume salt is neither created or destroyed in tank, and distribution of salt in tank is uniform (stirred). Then

$$dQ/dt = \text{rate in} - \text{rate out}$$

- Rate in: (1/4 lb salt/gal)(r gal/min) = (r/4) lb/min
- Rate out: If there is $Q(t)$ lbs salt in tank at time t, then concentration of salt is $Q(t)$ lb/100 gal, and it flows out at rate of $[Q(t)r/100]$ lb/min.
- Thus our IVP is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \quad Q(0) = Q_0$$

Example 1: (b) Find Solution $Q(t)$

- To find amount of salt $Q(t)$ in tank at any given time t, we need to solve the initial value problem

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}, \quad Q(0) = Q_0$$

- To solve, we use the method of integrating factors:

$$\mu(t) = e^{at} = e^{rt/100}$$

$$Q(t) = e^{-rt/100}\left[\int \frac{re^{rt/100}}{4}\,dt\right] = e^{-rt/100}\left[25e^{rt/100} + c\right] = 25 + ce^{-rt/100}$$

$$Q(t) = 25 + \left[Q_0 - 25\right] e^{-rt/100}$$

or

$$Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0\, e^{-rt/100}$$

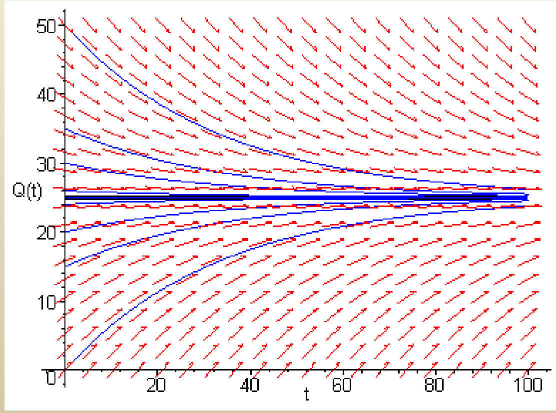
Example 1:
(c) Find Limiting Amount Q_L (4 of 7)

- Next, we find the limiting amount Q_L of salt $Q(t)$ in tank after a very long time:

$$Q_L = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(25 + [Q_0 - 25]e^{-rt/100}\right) = 25\text{ lb}$$

- This result makes sense, since over time the incoming salt solution will replace original salt solution in tank. Since incoming solution contains 0.25 lb salt / gal, and tank is 100 gal, eventually tank will contain 25 lb salt.
- The graph shows integral curves for $r = 3$ and different values of Q_0.

$$Q(t) = 25\left(1 - e^{-rt/100}\right) + Q_0\, e^{-rt/100}$$



Example 1: (d) Find Time T (5 of 7)

- Suppose $r = 3$ and $Q_0 = 2Q_L$. To find time T after which $Q(t)$ is within 2% of Q_L, first note $Q_0 = 2Q_L = 50$ lb, hence

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100} = 25 + 25e^{-0.03t}$$

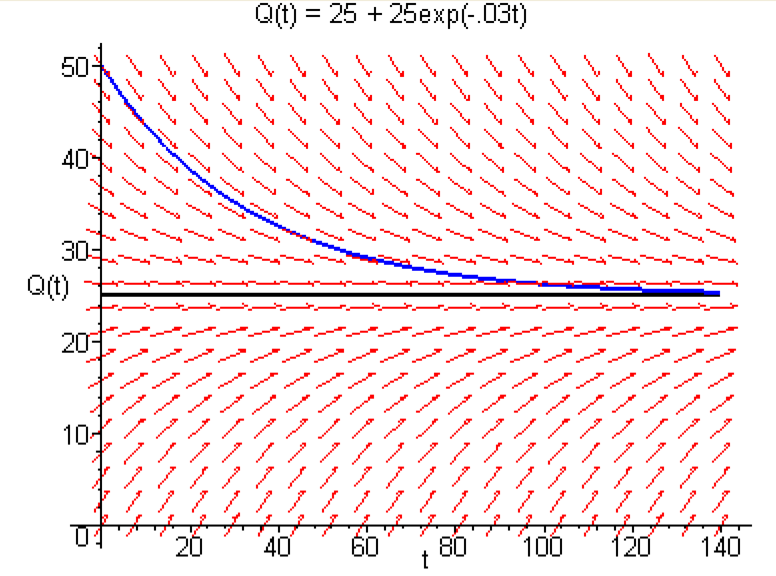
- Next, 2% of 25 lb is 0.5 lb, and thus we solve

$$25.5 = 25 + 25e^{-0.03T}$$

$$0.02 = e^{-0.03T}$$

$$\ln(0.02) = -0.03T$$

$$T = \frac{\ln(0.02)}{-0.03} \approx 130.4 \text{ min}$$



Example 1: (e) Find Flow Rate (6 of 7)

- To find flow rate *r* required if *T* is not to exceed 45 minutes, recall from part (d) that $Q_0 = 2Q_L = 50$ lb, with

$$Q(t) = 25 + 25e^{-rt/100}$$

and solution curves decrease from 50 to 25.5.

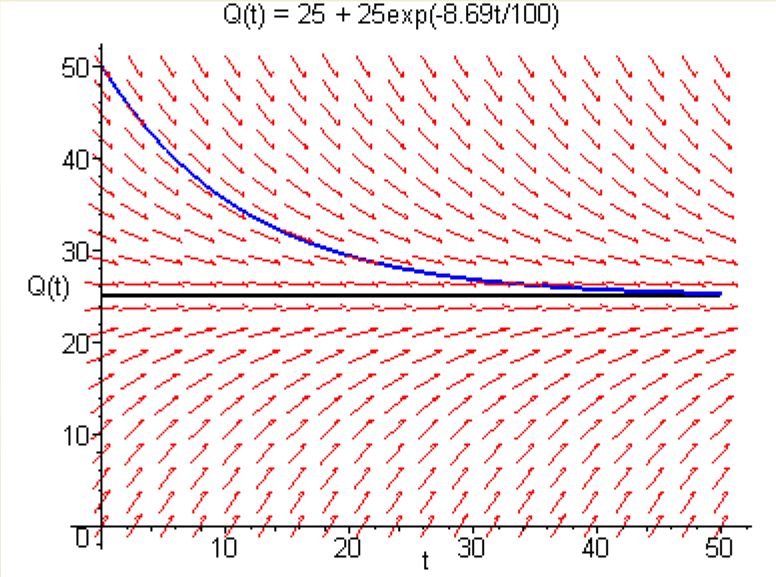
- Thus we solve

$$25.5 = 25 + 25e^{-\frac{45}{100}r}$$

$$0.02 = e^{-0.45r}$$

$$\ln(0.02) = -.45r$$

$$r = \frac{\ln(0.02)}{-0.45} \approx 8.69 \text{ gal/min}$$



Example 1: Discussion (7 of 7)

- Since this situation is hypothetical, the model is valid.
- As long as flow rates are accurate, and concentration of salt in tank is uniform, then differential equation is accurate description of the flow process.
- Models of this kind are often used for pollution in lake, drug concentration in organ, etc. Flow rates may be harder to determine, or may be variable, and concentration may not be uniform. Also, rates of inflow and outflow may not be same, so variation in amount of liquid must be taken into account.

Example 2: Compound Interest (1 of 3)

- If a sum of money is deposited in a bank that pays interest at an annual rate, *r*, compounded **continuously**, the amount of money (*S*) at any time in the fund will satisty the differential equation:

$$\frac{dS}{dt} = rS, \quad S(0) = S_0 \text{ where } S_0 \text{ represents the initial investment.}$$

- The solution to this differential equation, found by separating the variables and solving for *S*, becomes:

$$S(t) = S_0 e^{rt}, \text{ where } t \text{ is measured in years}$$

- Thus, with continuous compounding, the amount in the account grows exponentially over time.

$$S(t) = S_0 e^{rt}$$

Example 2: Compound Interest (2 of 3)

- In general, if interest in an account is to be compounded *m* times a year, rather than continuously, the equation describing the amount in the account for any time *t*, measured in years, becomes: $S(t) = S_0(1 + \frac{r}{m})^{mt}$
- The relationship between these two results is clarified if we recall from calculus that $\lim_{m \to \infty} S_0(1 + \frac{r}{m})^{mt} = S_0 e^{rt}$

Growth of Capital at a Return Rate of *r* = 8% For Several Modes of Compounding: *S(t)/S(0)*			
t	*m = 4*	*m = 365*	*exp(rt)*
Years	Compounded Quarterly	Compounded Daily	Compounded Continuously
1	1.082432	1.083278	1.083287
2	1.171659	1.17349	1.173511
5	1.485947	1.491759	1.491825
10	2.20804	2.225346	2.225541
20	4.875439	4.952164	4.953032
30	10.76516	11.02028	11.02318
40	23.76991	24.52393	24.53253

A comparison of the accumulation of funds for quarterly, daily, and continuous compounding is shown for short-term and long-term periods.

Example 2: Deposits and Withdrawals (3 of 3)

- Returning now to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate *k*, this is described by the differential equation:

  $$\frac{dS}{dt} = rS + k \text{ or in standard form } \frac{dS}{dt} - rS = k \text{ and } S(0) = S_0$$

  where k is positive for deposits and negative for withdrawals.
- We can solve this as a general linear equation to arrive at the solution: $S(t) = S_0 e^{rt} + (k/r)(e^{rt} - 1)$
- To apply this equation, suppose that one opens an IRA at age 25 and makes annual investments of \$2000 thereafter with r = 8%.
- At age 65, $S(40) = 0 * e^{0.08*40} + (2000/0.08)(e^{0.08*40} - 1) \approx \$588{,}313$

Example 3: Pond Pollution (1 of 7)

- Consider a pond that initially contains 10 million gallons of fresh water. Water containing toxic waste flows into the pond at the rate of 5 million gal/year, and exits at same rate. The concentration $c(t)$ of toxic waste in the incoming water varies periodically with time:

$$c(t) = 2 + \sin(2t) \text{ g/gal}$$

(a) Construct a mathematical model of this flow process and determine amount $Q(t)$ of toxic waste in pond at time t.

(b) Plot solution and describe in words the effect of the variation in the incoming concentration.

Example 3: (a) Initial Value Problem (2 of 7)

- Pond initially contains 10 million gallons of fresh water. Water containing toxic waste flows into pond at rate of 5 million gal/year, and exits pond at same rate. Concentration is $c(t) = 2 + \sin 2t$ g/gal of toxic waste in incoming water.
- Assume toxic waste is neither created or destroyed in pond, and distribution of toxic waste in pond is uniform (stirred).
- Then $dQ/dt = \text{rate in} - \text{rate out}$
- Rate in: $(2 + \sin(2t))\text{g/gal}(5 \times 10^6)\text{gal/year}$
- If there is $Q(t)$ g of toxic waste in pond at time t, then concentration of salt is $Q(t)$ lb/10^7 gal, thus
- Rate out: $(5 \times 10^6)\text{gal/year}(Q(t)/10^7)\text{g/gal} = Q(t)/2 \text{ g/yr}$

Example 3:
(a) Initial Value Problem, Scaling (3 of 7)

- Recall from previous slide that
  - Rate in: (2 + sin 2t g/gal)(5 x 10^6 gal/year)
  - Rate out: ($Q(t)$ g/10^7 gal)(5 x 10^6 gal/year) = $Q(t)/2$ g/yr.
- Then initial value problem is

$$\frac{dQ}{dt} = (2 + \sin 2t)(5 \text{ x } 10^6) - \frac{Q(t)}{2}, \;\; Q(0) = 0$$

- Change of variable (scaling): Let $q(t) = Q(t)/10^6$. Then

$$\frac{dq}{dt} + \frac{q}{2} = 10 + 5 \sin 2t, \;\; q(0) = 0$$

Example 3:
(a) Solve Initial Value Problem (4 of 7)

- To solve the initial value problem

$$q' + q/2 = 10 + 5\sin 2t, \quad q(0) = 0$$

we use the method of integrating factors:

$$\mu(t) = e^{at} = e^{t/2}$$

$$q(t) = e^{-t/2} \int e^{t/2} \left(10 + 5\sin 2t\right) dt$$

- Using integration by parts (see next slide for details) and the initial condition, we obtain after simplifying,

$$q(t) = e^{-t/2}\left[20e^{t/2} - \frac{40}{17}e^{t/2}\cos(2t) + \frac{10}{17}e^{t/2}\sin(2t) + c\right]$$

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-t/2}$$

Example 3: (a) Integration by Parts

$$\int e^{t/2}\sin(2t)\,dt = \left[-\frac{1}{2}e^{t/2}\cos(2t)+\frac{1}{4}\left(\int e^{t/2}\cos(2t)\,dt\right)\right]$$

$$= \left[-\frac{1}{2}e^{t/2}\cos(2t)+\frac{1}{4}\left(\frac{1}{2}e^{t/2}\sin(2t)-\frac{1}{4}\int e^{t/2}\sin(2t)\,dt\right)\right]$$

$$= \left[-\frac{1}{2}e^{t/2}\cos(2t)+\frac{1}{8}e^{t/2}\sin(2t)-\frac{1}{16}\int e^{t/2}\sin(2t)\,dt\right]$$

$$\frac{17}{16}\int e^{t/2}\sin(2t)\,dt = -\frac{1}{2}e^{t/2}\cos(2t)+\frac{1}{8}e^{t/2}\sin(2t)+c$$

$$\int e^{t/2}\sin(2t)\,dt = -\frac{8}{17}e^{t/2}\cos(2t)+\frac{2}{17}e^{t/2}\sin(2t)+c$$

$$5\int e^{t/2}\sin(2t)\,dt = -\frac{40}{17}e^{t/2}\cos(2t)+\frac{10}{17}e^{t/2}\sin(2t)+c$$

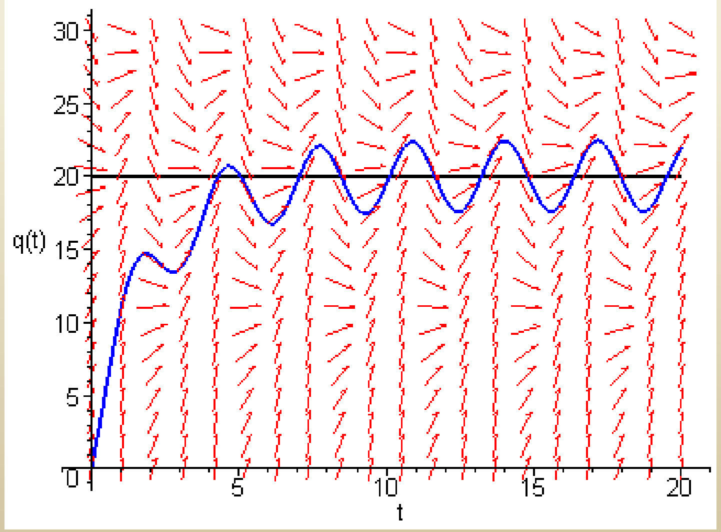
Example 3: (b) Analysis of solution

- Thus our initial value problem and solution is

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin(2t),\ \ q(0) = 0$$

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-t/2}$$

- A graph of solution along with direction field for differential equation is given below.
- Note that exponential term is important for small t, but decays away for large t. Also, $y = 20$ would be equilibrium solution if not for $\sin(2t)$ term.



Example 3:
(b) Analysis of Assumptions (7 of 7)

- Amount of water in pond controlled entirely by rates of flow, and none is lost by evaporation or seepage into ground, or gained by rainfall, etc.
- Amount of pollution in pond controlled entirely by rates of flow, and none is lost by evaporation, seepage into ground, diluted by rainfall, absorbed by fish, plants or other organisms, etc.
- Distribution of pollution throughout pond is uniform.

Example 4: Escape Velocity (1 of 2)

- A body of mass m is projected away from the earth in a direction perpendicular to the earth's surface with initial velocity v_0 and no air resistance. Taking into account the variation of the earth's gravitational field with distance, the gravitational force acting on the mass is

$$w(x) = -\frac{mgR^2}{(R+x)^2} \quad \text{where } x \text{ is the distance above the earth's surface}$$

R is the radius of the earth and g is the acceleration due to gravity at the earth's surface. Using Newton's law F = ma,

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \; v(0) = v_0$$

- Since $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$ and cancelling the m's, the differential equation becomes $v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}$, since $x = 0$ when $t = 0$, $v(0) = v_0$

$$v\frac{dv}{dx} = \frac{gR^2}{(R+x)^2},\ v(0) = v_0$$

Example 4: Escape Velocity (2 of 2)

- We can solve the differential equation by separating the variables and integrating to arrive at:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c = \frac{gR^2}{R+x} + \frac{v_0^2}{2} - gR$$

- The maximum height (altitude) will be reached when the velocity is zero. Calling that maximum height A_{max}, we have

$$A_{max} = \frac{v_0^2 R}{2gR - v_0^2}$$

- We can now find the initial velocity required to lift a body to a height A_{max}: $v_0 = \sqrt{2gR\frac{A_{max}}{R + A_{max}}}$

- and, taking the limit as $A_{max} \to \infty$, we get $v_0 = \sqrt{2gR}$ the escape velocity.
- Notice that this does not depend on the mass of the body.

Boyce/DiPrima/Meade 11th ed, Ch 2.4: Differences Between Linear and Nonlinear Equations

- Recall that a first order ODE has the form $y' = f(t,y)$, and is linear if f is linear in y, and nonlinear if f is nonlinear in y.
- Examples: $y' = ty - e^t$, $y' = ty^2$.
- In this section, we will see that first order linear and nonlinear equations differ in a number of ways, including:
  - The theory describing existence and uniqueness of solutions, and corresponding domains, are different.
  - Solutions to linear equations can be expressed in terms of a general solution, which is not usually the case for nonlinear equations.
  - Linear equations have explicitly defined solutions while nonlinear equations typically do not, and nonlinear equations may or may not have implicitly defined solutions.
- For both types of equations, numerical and graphical construction of solutions are important.

Theorem 2.4.1

- Consider the linear first order initial value problem:

$$y' + p(t)y = g(t), \quad y(0) = y_0$$

If the functions p and g are continuous on an open interval $I: a < t < b$ containing the point $t = t_0$, then there exists a unique function $y = f(t)$ that satisfies the IVP for each t in I.

- **Proof outline:** Use Ch 2.1 discussion and results:

$$y = \frac{\int_{t_0}^{t} \mu(t)g(t)dt + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

Theorem 2.4.2

- Consider the nonlinear first order initial value problem:

$$y' = f(t, y), \;\; y(0) = y_0$$

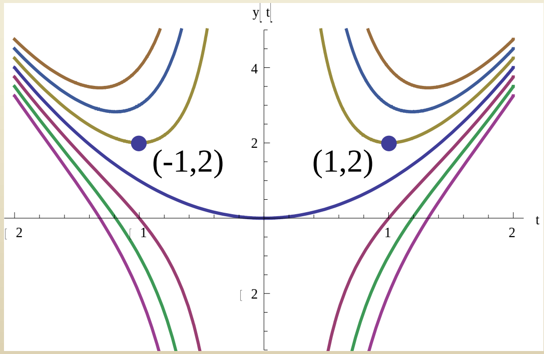
- Let the functions *f* and $\P f / \P y$ be continuous in some rectangle $a < t < b$, $g < y < d$ containing the point (t_0, y_0).
- Then in some interval $t_0 - h < t < t_0 + h$ in the rectangle there is a unique solution $y = f(t)$ of the initial value problem.
-
- **Proof discussion:** Since there is no general formula for the solution of arbitrary nonlinear first order IVPs, this proof is difficult, and is beyond the scope of this course.
- It turns out that conditions stated in Thm 2.4.2 are sufficient but not necessary to guarantee existence of a solution, and continuity of *f* ensures existence but not uniqueness of . $y = f(t)$

Example 1: Linear IVP

- Recall the initial value problem from Chapter 2.1 slides:

$$ty' + 2y = 4t^2, \quad y(1) = 2 \quad \Rightarrow \quad y = t^2 + \frac{1}{t^2}$$

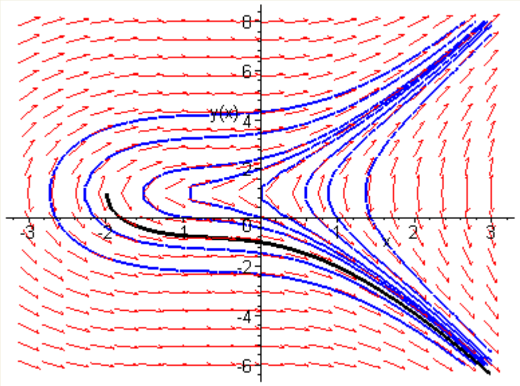
- The solution to this initial value problem is defined for $t > 0$, the interval on which $p(t) = 2/t$ is continuous.
- If the initial condition is $y(-1) = 2$, then the solution is given by same expression as above, but is defined on $t < 0$.
- In either case, Theorem 2.4.1 guarantees that solution is unique on corresponding interval.



Example 2: Nonlinear IVP (1 of 2)

- Consider nonlinear initial value problem from Ch 2.2:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$



- The functions *f* and ¶*f* / ¶*y* are given by

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

  and are continuous except on line *y* = 1.

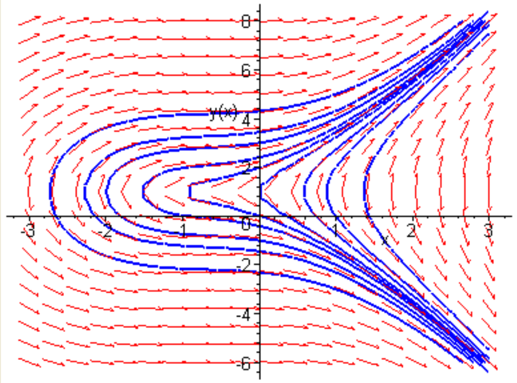
- Thus we can draw an open rectangle about (0, –1) in which *f* and ¶*f* / ¶*y* are continuous, as long as it doesn't cover *y* = 1.
- How wide is the rectangle? Recall solution defined for *x* > –2, with

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Example 2: Change Initial Condition (2 of 2)

- Our nonlinear initial value problem is

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$



with

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2},$$

which are continuous except on line $y = 1$.

- If we change initial condition to $y(0) = 1$, then Theorem 2.4.2 is not satisfied. Solving this new IVP, we obtain

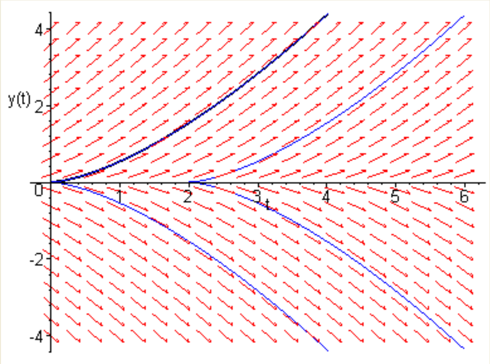
$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}, \quad x > 0$$

- Thus a solution exists but is not unique.

Example 3: Nonlinear IVP

- Consider nonlinear initial value problem

$$y' = y^{1/3}, \ y(0) = 0 \qquad (t \geq 0)$$



- The functions f and $\partial f / \partial y$ are given by

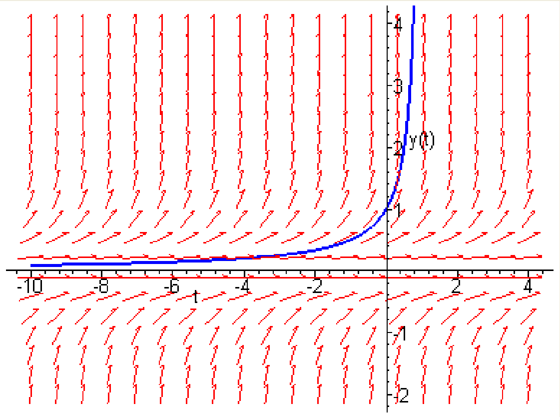
$$f(t, y) = y^{1/3}, \ \frac{\partial f}{\partial y}(t, y) = \frac{1}{3} y^{-2/3}$$

- Thus f continuous everywhere, but $\partial f / \partial y$ doesn't exist at $y = 0$, and hence Theorem 2.4.2 does not apply. Solutions exist but are not unique. Separating variables and solving, we obtain

$$y^{-1/3} dy = dt \ \Rightarrow \ \frac{3}{2} y^{2/3} = t + c \ \Rightarrow \ y = \pm\left(\frac{2}{3}t\right)^{3/2}, \ t \geq 0$$

- If initial condition is not on t-axis, then Theorem 2.4.2 does guarantee existence and uniqueness.

Example 4: Nonlinear IVP



- Consider nonlinear initial value problem

$$y' = y^2, \; y(0) = 1$$

- The functions f and $\partial f / \partial y$ are given by

$$f(t, y) = y^2, \; \frac{\partial f}{\partial y}(t, y) = 2y$$

- Thus f and $\partial f / \partial y$ are continuous at $t = 0$, so Theorem 2.4.2 guarantees that solutions exist and are unique.
- Separating variables and solving, we obtain

$$y^{-2} dy = dt \;\Rightarrow\; -y^{-1} = t + c \;\Rightarrow\; y = -\frac{1}{t+c} \;\Rightarrow\; y = \frac{1}{1-t}$$

- The solution $y(t)$ is defined on $(-\infty, 1)$. Note that the singularity at $t = 1$ is not obvious from original IVP statement.

Interval of Existence: Linear Equations

- By Theorem 2.4.1, the solution of a linear initial value problem

$$y' + p(t)y = g(t), \;\; y(0) = y_0$$

exists throughout any interval about $t = t_0$ on which p and g are continuous.

- Vertical asymptotes or other discontinuities of solution can only occur at points of discontinuity of p or g.
- However, solution may be differentiable at points of discontinuity of p or g. See Chapter 2.1: Example 3 of text.
- Compare these comments with Example 1 and with previous linear equations in Chapter 1 and Chapter 2.

Interval of Existence: Nonlinear Equations

- In the nonlinear case, the interval on which a solution exists may be difficult to determine.
- The solution $y = f(t)$ exists as long as $[t, f(t)]$ remains within a rectangular region indicated in Theorem 2.4.2. This is what determines the value of h in that theorem. Since $f(t)$ is usually not known, it may be impossible to determine this region.
- In any case, the interval on which a solution exists may have no simple relationship to the function f in the differential equation $y' = f(t, y)$, in contrast with linear equations.
- Furthermore, any singularities in the solution may depend on the initial condition as well as the equation.
- Compare these comments to the preceding examples.

General Solutions

- For a first order linear equation, it is possible to obtain a solution containing one arbitrary constant, from which all solutions follow by specifying values for this constant.
- For nonlinear equations, such general solutions may not exist. That is, even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by specifying values for this constant.
- Consider Example 4: The function $y = 0$ is a solution of the differential equation, but it cannot be obtained by specifying a value for c in solution found using separation of variables:

$$\frac{dy}{dt} = y^2 \Rightarrow y = -\frac{1}{t+c}$$

Explicit Solutions: Linear Equations

- By Theorem 2.4.1, a solution of a linear initial value problem

$$y' + p(t)y = g(t), \ \ y(0) = y_0$$

exists throughout any interval about $t = t_0$ on which p and g are continuous, and this solution is unique.

- The solution has an explicit representation,

$$y = \frac{\int_{t_0}^{t} \mu(t) g(t) dt + y_0}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s) ds},$$

and can be evaluated at any appropriate value of t, as long as the necessary integrals can be computed.

Explicit Solution Approximation

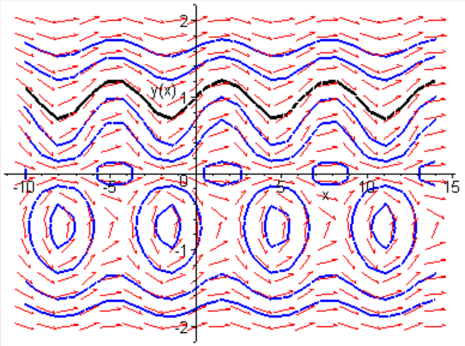
- For linear first order equations, an explicit representation for the solution can be found, as long as necessary integrals can be solved.
- If integrals can't be solved, then numerical methods are often used to approximate the integrals.

$$y = \frac{\int_{t_0}^{t} \mu(t)g(t)dt + C}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

$$\int_{t_0}^{t} \mu(t)g(t)dt \approx \sum_{k=1}^{n} \mu(t_k)g(t_k)\Delta t_k$$

Implicit Solutions: Nonlinear Equations

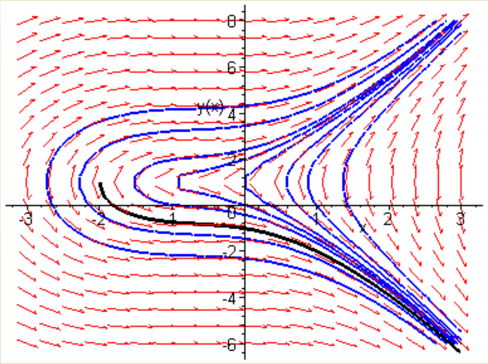
- For nonlinear equations, explicit representations of solutions may not exist.
- As we have seen, it may be possible to obtain an equation which implicitly defines the solution. If equation is simple enough, an explicit representation can sometimes be found.
- Otherwise, numerical calculations are necessary in order to determine values of *y* for given values of *t*. These values can then be plotted in a sketch of the integral curve.
- Recall the examples from earlier in the chapter and consider the following example

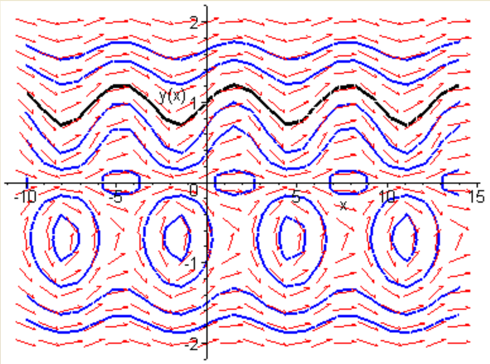


$$y' = \frac{y \cos x}{1+3y^3}, \quad y(0)=1 \;\Rightarrow\; \ln y + y^3 = \sin x + 1$$

Direction Fields

- In addition to using numerical methods to sketch the integral curve, the nonlinear equation itself can provide enough information to sketch a direction field.
- The direction field can often show the qualitative form of solutions, and can help identify regions in the *ty*-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigations.
- Chapter 2.7 and Chapter 8 focus on numerical methods.





Boyce/DiPrima/Meade 11th ed, Ch 2.5: Autonomous Equations and Population Dynamics

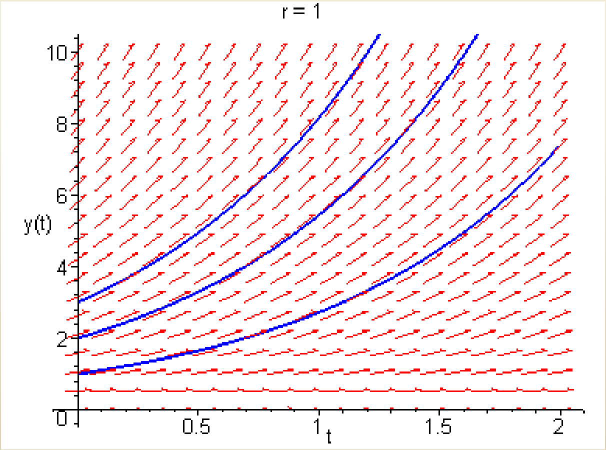
Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade

- In this section we examine equations of the form $dy/dt = f(y)$, called **autonomous** equations, where the independent variable t does not appear explicitly.
- The main purpose of this section is to learn how geometric methods can be used to obtain qualitative information directly from a differential equation without solving it.
- Example (Exponential Growth):

$$\frac{dy}{dt} = ry, \quad r > 0$$

- Solution:

$$y = y_0 e^{rt}$$



Logistic Growth

- An exponential model $y' = ry$, with solution $y = e^{rt}$, predicts unlimited growth, with rate $r > 0$ independent of population.
- Assuming instead that growth rate depends on population size, replace r by a function $h(y)$ to obtain $\frac{dy}{dt} = h(y)y$.
- We want to choose growth rate $h(y)$ so that
  - $h(y) \cong r > 0$ when y is small,
  - $h(y)$ decreases as y grows larger, and
  - $h(y) < 0$ when y is sufficiently large.

  The simplest such function is $h(y) = r - ay$, where $a > 0$.
- Our differential equation then becomes

$$\frac{dy}{dt} = (r - ay)y, \quad r, a > 0$$

- This equation is known as the Verhulst, or **logistic**, equation.

Logistic Equation

- The logistic equation from the previous slide is

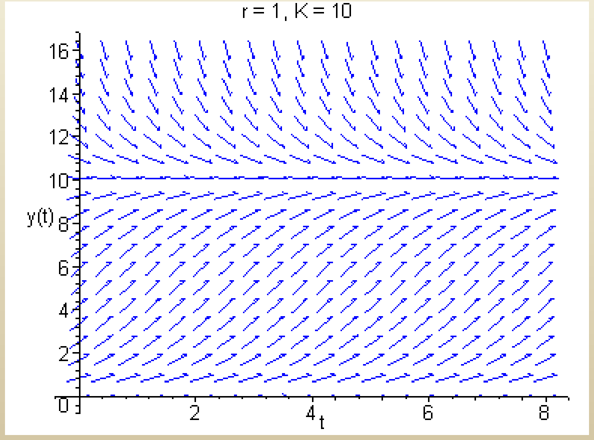
$$\frac{dy}{dt} = (r - ay)y, \quad r, a > 0$$

- This equation is often rewritten in the equivalent form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y,$$

  where $K = r/a$. The constant r is called the **intrinsic growth rate**, and as we will see, K represents the **carrying capacity** of the population.

- A direction field for the logistic equation with $r = 1$ and $K = 10$ is given here.



Logistic Equation: Equilibrium Solutions

- Our logistic equation is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r, K > 0$$

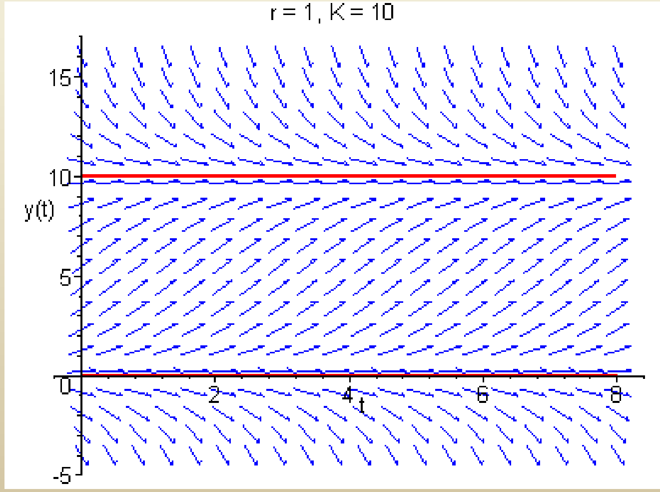
- Two **equilibrium solutions** are clearly present:

$$y = \phi_1(t) = 0, \quad y = \phi_2(t) = K$$

- In direction field below, with $r = 1$, $K = 10$, note behavior of solutions near equilibrium solutions:

  $y = 0$ is **unstable**,

  $y = 10$ is **asymptotically stable**.

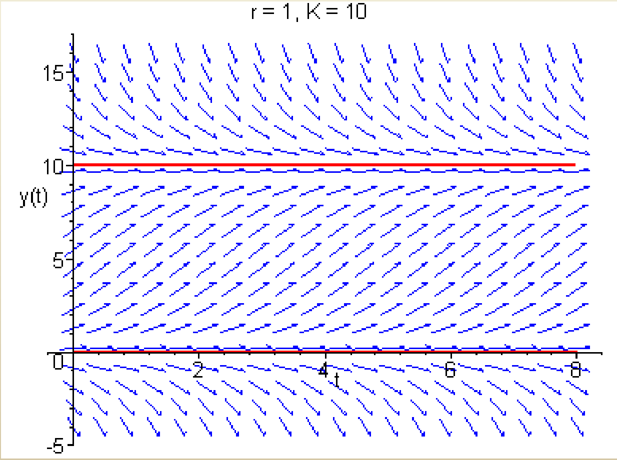


Autonomous Equations: Equilibrium Solns

- Equilibrium solutions of a general first order autonomous equation $y' = f(y)$ can be found by locating roots of $f(y) = 0$.
- These roots of $f(y)$ are called **critical points**.
- For example, the critical points of the logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

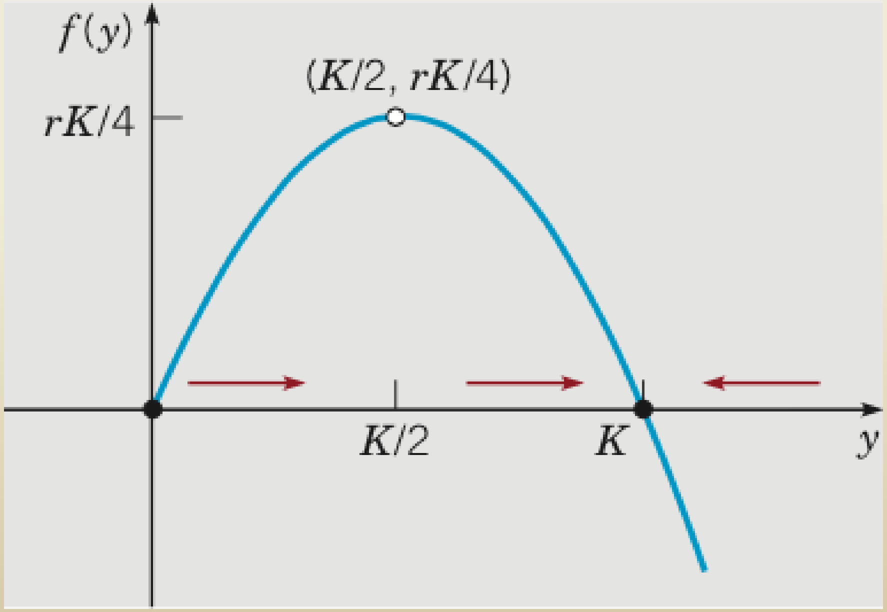
- are $y = 0$ and $y = K$.
- Thus critical points are constant functions (equilibrium solutions) in this setting.



Logistic Equation: Qualitative Analysis and Curve Sketching (1 of 7)

- To better understand the nature of solutions to autonomous equations, we start by graphing $f(y)$ vs. y.
- In the case of logistic growth, that means graphing the following function and analyzing its graph using calculus.

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$



Logistic Equation: Critical Points (2 of 7)

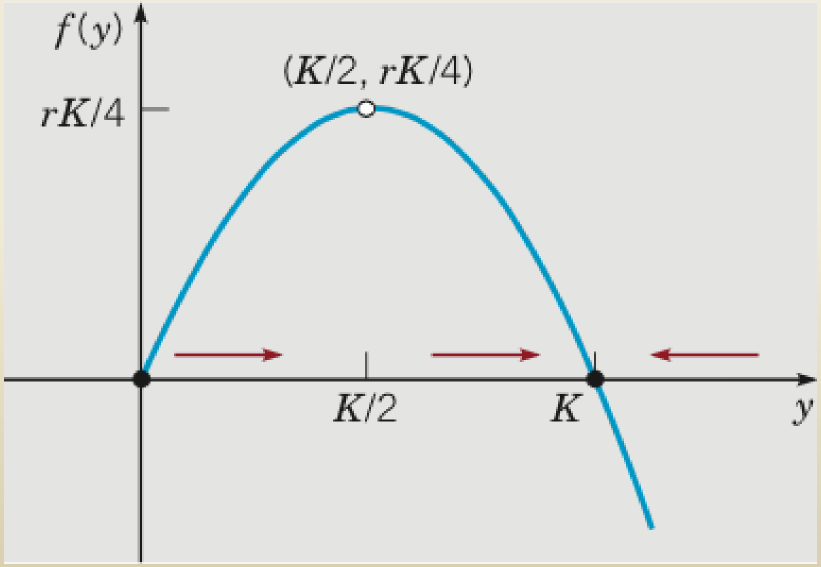
- The intercepts of *f* occur at *y* = 0 and *y* = *K*, corresponding to the critical points of logistic equation.
- The vertex of the parabola is (*K*/2, *rK*/4), as shown below.

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$

$$f'(y) = r\left[\left(-\frac{1}{K}\right)y + \left(1 - \frac{y}{K}\right)\right]$$

$$= -\frac{r}{K}\left[2y - K\right] \overset{set}{=} 0 \Rightarrow y = \frac{K}{2}$$

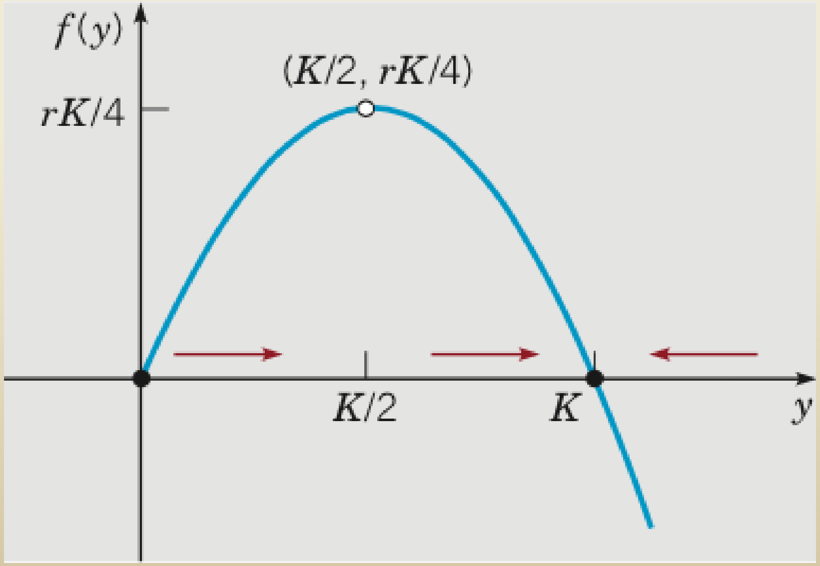
$$f\left(\frac{K}{2}\right) = r\left(1 - \frac{K}{2K}\right)\left(\frac{K}{2}\right) = \frac{rK}{4}$$



Logistic Solution: Increasing, Decreasing

- Note $dy/dt > 0$ for $0 < y < K$, so y is an increasing function of t there (indicate with right arrows along y-axis on $0 < y < K$).
- Similarly, y is a decreasing function of t for $y > K$ (indicate with left arrows along y-axis on $y > K$).
- In this context the y-axis is often called the **phase line**.

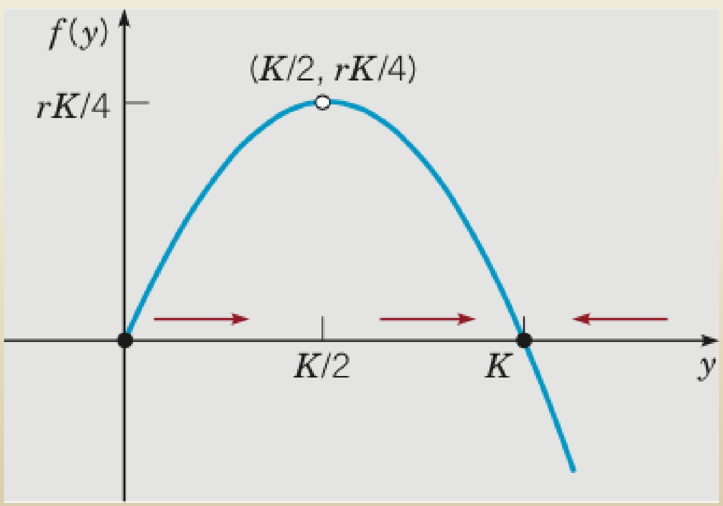
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r > 0$$

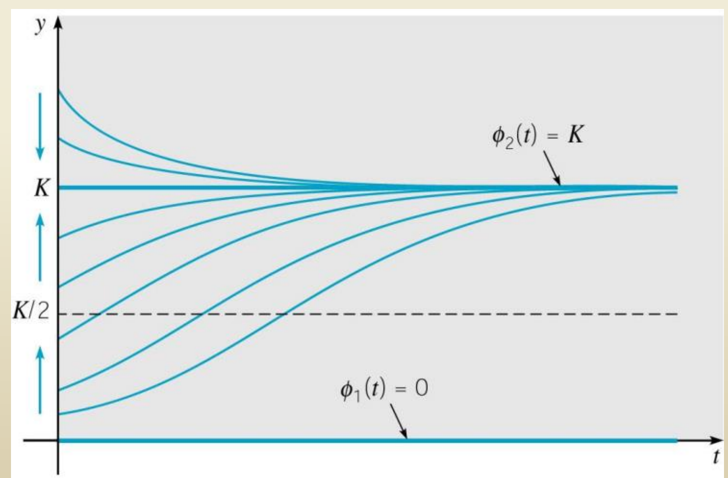


Logistic Solution: Steepness, Flatness (4 of 7)

- Note $dy/dt \cong 0$ when $y \cong 0$ or $y \cong K$, so y is relatively flat there, and y gets steep as y moves away from 0 or K.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$



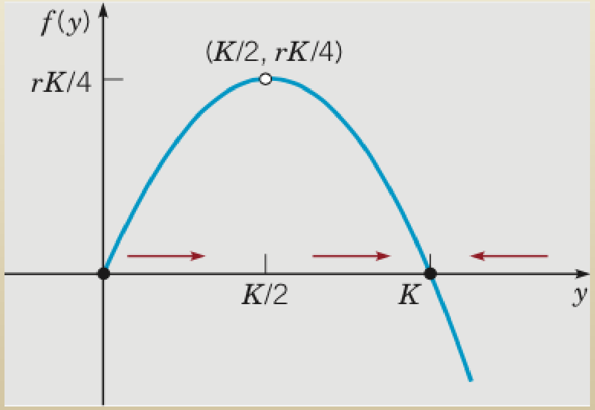


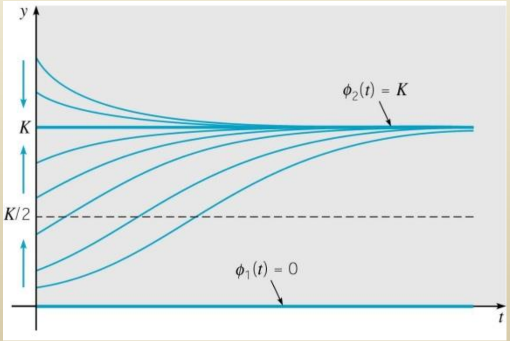
Logistic Solution: Concavity (5 of 7)

- Next, to examine concavity of $y(t)$, we find y'':

$$\frac{dy}{dt} = f(y) \Rightarrow \frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

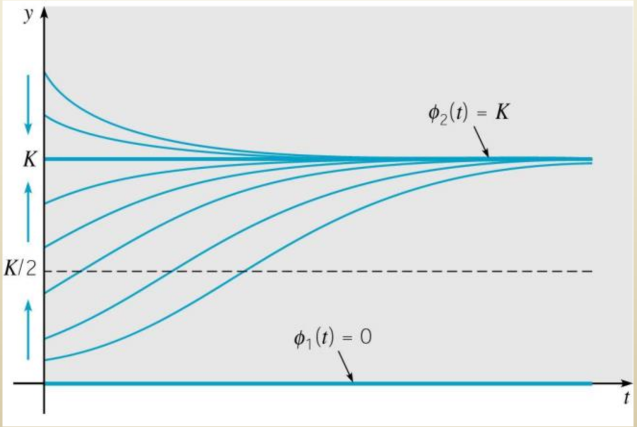
- Thus the graph of y is concave up when f and f' have same sign, which occurs when $0 < y < K/2$ and $y > K$.
- The graph of y is concave down when f and f' have opposite signs, which occurs when $K/2 < y < K$.
- Inflection point occurs at intersection of y and line $y = K/2$.





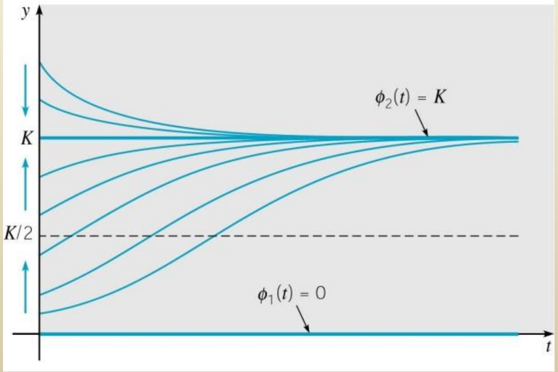
Logistic Solution: Curve Sketching (6 of 7)

- Combining the information on the previous slides, we have:
  - Graph of y increasing when $0 < y < K$.
  - Graph of y decreasing when $y > K$.
  - Slope of y approximately zero when $y \cong 0$ or $y \cong K$.
  - Graph of y concave up when $0 < y < K/2$ and $y > K$.
  - Graph of y concave down when $K/2 < y < K$.
  - Inflection point when $y = K/2$.
- Using this information, we can sketch solution curves y for different initial conditions.



Logistic Solution: Discussion (7 of 7)

- Using only the information present in the differential equation and without solving it, we obtained qualitative information about the solution *y*.
- For example, we know where the graph of *y* is the steepest, and hence where *y* changes most rapidly. Also, *y* tends asymptotically to the line $y = K$, for large *t*.
- The value of *K* is known as the **environmental carrying capacity**, or **saturation level**, for the species.
- Note how solution behavior differs from that of exponential equation, and thus the decisive effect of nonlinear term in logistic equation.



Solving the Logistic Equation (1 of 3)

- Provided $y \neq 0$ and $y \neq K$, we can rewrite the logistic ODE:

$$\frac{dy}{\left(1 - y/K\right)y} = r\,dt$$

- Expanding the left side using partial fractions,

$$\frac{1}{\left(1 - y/K\right)y} = \frac{A}{1 - y/K} + \frac{B}{y} \Rightarrow 1 = Ay + B\left(1 - y/K\right) \Rightarrow B = 1,\ A = y/K$$

- Thus the logistic equation can be rewritten as

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = r\,dt$$

- Integrating the above result, we obtain

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

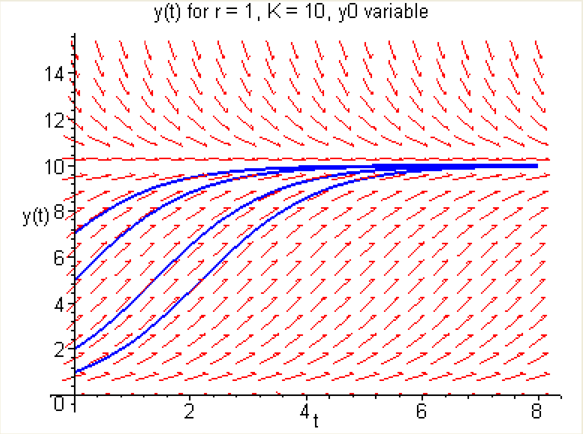
Solving the Logistic Equation (2 of 3)

- We have:

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

- If $0 < y_0 < K$, then $0 < y < K$ and hence

$$\ln y - \ln\left(1 - \frac{y}{K}\right) = rt + C$$



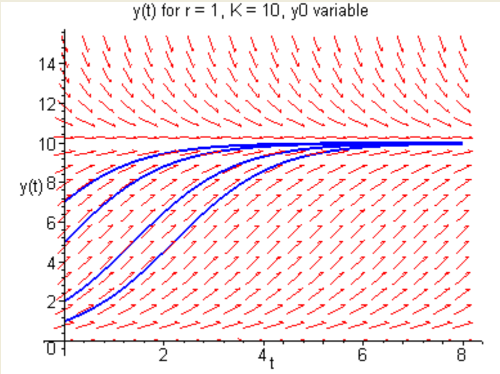
- Rewriting, using properties of logs:

$$\ln\left[\frac{y}{1 - y/K}\right] = rt + C \iff \frac{y}{1 - y/K} = e^{rt+C} \iff \frac{y}{1 - y/K} = ce^{rt}$$

or $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$, where $y_0 = y(0)$

Solution of the Logistic Equation (3 of 3)

- We have:

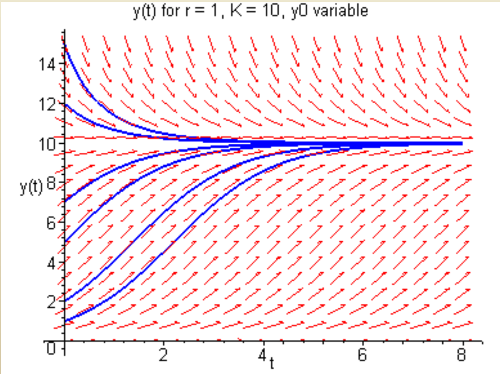


$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

for $0 < y_0 < K$.

- It can be shown that solution is also valid for $y_0 > K$. Also, this solution contains equilibrium solutions $y = 0$ and $y = K$.
- Hence solution to logistic equation is

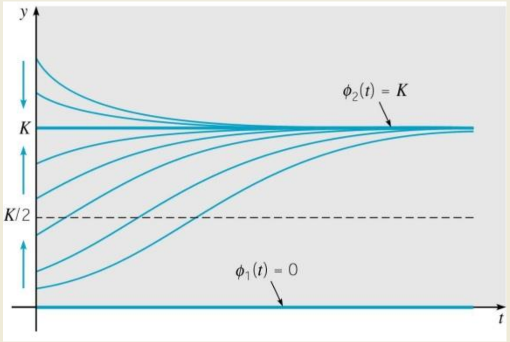
$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$



Logistic Solution: Asymptotic Behavior

- The solution to logistic ODE is

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$



- We use limits to confirm asymptotic behavior of solution:

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} = \lim_{t\to\infty} \frac{y_0 K}{y_0} = K$$

- Thus we can conclude that the equilibrium solution $y(t) = K$ is **asymptotically stable**, while equilibrium solution $y(t) = 0$ is **unstable**.
- The only way to guarantee that the solution remains near zero is to make $y_0 = 0$.

$$y = \frac{y_0 K}{y_0 + \left(K - y_0\right)e^{-rt}}$$

Example 1: Pacific Halibut (1 of 2)

- Let y be biomass (in kg) of halibut population at time t, with r = 0.71/year and K = 80.5 x 10^6 kg. If y_0 = 0.25K, find
  - (a) biomass 2 years later
  - (b) the time t such that y(t) = 0.75K.

(a) For convenience, scale equation:

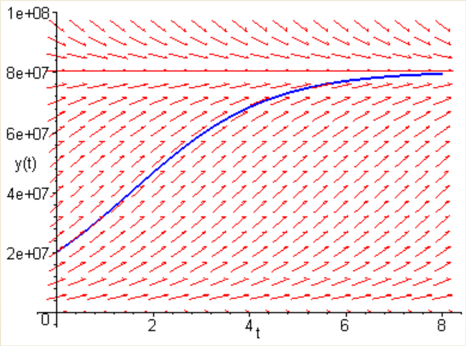
$$\frac{y}{K} = \frac{y_0/K}{y_0/K + [1 - y_0/K]e^{-rt}}$$

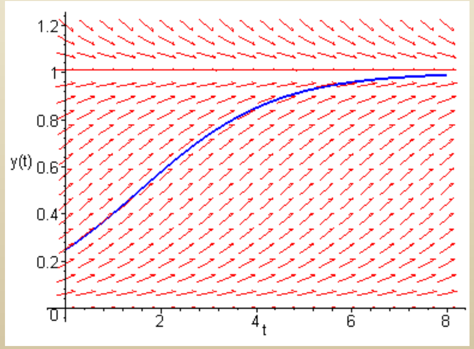
Then

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-(0.71)(2)}} \text{@}0.5797$$

and hence

$$y(2) \approx 0.5797K \approx 46.7 \times 10^6 \text{ kg}$$





Example 1: Pacific Halibut, Part (b) (2 of 2)

(b) Find time t for which y(t) = 0.75K.

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$

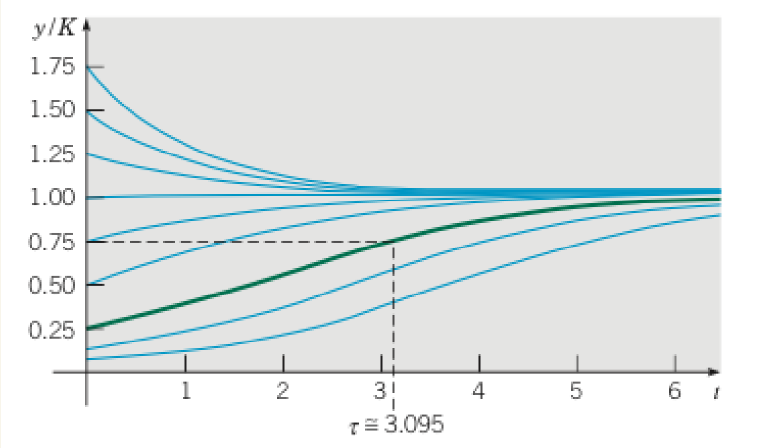
$$0.75 = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$

$$0.75\left[\frac{y_0}{K} + \left(1 - \frac{y_0}{K}\right)e^{-rt}\right] = \frac{y_0}{K}$$

$$0.75\,y_0/K + 0.75(1 - y_0/K)e^{-rt} = y_0/K$$

$$e^{-rt} = \frac{0.25\,y_0/K}{0.75(1 - y_0/K)} = \frac{y_0/K}{3(1 - y_0/K)}$$

$$t = \frac{-1}{0.71}\ln\left(\frac{0.25}{3(0.75)}\right) \approx 3.095 \text{ years}$$

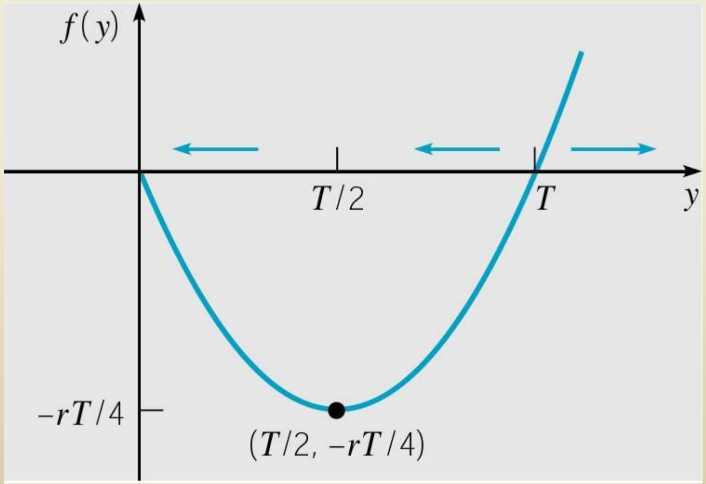


Critical Threshold Equation (1 of 2)

- Consider the following modification of the logistic ODE:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

- The graph of the right hand side $f(y)$ is given below.



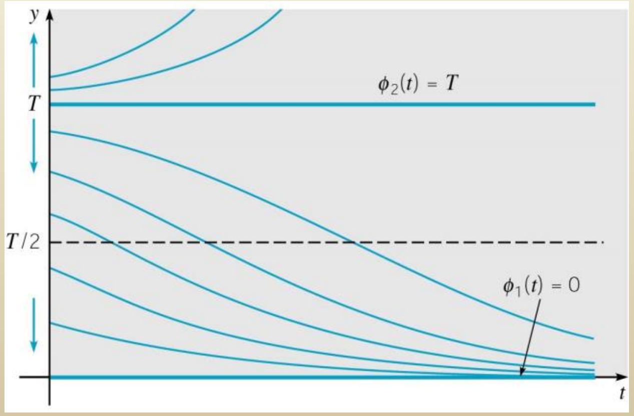
Critical Threshold Equation: Qualitative Analysis and Solution (2 of 2)

- Performing an analysis similar to that of the logistic case, we obtain a graph of solution curves shown below.
- T is a **threshold level** for y_0, in that population dies off or grows unbounded, depending on which side of T the initial value y_0 is.
- See also laminar flow discussion in text.
- It can be shown that the solution to the threshold equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

is

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

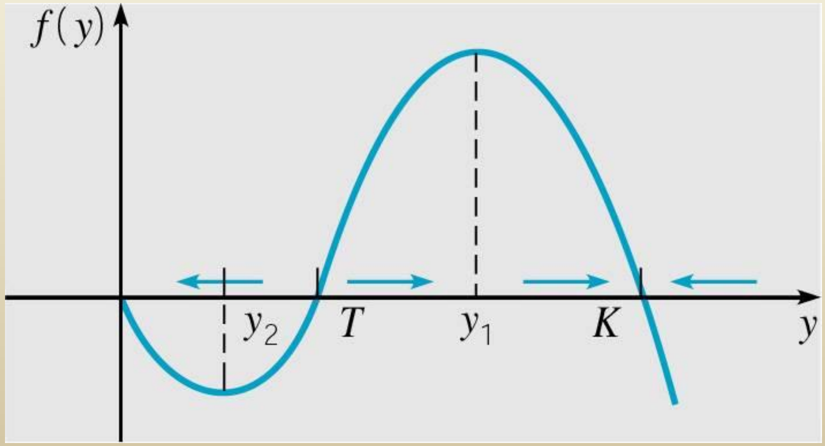


Logistic Growth with a Threshold (1 of 2)

- In order to avoid unbounded growth for $y > T$ as in previous setting, consider the following modification of the logistic equation:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y, \quad r > 0 \text{ and } 0 < T < K$$

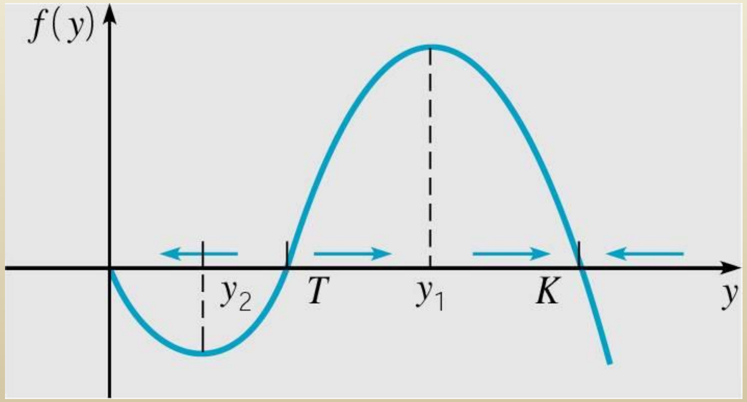
- The graph of the right hand side $f(y)$ is given below.

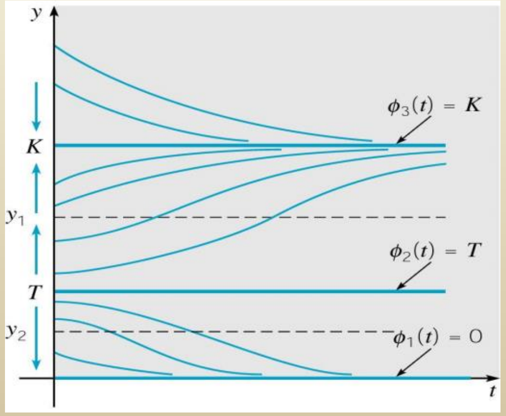


Logistic Growth with a Threshold (2 of 2)

- Performing an analysis similar to that of the logistic case, we obtain a graph of solution curves shown below right.
- T is threshold value for y_0, in that population dies off or grows towards K, depending on which side of T y_0 is.
- K is the carrying capacity level.
- Note: $y = 0$ and $y = K$ are stable equilibrium solutions,

  and $y = T$ is an unstable equilibrium solution.





Boyce/DiPrima/Meade 11th ed, Ch 2.6: Exact Equations and Integrating Factors

- Consider a first order ODE of the form

$$M(x,y)+N(x,y)y'=0$$

- Suppose there is a function $\psi(x,y)$ such that

$$\psi_x(x,y)=M(x,y),\ \psi_y(x,y)=N(x,y)$$

and such that $\psi(x,y)=c$ defines $y=\phi(x)$ implicitly. Then

$$M(x,y)+N(x,y)y'=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=\frac{d}{dx}\psi(x,\phi(x))$$

and hence the original ODE becomes

$$\frac{d}{dx}\psi(x,\phi(x))=0$$

- Thus $\psi(x,y)=c$ defines a solution implicitly.
- In this case, the ODE is said to be an **exact differential equation**.

Example 1: Exact Equation

- Consider the equation:

$$2x + y^2 + 2xyy' = 0$$

- It is neither linear nor separable, but there is a function φ such that

$$2x + y^2 = \frac{\partial \psi}{\partial y} \text{ and } 2xy = \frac{\partial \psi}{\partial x}$$

- The function that works is $\psi(x, y) = x^2 + xy^2$
- Thinking of y as a function of x and calling upon the chain rule, the differential equation and its solution become

$$\frac{d\psi}{dx} = \frac{d}{dx}(x^2 + xy^2) = 0 \Rightarrow \psi(x, y) = x^2 + xy^2 = c$$

Theorem 2.6.1

- Suppose an ODE can be written in the form

$$M(x, y) + N(x, y)y' = 0 \qquad (1)$$

where the functions M, N, M_y and N_x are all continuous in the rectangular region R: $a < x < b,\ g < y < d$. Then Eq. (1) is an **exact** differential equation if and only if

$$M_y(x, y) = N_x(x, y),\ \ \forall (x, y) \in R \qquad (2)$$

- That is, there exists a function ψ satisfying the conditions

$$\psi_x(x, y) = M(x, y),\ \ \psi_y(x, y) = N(x, y) \qquad (3)$$

if and only if M and N satisfy Equation (2).

Example 2: Exact Equation (1 of 3)

- Consider the following differential equation.

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0$$

- Then

$$M(x,y) = y\cos x + 2xe^y,\; N(x,y) = \sin x + x^2 e^y - 1$$

and hence

$$M_y(x,y) = \cos x + 2xe^y = N_x(x,y) \;\Rightarrow\; \text{ODE is exact}$$

- From Theorem 2.6.1,

$$\psi_x(x,y) = M = y\cos x + 2xe^y,\;\; \psi_y(x,y) = N = \sin x + x^2 e^y - 1$$

- Thus

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2 e^y + h(y)$$

Example 2: Solution (2 of 3)

- We have

$$\psi_x(x,y) = M = y\cos x + 2xe^y, \ \ \psi_y(x,y) = N = \sin x + x^2 e^y - 1$$

and

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2 e^y + h(y)$$

- It follows that

$$\psi_y(x,y) = \sin x + x^2 e^y - 1 = \sin x + x^2 e^y + h'(y)$$

$$\Rightarrow \ \ h'(y) = -1 \ \ \Rightarrow \ \ h(y) = -y + k$$

- Thus

$$\psi(x,y) = y\sin x + x^2 e^y - y + k$$

- By Theorem 2.6.1, the solution is given implicitly by

$$y\sin x + x^2 e^y - y = c$$

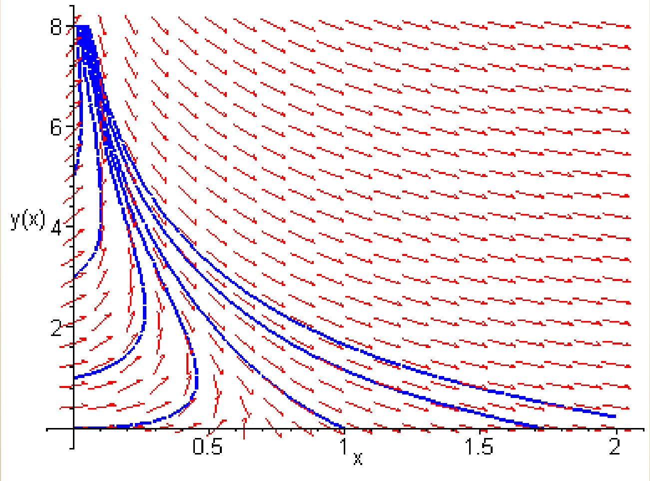
Example 2:
Direction Field and Solution Curves (3 of 3)

- Our differential equation and solutions are given by

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)y' = 0,$$

$$y\sin x + x^{2}e^{y} - y = c$$

- A graph of the direction field for this differential equation, along with several solution curves, is given below.



Example 3: Non-Exact Equation (1 of 2)

- Consider the following differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Then

$$M(x, y) = 3xy + y^2, \; N(x, y) = x^2 + xy$$

and hence

$$M_y(x, y) = 3x + 2y \neq 2x + y = N_x(x, y) \;\Rightarrow\; \text{ODE is not exact}$$

- To show that our differential equation cannot be solved by this method, let us seek a function ψ such that

$$\psi_x(x, y) = M = 3xy + y^2, \; \psi_y(x, y) = N = x^2 + xy$$

- Thus

$$\psi(x, y) = \int \psi_x(x, y)\,dx = \int \left(3xy + y^2\right)dx = \frac{3}{2}x^2 y + xy^2 + h(y)$$

Example 3: Non-Exact Equation **(2 of 2)**

- We seek ψ such that

$$\psi_x(x,y) = M = 3xy + y^2, \;\; \psi_y(x,y) = N = x^2 + xy$$

and

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int \left(3xy + y^2\right) dx = 3x^2 y/2 + xy^2 + C(y)$$

- Then

$$\psi_y(x,y) = x^2 + xy = \frac{3}{2}x^2 + 2xy + h'(y)$$

$$\Rightarrow \;\; h'(y) \overset{?}{=} -\frac{1}{2}x^2 - xy$$

- Because $h'(y)$ depends on x as well as y, there is no such function $\psi(x, y)$ such that

$$\frac{d\psi}{dx} = (3xy + y^2) + (x^2 + xy)y'$$

Integrating Factors

- It is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x,y)$:

$$M(x,y) + N(x,y)y' = 0$$

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

- For this equation to be exact, we need

$$(\mu M)_y = (\mu N)_x \Leftrightarrow M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

- This partial differential equation may be difficult to solve. If μ is a function of x alone, then $\mu_y = 0$ and hence we solve

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

provided right side is a function of x only. Similarly if μ is a function of y alone. See text for more details.

Example 4: Non-Exact Equation

- Consider the following non-exact differential equation.

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

- Seeking an integrating factor, we solve the linear equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \iff \frac{d\mu}{dx} = \frac{\mu}{x} \implies \mu(x) = x$$

- Multiplying our differential equation by μ, we obtain the exact equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0,$$

which has its solutions given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = c$$

Boyce/DiPrima/Meade 11th ed, Ch 2.7: Numerical Approximations: Euler's Method

- Recall that a first order initial value problem has the form

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

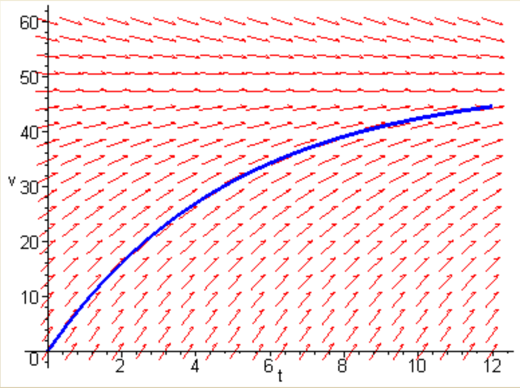
- If f and $\partial f / \partial y$ are continuous, then this IVP has a unique solution $y = f(t)$ in some interval about t_0.
- When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations.
- However, the solutions for most differential equations of this form cannot be found by analytical means.
- Therefore it is important to be able to approach the problem in other ways.

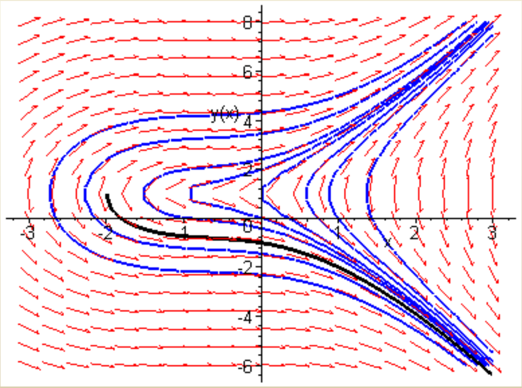
Direction Fields

- For the first order initial value problem

$$y' = f(t, y), \;\; y(t_0) = y_0,$$

we can sketch a direction field and visualize the behavior of solutions. This has the advantage of being a relatively simple process, even for complicated equations. However, direction fields do not lend themselves to quantitative computations or comparisons.





Numerical Methods

- For our first order initial value problem
$$y' = f(t, y), \ \ y(t_0) = y_0,$$
an alternative is to compute approximate values of the solution $y = f(t)$ at a selected set of t-values.
- Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
- There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8.
- In this section, we examine the **tangent line method**, which is also called **Euler's Method**.

Euler's Method: Tangent Line Approximation

- For the initial value problem

$$y' = f(t, y), \;\; y(t_0) = y_0,$$

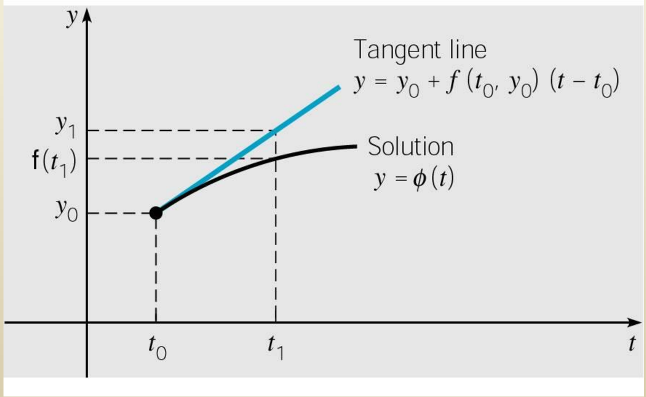
we begin by approximating solution $y = f(t)$ at initial point t_0.

- The solution passes through initial point (t_0, y_0) with slope $f(t_0,y_0)$. The line tangent to the solution at this initial point is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- The tangent line is a good approximation to solution curve on an interval short enough.
- Thus if t_1 is close enough to t_0, we can approximate $y = f(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$



Euler's Formula

- For a point t_2 close to t_1, we approximate $y = f(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- Thus we create a sequence y_n of approximations $y = f(t_n)$:

$$\begin{aligned} y_1 &= y_0 + f_0 \cdot (t_1 - t_0) \\ y_2 &= y_1 + f_1 \cdot (t_2 - t_1) \\ &\vdots \\ y_{n+1} &= y_n + f_n \cdot (t_{n+1} - t_n) \end{aligned}$$

where $f_n = f(t_n, y_n)$.

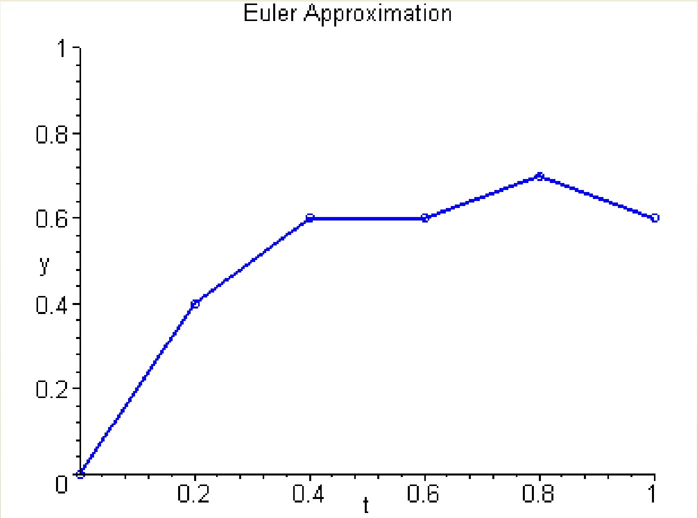
- For a uniform step size $t_{n+1} = -t_n + h$, Euler's formula becomes

$$y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots$$

Euler Approximation

- To graph an Euler approximation, we plot the points $(t_0, y_0), (t_1, y_1), \ldots, (t_n, y_n)$, and then connect these points with line segments.

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n), \text{ where } f_n = f(t_n, y_n)$$



Example 1: Euler's Method (1 of 3)

- For the initial value problem

$$\frac{dy}{dt} = 3 - 2t - 0.5y, \quad y(0) = 1$$

we can use Euler's method with h = 0.2 to approximate the solution at t = 0.2, 0.4, 0.6, 0.8, and 1.0 as shown below.

$$y_1 = y_0 + f_0 \cdot h = 1 + (3 - 0 - 0.5)(0.2) = 1 + 2.5(0.2) = 1.5$$

$$y_2 = y_1 + f_1 \cdot h = 1.5 + \left(3 - 2(0.2) - 0.5(1.5)\right)(0.2) \approx 1.87$$

$$y_3 = y_2 + f_2 \cdot h = 1.87 + \left(3 - 2(0.4) - 0.5(1.87)\right)(0.2) \approx 2.123$$

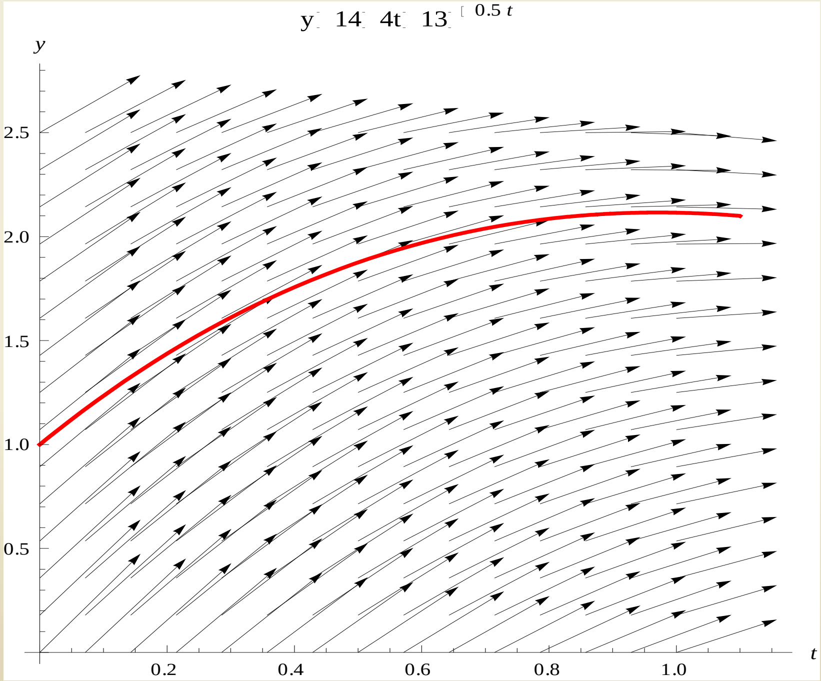
$$y_4 = y_3 + f_3 \cdot h = 2.123 + \left(3 - 2(0.6) - 0.5(2.123)\right)(0.2) \approx 2.2707$$

$$y_5 = y_4 + f_4 \cdot h = 2.2707 + \left(3 - 2(0.8) - 0.5(2.2707)\right)(0.2) \approx 2.32363$$

Example 1: Exact Solution (2 of 3)

- We can find the exact solution to our IVP, as in Chapter 2.1:

$$y' = 3 - 2t - 0.5y, \quad y(0) = 1$$
$$y' + 0.5y = 3 - 2t$$
$$e^{0.5t}y' + 0.5e^{0.5t}y = 3e^{0.5t} - 2te^{0.5t}$$
$$e^{0.5t}y = 14e^{0.5t} - 4te^{0.5t} + k$$
$$y = 14 - 4t + ke^{-0.5t}$$
$$y(0) = 1 \Rightarrow k = -13$$
$$\Rightarrow y = 14 - 4t - 13e^{-0.5t}$$



Example 1: Error Analysis (3 of 3)

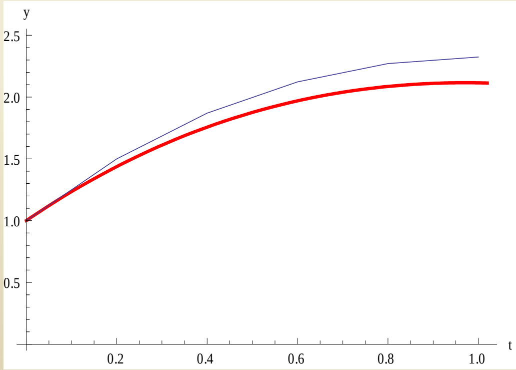
- From table below, we see that the errors start small, but get larger. This is most likely due to the fact that the exact solution is not linear on [0, 1]. Note:

$$\text{Percent Relative Error} = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

t	Exact y	Approx y	Error	% Rel Error
0	1	1	0	0
0.2	1.43711	1.5	-0.06	-4.38
0.4	1.7565	1.87	-0.11	-6.46
0..6	1.96936	2.123	-0.15	-7.8
0.8	2.08584	2.2707	-0.18	-8.86
1	2.1151	2.32363	-0.2085	-9.8591083

Exact y in red

Approximate y in blue



Example 2: Euler's Method (1 of 3)

- For the initial value problem

$$\frac{dy}{dt} = 3 - 2t - 0.5y, \quad y(0) = 1$$

we can use Euler's method with various step sizes to approximate the solution at t = 1.0, 2.0, 3.0, 4.0, and 5.0 and compare our results to the exact solution

$$y = 14 - 4t - 13e^{-0.5t}$$

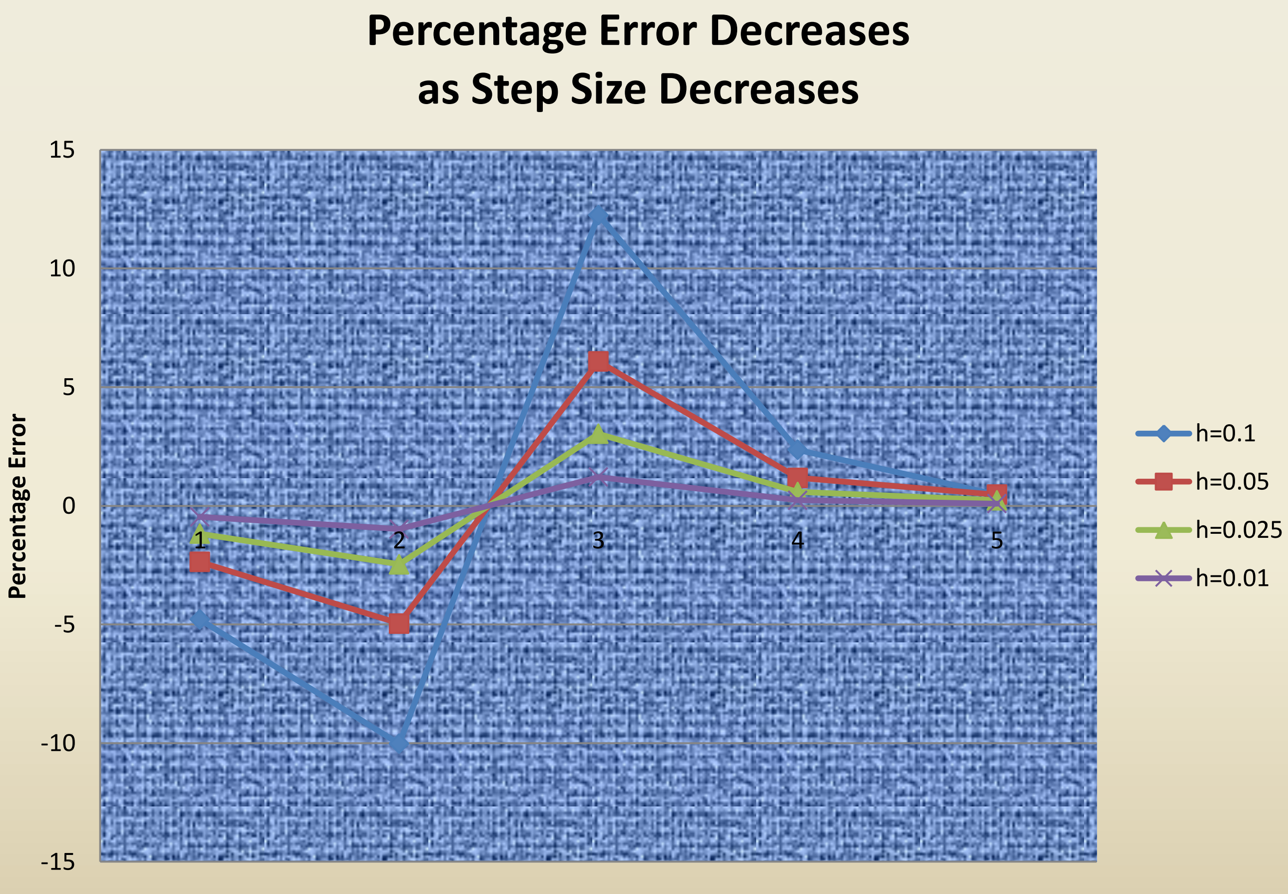
at those values of t.

Example 2: Euler's Method (2 of 3)

- Comparison of exact solution with Euler's Method for h = 0.1, 0.05, 0.25, 0.01

t	**h = 0.1**	**h = 0.05**	**h = 0.025**	**h = 0.01**	**EXACT**
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	2.2164	2.1651	2.1399	2.1250	2.1151
2.0	1.3397	1.2780	1.2476	1.2295	1.2176
3.0	–0.7903	–0.8459	–0.8734	–0.8898	–0.9007
4.0	–3.6707	–3.7152	–3.7373	–3.7506	–3.7594
5.0	–7.0003	–7.0337	–7.0504	–7.0604	–7.0671

Example 2: Euler's Method (3 of 3)



Example 3: Euler's Method (1 of 3)

- For the initial value problem

$$\frac{dy}{dt} = 4 - t + 2y, \quad y(0) = 1$$

we can use Euler's method with $h = 0.1$ to approximate the solution at $t = 1, 2, 3,$ and 4, as shown below.

$$\begin{aligned}
y_1 &= y_0 + f_0 \cdot h = 1 + \big(4 - 0 + (2)(1)\big)(0.1) = 1.6 \\
y_2 &= y_1 + f_1 \cdot h = 1.6 + \big(4 - 0.1 + (2)(1.6)\big)(0.1) = 2.31 \\
y_3 &= y_2 + f_2 \cdot h = 2.31 + \big(4 - 0.2 + (2)(2.31)\big)(0.1) \approx 3.15 \\
y_4 &= y_3 + f_3 \cdot h = 3.15 + \big(4 - 0.3 + (2)(3.15)\big)(0.1) \approx 4.15 \\
&\vdots
\end{aligned}$$

- Exact solution (see Chapter 2.1):

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

Example 3: Error Analysis (2 of 3)

- The first ten Euler approximationss are given in table below on left. A table of approximations for $t = 0, 1, 2, 3$ is given on right for $h = 0.1$. See text for numerical results with $h = 0.05, 0.025, 0.01$.
- The errors are small initially, but quickly reach an unacceptable level. This suggests a nonlinear solution.

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
0.10	1.66	1.60	0.06	3.55
0.20	2.45	2.31	0.14	5.81
0.30	3.41	3.15	0.26	7.59
0.40	4.57	4.15	0.42	9.14
0.50	5.98	5.34	0.63	10.58
0.60	7.68	6.76	0.92	11.96
0.70	9.75	8.45	1.30	13.31
0.80	12.27	10.47	1.80	14.64
0.90	15.34	12.89	2.45	15.96
1.00	19.07	15.78	3.29	17.27

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

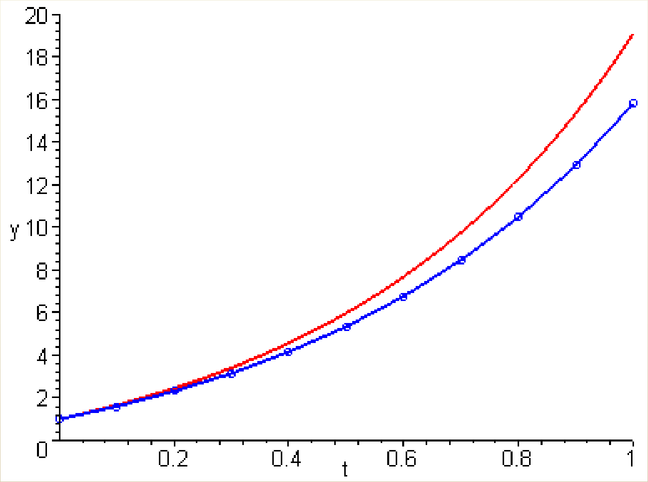
Example 3: Error Analysis & Graphs (3 of 3)

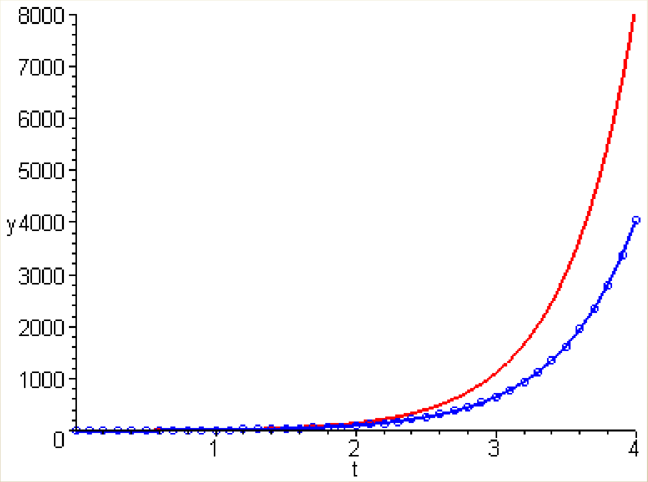
- Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$





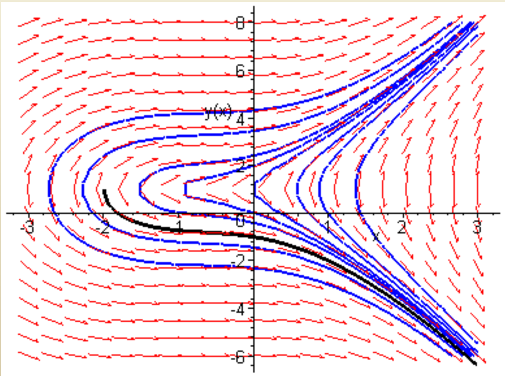
General Error Analysis Discussion **(1 of 2)**

- Recall that if f and $\partial f/\partial y$ are continuous, then our first order initial value problem

$$\frac{dy}{dt} = f(t, y), \;\; y(t_0) = y_0$$

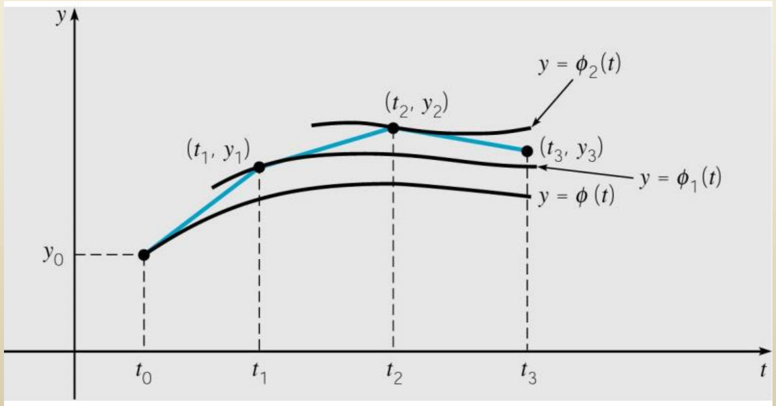
has a solution $y = \phi(t)$ in some interval about t_0.

- In fact, the equation has infinitely many solutions, each one indexed by a constant c determined by the initial condition.
- Thus $\phi(t)$ is the member of an infinite family of solutions that satisfies $\phi(t_0) = y_0$.



General Error Analysis Discussion (2 of 2)

- The first step of Euler's method uses the tangent line to f at the point (t_0, y_0) in order to estimate $f(t_1)$ with y_1.
- The point (t_1, y_1) is typically not on the graph of f, because y_1 is an approximation of $f(t_1)$.
- Thus the next iteration of Euler's method does not use a tangent line approximation to f, but rather to a nearby solution f_1 that passes through the point (t_1, y_1).
- Thus Euler's method uses a succession of tangent lines to a sequence of different solutions $f(t)$, $f_1(t)$, $f_2(t)$,... of the differential equation.



Error Bounds and Numerical Methods

- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- In our examples, we compared approximations with exact solutions. However, numerical procedures are usually used when an exact solution is not available. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- Also, direction fields are often a relatively easy first step in understanding behavior of solutions.

Boyce/DiPrima/Meade 11th ed, Ch 2.8: The Existence and Uniqueness Theorem

- The purpose of this section is to prove Theorem 2.4.2, the fundamental existence and uniqueness theorem for first order initial value problems. This theorem states that under certain conditions on $f(t, y)$, the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

  has a unique solution in some interval containing t_0 .
- First, we note that it is sufficient to consider the problem in which the point (t_0, y_0) is the origin. If some other initial point is given, we can always make a preliminary change of variables, corresponding to a translation of the coordinate axes, that will take the given point into the origin.

Theorem 2.8.1

- If f and ¶f / ¶y are continuous in a rectangle R: $|t| \leq a$, $|y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(0) = 0$$

- We will begin the proof by transforming the differential equation into an integral equation. If we suppose that there is a differentiable function $y = \phi(t)$ that satisfies the initial value problem, then $f[t, \phi(t)]$ is a continuous function of t only. Hence we can integrate $y' = f(t, y) = f(t, \phi(t))$ from the initial value $t = 0$ to an arbitrary value t, obtaining

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds$$

Proving the Theorem for the Integral Equation

- It is more convenient to show that there is a unique solution to the integral equation in a certain interval $|t| \leq h$ than to show that there is a unique solution to the corresponding differential equation. The integral equation also satisfies the initial condition.

$$\phi(t) = \int_0^t f(s, \phi(s))\,ds \Rightarrow \phi(0) = 0 \quad (s \text{ is a dummy variable})$$

- The same conclusion will then hold for the initial value problem

$$y' = f(t, y), \quad y(0) = 0$$

as holds for the integral equation.

The Method of Successive Approximations

- One method of showing that the integral equation has a unique solution is known as the method of successive approximations or Picard's iteration method. We begin by choosing an initial function that in some way approximates the solution. The simplest choice utilizes the initial condition

$$\phi_0(t) = 0$$

- The next approximation ϕ_1 is obtained by substituting $\phi_0(s)$ for $\phi(s)$ into the right side of the integral equation. Thus

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\,ds = \int_0^t f(s, 0)\,ds$$

- Similarly, $\phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds$
- And in general, $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\,ds$$

Examining the Sequence

- As described on the previous slide, we can generate the sequence $\{\phi_n\} = \phi_0, \phi_1, \phi_2, \ldots, \phi_n, \ldots$ with

$$\phi_0(t) = 0 \text{ and } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$$

- Each member of the sequence satisfied the initial condition, but in general none satisfies the differential equation. However, if for some *n* = *k*, we find $\phi_{k+1}(t) = \phi_k(t)$, then $\phi_k(t)$ is a solution of the integral equation and hence of the initial value problem, and the sequence is terminated.
- In general, the sequence does not terminate, so we must consider the entire infinite sequence. Then to prove the theorem, we answer four principal questions.

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds$$

Four Principal Questions about the Sequence

1. Do all members of the sequence $\{\phi_n\}$ exist, or may the process break down at some stage?
2. Does the sequence converge?
3. What are the properties of the limit function? In particular, does it satisfy the integral equation and hence the corresponding initial value problem?
4. Is this the only solution or may there be others?

To gain insight into how these questions can be answered, we will begin by considering a relatively simple example.

Example 1: An Initial Value Problem (1 of 6)

- We will use successive approximations to solve the initial value problem

$$y' = 2t(1+y), \quad y(0) = 0$$

- Note first that the corresponding integral equation becomes

$$\phi(t) = \int_0^t 2s(1+\phi(s)])\,ds$$

- The initial approximation $\phi_0(t) = 0$ generates the following:

$$\phi_1(t) = \int_0^t 2s(1+0)\,ds = \int_0^t 2s\;ds = t^2$$

$$\phi_2(t) = \int_0^t 2s(1+s^2)\,ds = \int_0^t (2s+2s^3)\,ds = t^2 + \frac{t^4}{2}$$

$$\phi_3(t) = \int_0^t 2s(1+s^2+s^4/2\;)\,ds = \int_0^t (2s+2s^3+s^5)\,ds = t^2 + \frac{t^4}{2} + \frac{t^6}{2\cdot 3}$$

$$\phi(t) = \int_0^t 2s[1+\phi(s)]\,ds$$

Example 1: An Inductive Proof (2 of 6)

- The evolving sequence suggests that

$$f_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \ldots + \frac{t^{2n}}{n!}$$

- This can be proved true for all $n \geq 1$ by mathematical induction. It was already established for $n = 1$ and if we assume it is true for $n = k$, we can prove it true for $n = k+1$:

$$\phi_{k+1}(t) = \int_0^t 2s\,(1+\phi_k(s)])\,ds$$

$$= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2!} + \cdots + \frac{s^{2k}}{k!}\right)ds$$

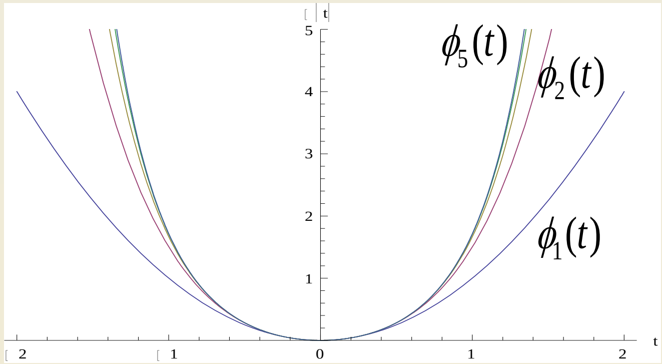
$$= t^2 + \frac{t^4}{2!} + \cdots + \frac{t^{2k}}{k!} + \frac{t^{2(k+1)}}{(k+1)!}$$

- Thus, the inductive proof is complete.

$$\phi_n(t) = t^2 + t^4/2! + t^6/3! + t^8/4! + \cdots + t^{2n}/n!$$

Example 1: The Limit of the Sequence (3 of 6)

- A plot of the first five iterates suggests eventual convergence to a limit function:



- Taking the limit as n→∞ and recognizing the Taylor series and the function to which it converges, we have:

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t^{2k}}{k!} = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1$$

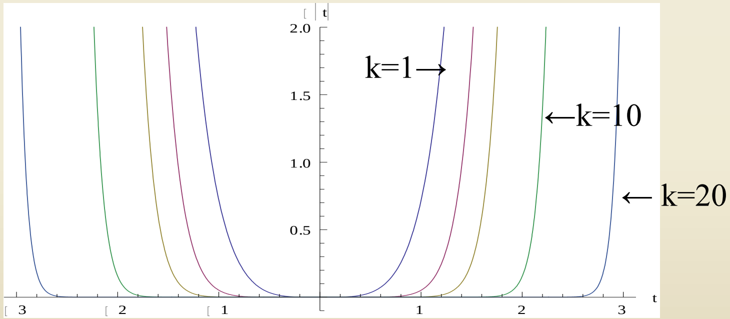
$$\lim_{n\to\infty} \phi_n(t) = e^{t^2} - 1$$

Example 1: The Solution (4 of 6)

- Now that we have an expression for

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t^{2k}}{k!} = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = e^{t^2} - 1$$

let us examine $\phi(t) - \phi_k(t)$ for increasing values of *k* in order to get a sense of the interval of convergence:



- The interval of convergence increases as *k* increases, so the terms of the sequence provide a good approximation to the solution about an interval containing $t = 0$.

$$\phi(t) = \int_0^t 2s[1+\phi(s)]\,ds$$

Example 1: The Solution Is Unique (5 of 6)

- To deal with the question of uniqueness, suppose that the IVP has two solutions $\phi(t)$ and $\psi(t)$. Both functions must satisfy the integral equation. We will show that their difference is zero:

$$|\phi(t)-\psi(t)| = \left| \int_0^t 2s[1+\phi(s)]\,ds - \int_0^t 2s[1+\psi(s)]\,ds \right|$$

$$= \left| \int_0^t 2s[\phi(s)-\psi(s)]\,ds \right| \le \int_0^t 2s|\phi(s)-\psi(s)|\,ds$$

$$\le A\int_0^t |\phi(s)-\psi(s)|\,ds$$

For the last inequality, we restrict t to $0 \le t \le A/2$, where A is arbitrary, then $2t \le A$.

$$\left|\phi(t)-\psi(t)\right| \le A\int_0^t \left|\phi(s)-\psi(s)\right| ds$$

Example 1: The Solution Is Unique (6 of 6)

- It is now convenient to define a function U such that

$$U(t)=\int_0^t \left|\phi(s)-\psi(s)\right| ds$$

- Notice that $U(0) = 0$ and $U(t) \ge 0$ for $t \ge 0$ and $U(t)$ is differentiable with $U'(t)=\left|\phi(t)-\psi(t)\right|$. This gives:

  $U'(t)$ - $AU(t)$ £ 0 and multiplying by e^{-At}

  $(e^{-At}U(t))'$ £ 0 Þ $e^{-At}U(t)$ £ 0 Þ $U(t)$ £ 0 for t ³ 0

- The only way for the function $U(t)$ to be both greater than and less than zero is for it to be identically zero. A similar argument applies in the case where $t \le 0$. Thus we can conclude that our solution is unique.

$$y' = f(t, y), \quad y(0) = 0$$

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\,ds$$

Theorem 2.8.1: The First Step in the Proof

- Returning to the general problem, do all members of the sequence exist? In the general case, the continuity of *f* and its partial with respect to *y* were assumed only in the rectangle *R*: $|t| \leq a$, $|y| \leq b$. Furthermore, the members of the sequence cannot usually be explicitly determined.
- A theorem from calculus states that a function continuous in a closed region is bounded there, so there is some positive number *M* such that $|f(t,y)| \leq M$ for (t, y) in R.
- Since $\phi_n(0) = 0$ and $\phi_n'(t) = f(t, \phi_n(t)) \leq M$, the maximum slope for any function in the sequence is *M*. The graphs on page 88 of the text indicate how this may impact the interval over which the solution is defined.

$$y' = f(t, y), \quad y(0) = 0$$

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds$$

Theorem 2.8.1: The Second Step in the Proof

- The terms in the sequence $\{\phi_n\}$ can be written in the form

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$$

and $\lim_{n \to \infty} \phi_n(t) = \phi_1(t) + \sum_{k=1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)]$

- The convergence of this sequence depends on being able to bound the value of $\left|\phi_{k+1}(t) - \phi_k(t)\right|$. This can be established based on the fact that $\partial f / \partial y$ is continuous over a closed region and hence bounded there. Problems 15 through 18 in the text lead you through this validation.

$$y' = f(t, y), \quad y(0) = 0$$

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\,ds$$

Theorem 2.8.1: The Third Step in the Proof

- There are details in this proof that are beyond the scope of the text. If we assume uniform convergence of our sequence over some interval $|t| \le h \le a$ and the continuity of f and its first partial derivative with respect to y for $|t| \le h \le a$, the following steps can be justified:

$$\phi(t) = \lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_n(s))\,ds$$

$$= \int_0^t \lim_{n\to\infty} f(s, \phi_n(s))\,ds = \int_0^t f(s, \lim_{n\to\infty} \phi_n(s))\,ds$$

$$= \int_0^t f(s, \phi(s))\,ds$$

$$y' = f(t, y), \quad y(0) = 0$$

$$y = \phi(t) = \int_0^t f[s, \phi(s)]\, ds$$

Theorem 2.8.1: The Fourth Step in the Proof

- The steps outlined establish the fact that the function $\phi(t)$ is a solution to the integral equation and hence to the initial value problem. To establish its uniqueness, we would follow the steps outlined in Example 1.
- We conjecture that the IVP has two solutions: $\phi(t)$ and $\psi(t)$. Both functions have to satisfy the integral equation and we show that their difference is zero using the inequality:

$$\left|\phi(t) - \psi(t)\right| \le A \int_0^t \left|\phi(s) - \psi(s)\right| ds$$

- If the assumptions of this theorem are not satisfied, you cannot be guaranteed a unique solution to the IVP. There may be no solution or there may be more than one solution.

Boyce/DiPrima/Meade 11th ed, Ch 2.9: First Order Difference Equations

Elementary Differential Equations and Boundary Value Problems, 11th edition, by William E. Boyce, Richard C. DiPrima, and Doug Meade ©2017 by John Wiley & Sons, Inc.

- Although a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more appropriate. Examples of this include accounts where interest is paid or charged monthly rather than continuously, applications involving drug dosages, and certain population growth problems where the population one year depends on the population in the previous year. For example,

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \ldots$$

- Notice here that the independent variable *n* is discrete. Such equations are classified according to order, as linear or nonlinear, as homogeneous or nonhomogeneous. There is frequently an initial condition describing the first term y_0 .

Difference Equation and Equilibrium Solution

- Assume for now that the state at year n +1 depends only on the state at year n, and not on the value of n itself

$$y_{n+1} = f(n, y_n), \qquad n = 0,\ 1,\ 2,\ \ldots$$

- Then

$$y_1 = f(y_0),\ y_2 = f(y_1) = f(f(y_0)),\ y_3 = f(y_2) = f^3(y_0), \ldots, y_n = f^n(y_0)$$

- This procedure is referred to as iterating the difference and it is often of interest to determine the behavior of y_n as $n \to \infty$.
- An **equilibrium solution** exists when

$$y_n = f(y_n)$$

and this is often of special interest, just as it is in differential equations.

Linear Homogeneous Difference Equations

- Suppose that the population of a certain species in a region in year $n+1$ is a positive multiple of the population in year n:

$$y_{n+1} = \rho_n y_n, \quad n = 0, 1, 2, \ldots$$

- Notice that the reproduction rate may differ from year to year.

$$y_1 = \rho_0 y_0,\ y_2 = \rho_1 y_1 = \rho_1 \rho_0 y_0,\ \cdots,\ y_n = \rho_{n-1} \cdots \rho_1 \rho_0 y_0$$

- If the reproduction rate has the same value ρ for all n:

$$y_n = \rho^n y_0$$

- If the initial value y_0 is zero, then the equilibrium solution = 0
- Otherwise

$$\lim_{n\to\infty} y_n = \begin{cases} 0, & \text{if } |r| < 1; \quad \Rightarrow \text{asymptotically stable} \\ y_0, & \text{if } r = 1; \\ \text{does not exist,} & \text{otherwise.} \quad \Rightarrow \text{asymptotically unstable} \end{cases}$$

Adding/Subtracting a Term to the Equation

- Suppose we have a net increase in population each year:

$$y_{n+1} = r\,y_n + b_n, \quad n = 0,\ 1,\ 2,\ \ldots$$

- Then iterating this:

$$y_1 = r\,y_0 + b_0,$$
$$y_2 = r\,y_1 + b_1 = r(r\,y_0 + b_0) = r^2 y_0 + r\,b_0 + b_1,$$
$$y_3 = r\,y_2 + b_2 = r(r^2 y_0 + r\,b_0 + b_1) + b_2 = r^3 y_0 + r^2 b_0 + r\,b_1 + b_2, \ldots$$

$$y_n = r^n y_0 + r^{n-1} b_0 + \ldots + r\,b_{n-2} + b_{n-1} = r^n y_0 + \sum_{j=0}^{n-1} r^{n-1-j} b_j$$

- If the migration is constant (b) each year:

$$y_n = \rho^n y_0 + (\rho^{n-1} + \cdots + \rho + 1)\,b$$

- And as long as $\rho \neq 1$, we can use the geometric series formula to get:

$$y_n = r^n y_0 + \frac{1 - r^n}{1 - r} b \quad = r^n\left(y_0 - \frac{b}{1 - r}\right) + \frac{b}{1 - r}$$

$$y_n = \rho^n\left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}$$

Conditions for an Equilibrium

- Letting $n \to \infty$ in the equation for y_n we get:

$$\lim_{n\to\infty} y_n = \left[\lim_{n\to\infty} \rho^n\right]\left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}$$

- Recall that $\rho \neq 1$. If it were, the sequence would become:

$$y_n = y_0 + nb \to \infty \text{ as } n \to \infty$$

- If $|\rho| < 1$, $\lim_{n\to\infty} \rho^n = 0$, so $y_n \to \frac{b}{1-\rho}$, an equilibrium solution.

- If $|\rho| > 1$ or if $\rho = -1$, $\lim_{n\to\infty} \rho^n$ does not exist, so the $\lim_{n\to\infty} y_n$ fails to exist unless

$$y_0 = \frac{b}{1-\rho} \Rightarrow \text{ the solution starts at its equilibrium and stays there.}$$

$$y_n = \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}$$

Example 1: Extending the Model

- If we have a \$10,000 car loan at an annual interest rate of 12%, and we wish to pay it off in four years by making **monthly payments (–*b*)**, we can adapt the previous result as follows:

$$y_n = \text{loan balance (\$) in the } n^{\text{th}} \text{ month, } y_0 = 10{,}000$$

$$r = 1 + \frac{0.12}{12} = 1.01,\ 1 - r = -0.01,\ \frac{b}{1-r} = -100b$$

$$y_n = r^n \left(y_0 - \frac{b}{1-r} \right) + \frac{b}{1-r} = 1.01^n (10{,}000 + 100b) - 100b$$

- To pay the loan off in four years, we set $y_{48} = 0$ and solve for ***b*:**

$$y_{48} = 1.01^{48}(10{,}000 + 100b) - 100b = 0 \Rightarrow$$

$$b = -100 \frac{1.01^{48}}{1.01^{48} - 1} \approx -263.34$$

- The total amount paid on the loan is 48(263.34)=\$12,640.32, so the amount of interest paid is \$2640.32.

Nonlinear Difference Equations

- As is the case with differential equations, nonlinear difference equations are much more complicated and have much more varied solutions than linear equations.
- We will analyze only the logistic equation, which is similar to the logistic differential equation discussed in 2.5.

$$y_{n+1} = r\, y_n (1 - \frac{y_n}{k}), \quad n = 0, 1, 2, \ldots$$

$$\text{Letting } u_n = y_n / k \;, \; u_{\text{n+1}} = r\, u_n (1 - u_n)$$

- Seeking the equilibrium solution yields:

$$\mathrm{u_n} = \rho u_n (1 - u_n) = \rho u_n - \rho u_n{}^2 \Rightarrow$$

$$\mathrm{u_n} = 0 \text{ or } \mathrm{u_n} = \frac{\rho - 1}{\rho}$$

- Are either of these equilibrium solutions asymptotically stable?

$$u_n = 0 \text{ or } u_n = \frac{\rho - 1}{\rho}$$

Examining Points Near Equilibrium Solutions

- For the first equilibrium solution of zero, the quadratic term ≈ 0:

$$\text{Near } u_n = 0,\ u_{n+1} = \rho u_n - \rho u_n^{2} \approx \rho u_n \Rightarrow u_{n+1} = \rho u_n$$

- We have already examined this equation and concluded that for $|\rho| < 1$, the solution is asymptotically stable.
- We will now consider solutions near the second equilibrium:

$$\text{Let } u_n = \frac{\rho - 1}{\rho} + v_n \text{ where } v_n \text{ is assumed to be small so quadratic term} \approx 0,$$

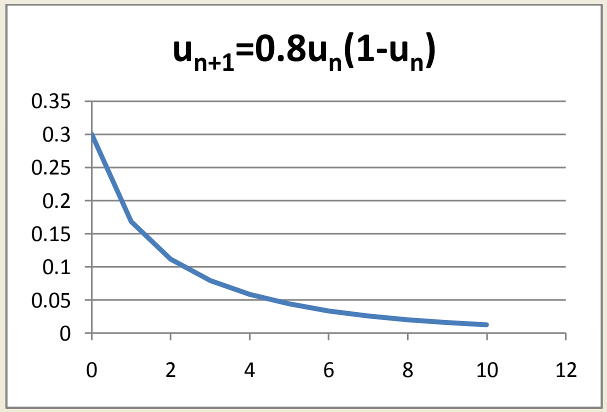
$$\Rightarrow v_{n+1} = (2 - \rho) v_n - \rho v_n^{2} \approx (2 - \rho) v_n \quad \left(\text{after simplifying } u_{n+1} \text{ expression}\right)$$

$$\Rightarrow v_{n+1} = (2 - \rho) v_n$$

- From our previous discussion, we can conclude that $v_n \to 0$ provided $|2 - \rho| < 1$ or $1 < \rho < 3$. So, for these values of ρ, we can conclude that the solution is asymptotically stable.

Solutions for Varying Initial States and Parameter Values Between 0 and 3 (1 of 2)

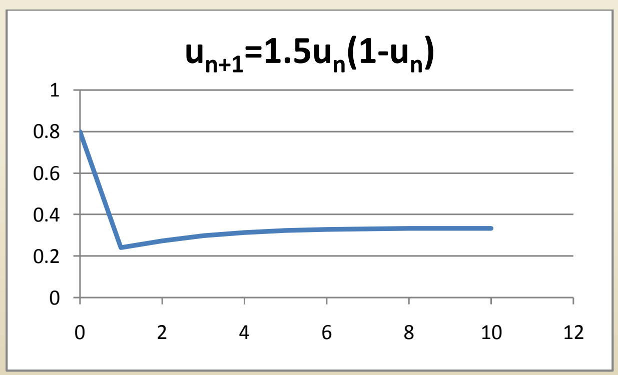
n	u_n
0	0.3
1	0.168
2	0.111821
3	0.079454
4	0.058513
5	0.044071
6	0.033703
7	0.026054
8	0.0203
9	0.01591
10	0.012526



$$y_0 = 0.3,\ \rho = 0.8$$

$$|\rho| < 1 \Rightarrow u_n \to 0$$

n	u_n
0	0.8
1	0.24
2	0.2736
3	0.298115
4	0.313863
5	0.32303
6	0.328022
7	0.330636
8	0.331974
9	0.332651
10	0.332991

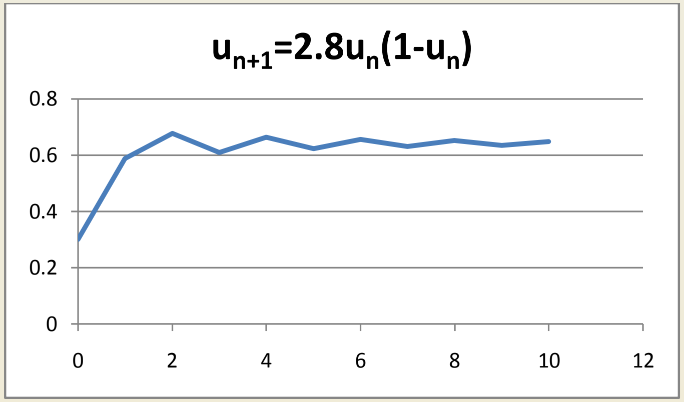


$$y_0 = 0.8,\ \rho = 1.5$$

$$1 < \rho < 3 \Rightarrow u_n \to \frac{\rho - 1}{\rho} = \frac{0.5}{1.5} = \frac{1}{3}$$

Solutions for Varying Initial States and Parameter Values Between 0 and 3 (2 of 2)

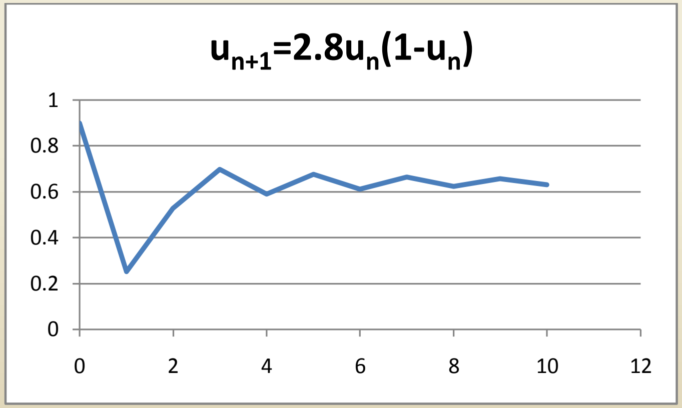
n	u_n
0	0.3
1	0.588
2	0.678317
3	0.610969
4	0.665521
5	0.623288
6	0.65744
7	0.630595
8	0.652246
9	0.6351
10	0.648895



$$y_0 = 0.3,\ \rho = 2.8$$

$$1 < \rho < 3 \Rightarrow u_n \to \frac{\rho - 1}{\rho} = \frac{1.8}{2.8} \approx 0.6429$$

n	u_n
0	0.9
1	0.252
2	0.527789
3	0.697838
4	0.590409
5	0.677114
6	0.612166
7	0.664773
8	0.62398
9	0.656961
10	0.631017

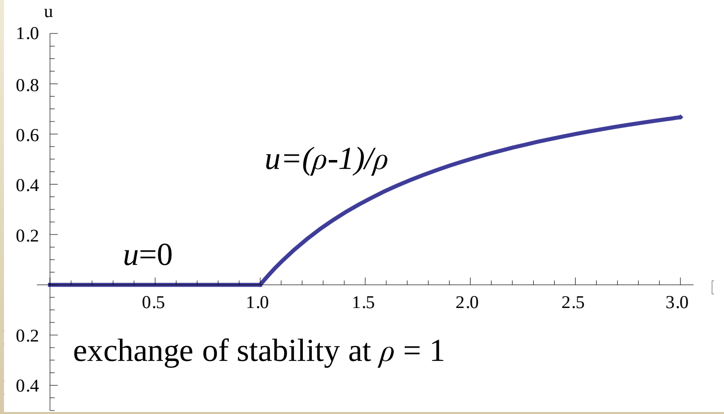


$$y_0 = 0.9,\ \rho = 2.8$$

$$\rho = 2.8 \text{ as above} \Rightarrow u_n \to 0.6429$$

Summary of Asymptotic Stability Intervals

- We found that the difference equation $u_{n+1} = \rho u_n (1 - u_n)$ has two equilibrium solutions: $u_n = 0$ or $u_n = \frac{r - 1}{r}$
- Considering nonnegative values of the parameter ρ, the first equilibrium solution required that $0 \leq \rho < 1$, while the second equilibrium solution required that $1 < \rho < 3$. there is an **exchange of stability** from one equilibrium solution to the other at $\rho = 1$. This is demonstrated in the chart below:

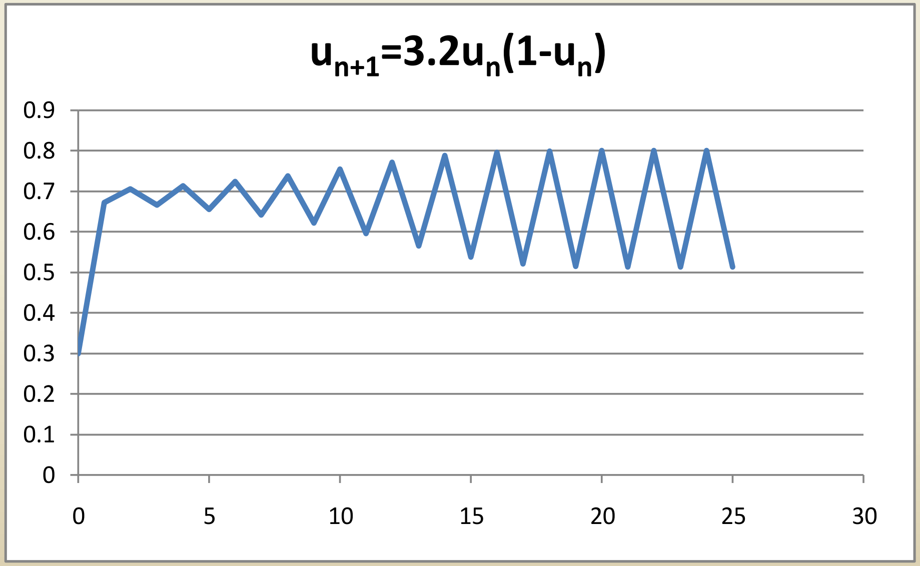


Solutions of the Difference Equation That Do Not Approach an Equilibrium (1 of 4)

n	u_n
0	0.3
1	0.672
2	0.705331
3	0.665085
4	0.71279
5	0.655105
6	0.723016
7	0.640845
8	0.736521
9	0.620986
10	0.75316
11	0.594912
12	0.771173
13	0.564688
14	0.78661
15	0.537136
16	0.795587
17	0.520411
18	0.798667
19	0.514554
20	0.799322
21	0.5133
22	0.799434
23	0.513086
24	0.799452

Below is an example of a 2-cycle. Notice how as *n* increases, the value of u_n alternates between two values (0513 and 0.799).

ρ = 3.2 and y_0 = 0.3

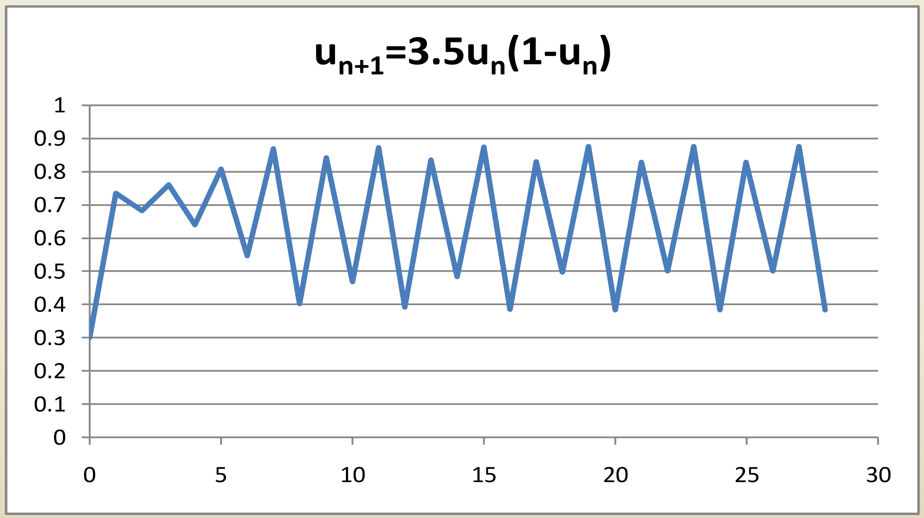


Solutions of the Difference Equation That Do Not Approach an Equilibrium (2 of 4)

n	u_n
n	***u(n)***
0	0.3
1	0.735
2	0.681713
3	0.759432
4	0.639433
…	…
12	0.392152
13	0.834291
14	0.483873
15	0.87409
16	0.385199
17	0.828873
18	0.49645
19	0.874956
20	0.382928
21	0.82703
22	0.50068
23	0.874998
24	0.382817
25	0.826938
26	0.50089
27	0.874997
28	0.38282

Below is an example of a 4-cycle. Notice how as *n* increases, the value of u_n alternates between four values (0.3828, 0.5009, 0.8269, 0.8750).

$\rho = 3.5$ and $y_0 = 0.3$

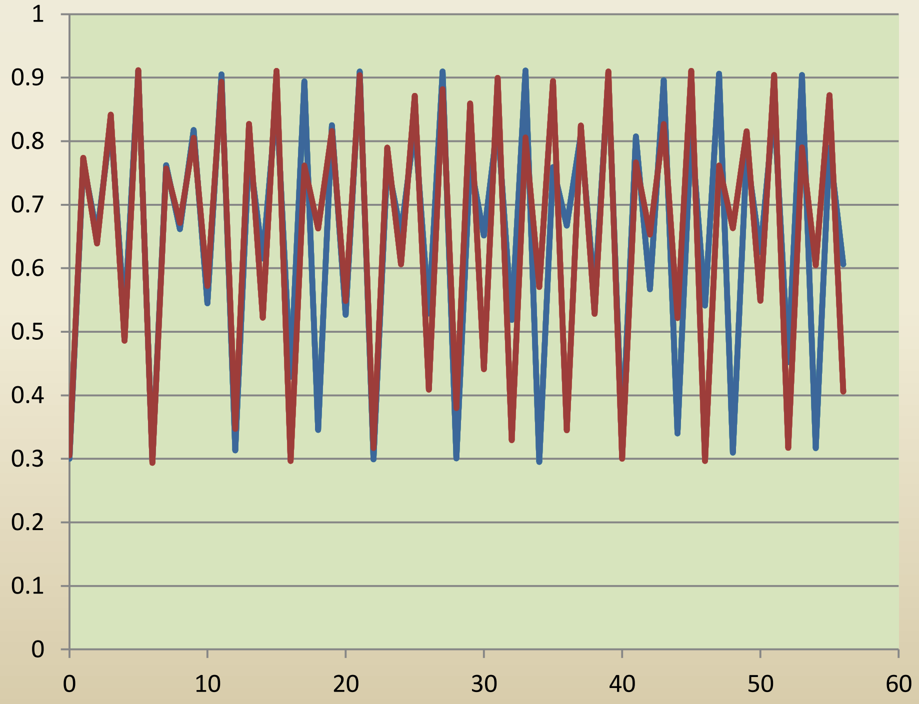


Solutions of the Difference Equation That Do Not Approach an Equilibrium (3 of 4)

- Notice from the preceding graphs how the behavior of the solution to the difference equation $u_{n+1} = r\,u_n(1 - u_n)$ behaves rather unpredictably when $\rho > 3$. First, at $\rho = 3.2$, we saw the sequence oscillate between two values, creating a period of two. Then, at $\rho = 3.5$, the terms in the sequence were oscillating among four values, creating a period of 4. It is actually around $\rho = 3.449$ that this doubling of the period occurs and this is called a point of **bifurcation**. As ρ increases slightly further, periodic solutions of period 8, 16, … occur.
- By the time we reach $\rho > 3.57$, the solutions possess some regularity, but no discernible detailed pattern is present for most values of ρ. The term **chaotic** is used to describe this situation. One of the features of chaotic solutions is extreme sensitivity to the initial conditions. This is demonstrated on the following slide.

Chaotic Solutions (4 of 4)

- Below are two solutions to $u_{n+1} = 3.65u_n(1-u_n)$
- The gray solution corresponds to the initial state $y_0 = 0.300$
- The brown solution corresponds to the initial state $y_0 = 0.305$



What Chaotic Solutions May Suggest

- On the basis of Robert May's analysis of the nonlinear equation we have considered

$$u_{n+1} = r\,u_n(1 - u_n) \quad \text{and similarly} \quad y' = ry(1 - y)$$

  as a model for the population of certain insect species, we might conclude that if the growth rate ρ is too large, it will be impossible to make effective long-range predictions about these insect populations.
- It is increasingly clear that chaotic solutions are much more common than was suspected at first, and that they may be part of the investigation of a wide range of phenomena.